Bohr's phenomenon for functions on the Boolean cube

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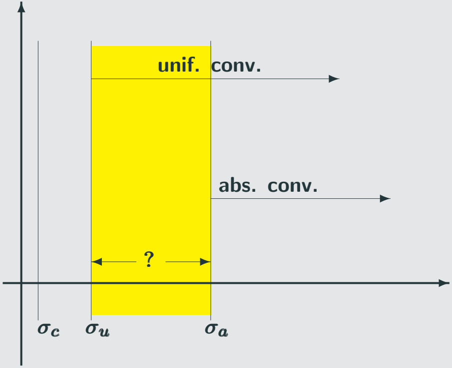
Workshop: Valencia 2017

Bohr radii and the BH-inequality on the polytorus

Point of departure: Bohr strips for Dirichlet series $D(s) = \sum_n a_n n^{-s}$



Bohr's power series theorem, 1914

For each $f \in H_\infty(\mathbb{D})$

$$\sum_{k=0}^{\infty} \frac{|f^{(k)}(0)|}{k!} \frac{1}{3^k} \leq \|f\|_{\mathbb{D}} \,,$$

and the radius $r = \frac{1}{3}$ is optimal.

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In terms of Fourier analysis ...

For each $f \in H_\infty(\mathbb{T})$

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Definition – Niels Bohr radius

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Definition – Harald Bohr radius: Given $N \in \mathbb{N}$

$$K^N := \sup\left\{0 < r < 1 : \sum_{\alpha \in \mathbb{N}_0^N} |\widehat{f}(\alpha)|\, r^\alpha \leq \|f\|_{\mathbb{T}^N},\ f \in H_\infty(\mathbb{T}^N)\right\}$$

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Bohr radius of $\mathcal{F} \subset H_\infty(\mathbb{T}^N)$

$$K(\mathcal{F}) := \sup \left\{ 0 < r < 1 : \sum_{\alpha \in \mathbb{N}_0^N} |\widehat{f}(\alpha)| \, r^\alpha \leq \|f\|_{\mathbb{T}^N}, \ f \in \mathcal{F} \right\}$$

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Let

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Then:

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- $\sum_{k=0}^{\infty} |\hat{f}(k)| \frac{1}{3^k} \leq |\hat{f}(0)| + 2(1 - |\hat{f}(0)|) \sum_{k=1}^{\infty} \frac{1}{3^k} = 1$

Challenges

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functions	Bohr radius
$H_\infty(\mathbb{T}^N)$	K^N
$\mathcal{P}(\mathbb{T}^N)$	K^N_{pol}
$P_{\mathsf{hom}}(\mathbb{T}^N)$	K^N_{hom}
$\mathcal{P}_{\leq d}(\mathbb{T}^N)$	$K^N_{\leq d}$
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Using Wiener's argument in several variables....

- $K^N \sim K^N_{\mathrm{pol}} \sim K^N_{\mathrm{hom}}$
- $K^N_{\leq d} \sim K^N_{=d}$

$$\lim_{N\to\infty} \frac{K^N_{\mathsf{hom}}}{\sqrt{\frac{\log N}{N}}} = 1$$

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Crucial step

$$K^N_{=d} \sim \left(\frac{d}{N}\right)^{\frac{d-1}{2d}}$$

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In terms of Fourier analysis ...

Let $S(N,d) =$ Sidon constant of all characters z^α, $|\alpha| = d$ on the group $\mathbb{T}^N$. Then

$$S(n,d) \;\sim\; \left(\frac{d}{N}\right)^{\frac{d-1}{2d}}$$

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$$K^N_{\leq d} \sim \left(\frac{d}{N}\right)^{\frac{d-1}{2d}}$$

Crucial tool – BH-inequality, 1931

For each $d \in \mathbb{N}$ there is a (best) constant $\mathrm{BH}_{\mathbb{T}}^{\leq d}$ such that every degree-d polynomial $f : \mathbb{C}^N \to \mathbb{C}$

$$\Big(\sum_{|\alpha| \leq d} |\widehat{f}(\alpha)|^{\frac{2d}{d+1}} \Big)^{\frac{d+1}{2d}} \leq \mathrm{BH}_{\mathbb{T}}^{\leq d} \, \|f\|_{\mathbb{T}^N} .$$

Moreover, the exponent is optimal.

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Recall

$$\lim_{N \to \infty} \frac{K^N}{\sqrt{\frac{\log N}{N}}} = 1$$

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Uppper estimate: Kahane-Salem-Zygmund inequality

$$\overline{\lim}_{N\to\infty} \frac{K^N}{\sqrt{\frac{\log N}{N}}} \leq 1$$

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Moreover, the exponent is optimal.

Lower estimate: BH-inequality plus Wiener's technique...

$$\frac{1}{\overline{\lim}_{d \to \infty} \sqrt[d]{\mathrm{BH}_{\mathbb{T}}^{\leq d}}} \leq \underline{\lim}_{N \to \infty} \frac{K^N}{\sqrt{\frac{\log N}{N}}}$$

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The quality of these constants improved over the recent years:

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Queffelec, 1995

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Defant-Frerick-Ortega-Ounaies-Seip, 2011

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Bayart-Pellegrino-Seoane, 2014

$$\mathrm{BH}_{\mathbb{T}}^{\leq d} \leq C^{\sqrt{d \log d}}$$

In particular,

$$\overline{\lim}_d \sqrt[d]{\mathrm{BH}_{\mathbb{T}}^{\leq d}} = 1$$

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The real case? To focus on constants means new difficulties...

Real BH-inequality

For each $d \in \mathbb{N}$ there is a (best) constant $\mathrm{BH}^{\leq d}_{[-1,1]}$ such that every degree-d polynomial $f : \mathbb{R}^N \to \mathbb{R}$

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How to prove this?

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Constants?

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Surprising?

$$\mathrm{BH}^{\leq d}_{[-1,1]} \neq \mathrm{BH}^{=d}_{[-1,1]}$$

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More precisely:

- $\overline{\lim}_d \sqrt[d]{\mathrm{BH}^{=d}_{[-1,1]}} = 2$
- $\overline{\lim}_d \sqrt[d]{\mathrm{BH}^{\leq d}_{[-1,1]}} = 1 + \sqrt{2}$

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Comparison of sup norms

- $\|P\|_{\mathbb{T}^N} \leq (1+\sqrt{2})^d \|P\|_{[-1,1]^N}$
- $\|P\|_{\mathbb{T}^N} \leq 2^{d-1} \|P\|_{[-1,1]^N}$ if P is d-homogeneous

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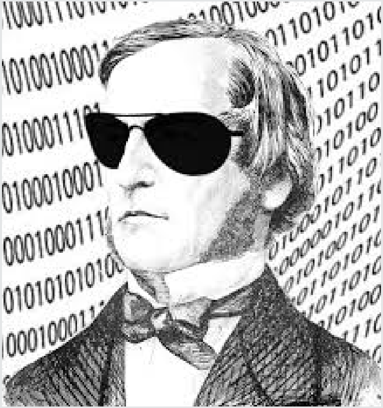
Cauchy type estimates

For the m-homogeneous part P_m of P

$$\|P_m\|_{[-1,1]^N} \leq (1+\sqrt{2})^d \|P\|_{[-1,1]^N}$$

Boolean radii and the BH-inequality on the Boolean cube

George



Analysis of functions on the Boolean cube

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- Theoretical computer sciences
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Example – majority function

$$\mathsf{Maj}(x) = \mathsf{sign}(x_1 + \ldots + x_N)$$

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- Fourier expansion: $f(x) = \sum_{S \subset [N]} \widehat{f}(S)\, x^S$ with $\widehat{f}(S) = \mathbb{E}[f \cdot x^S]$

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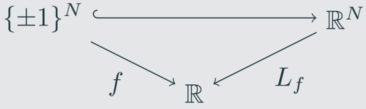
Degree-d functions and d-homogeneous functions on $\{\pm 1\}^N$...

$$d = \max\{|S| \colon \hat{f}(S) \neq 0\}$$

Tetrahedral polynomials



$$\sup_{x \in \{\pm 1\}^N} |f(x)| = \sup_{x \in [-1,1]^N} |L_f(x)|$$

Compare – Fourier analysis of functions on the polytorus $\mathbb{T}^N$

- $G = \mathbb{T}^N$ compact abelian group
- Haar measure = normalized Lebesgue measure
- Dual group: $z^\alpha : \mathbb{T}^N \to \mathbb{T}\,,\ z \mapsto \prod_{n\in\mathbb{N}} z_n^{\alpha_n}$ with $\alpha \in \mathbb{Z}_0^N$
- Expectation: $\mathbb{E}[f] := \int_{\mathbb{T}^N} f(z)dz$ for $f \in L_1(\mathbb{T}^N)$
- Fourier expansion: $f(z) \sim \sum_\alpha \widehat{f}(\alpha)\, z^\alpha$ with $\widehat{f}(\alpha) = \mathbb{E}[f \cdot z^{-\alpha}]$

Finding the coefficients may not be easy:

For N odd

$$\hat{\mathsf{Maj}}(S) = \begin{cases} 0 & |S| \quad \text{even} \\ (-1)^{\frac{k-1}{2}} \frac{1}{2^{N-1}} \binom{N-1}{\frac{N-1}{2}} \binom{\frac{N-1}{2}}{\frac{k-1}{2}} \binom{N-1}{k-1}^{-1} & |S| = k \ \text{odd} \end{cases}$$

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Definition – Boolean radius

$\mathcal{F}$ a subset of functions on $f : \{\pm 1\}^N \to \mathbb{R}$

$$\rho(\mathcal{F}) := \sup \Big\{ 0 < \rho : \sum_{S \subset [N]} |\widehat{f}(S)| \, \rho^{|S|} \leq \|f\|_{\{\pm 1\}^N} \ \text{ for all } f \in \mathcal{F} \Big\}$$

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  More precisely, are Wiener's techniques or BH-inequalities still useful in the Boolean world? As e.g. hypercontractivity arguments are essential in both worlds!

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  More precisely, are Wiener's techniques or BH-inequalities still useful in the Boolean world? As e.g. hypercontractivity arguments are essential in both worlds!
- Is there hope to connect Bohr's world with the hot topic of quantum information theory, e.g., XOR games, AA-conjecture ....

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- $\mathcal{B}^N_{\leq d} :=$ all degree-d fct.
- $\mathcal{B}^N_{=d} :=$ all d- homo. fct.
- $\mathcal{B}^N_{\text{hom}} :=$ all homo. fct.

Moreover, for $0 \leq \delta \leq 1$

- $\mathcal{B}^N_{\delta} :=$ all fct. on $\{\pm 1^N\}$ with $|\mathbb{E}[f]| \leq (1-\delta)\|f\|_\infty$

Two out of four...

Theorem

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Take $f : \{\pm 1\}^N \to [-1, 1]$, and show first that $\hat{f}(S) \leq 1 - \hat{f}(\emptyset)$ for every $\emptyset \neq S \subset [N]$.

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Sketch of proof – an argument à la Wiener:

Take $f : \{\pm 1\}^N \to [-1, 1]$, and show first that $\hat{f}(S) \leq 1 - \hat{f}(\emptyset)$ for every $\emptyset \neq S \subset [N]$. Then for every $\rho > 0$

$$\begin{aligned}\sum_{S \subset [N]} |\widehat{f}(S)| \rho^{|S|} &= |\widehat{f}(\emptyset)| + \sum_{S \neq \emptyset} |\widehat{f}(S)| \rho^{|S|} \\ &\leq |\widehat{f}(\emptyset)| + (1 - |\widehat{f}(\emptyset)|)\big((1+\rho)^N - 1\big).\end{aligned}$$

This shows that $2^{\frac{1}{N}} - 1 \leq \rho(\mathcal{B}^N)$.

Theorem

$$\rho(\mathcal{B}^N) = 2^{\frac{1}{N}} - 1 = \frac{1}{N}\,\big(\log 2 + o(1)\big)$$

Sketch of proof – an argument à la Wiener:

Take $f : \{\pm 1\}^N \to [-1,1]$, and show first that $\hat{f}(S) \leq 1 - \hat{f}(\emptyset)$ for every $\emptyset \neq S \subset [N]$. Then for every $\rho > 0$

$$\begin{aligned} \sum_{S\subset[N]} |\widehat{f}(S)|\rho^{|S|} &= |\widehat{f}(\emptyset)| + \sum_{S\neq\emptyset} |\widehat{f}(S)|\rho^{|S|} \\ &\leq |\widehat{f}(\emptyset)| + (1 - |\widehat{f}(\emptyset)|)\big((1+\rho)^N - 1\big). \end{aligned}$$

This shows that $2^{\frac{1}{N}} - 1 \leq \rho(\mathcal{B}^N)$. For the converse inequality consider

$$f\colon \{\pm1\}^N \to \{\pm 1\}\,,\ f(x) = \begin{cases} -1 & x = \mathbf{1} \\ 1 & x \neq \mathbf{1} \end{cases} \qquad \square$$

Recall

$$\mathcal{B}^N_{\leq d} := \text{all functions on } \{\pm 1\}^N \text{ with of degree} \leq d$$

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The upper bound uses the functions

$$f(x) = 1 - \frac{(x_1 + \ldots + x_N)^d}{N}$$

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Theorem

There is $C > 0$ such that for all N and $\frac{1}{2^N} \leq \delta \leq 1$

$$\frac{C^{-1}}{\sqrt{N}\sqrt{\log(2/\delta)}} \leq \rho(\mathcal{B}_\delta^N) \leq \frac{C}{\sqrt{N}\sqrt{\log(2/\delta)}}$$

The upper bound uses so-called threshold functions

$$\psi_{N,\alpha} : \{\pm 1\}^N \to \{\pm 1\}, \quad \psi_{N,\alpha}(x) = \mathsf{sign}(x_1 + \ldots + x_N - \alpha)$$

Theorem

$$\rho(\mathcal{B}^N_{=d}) \sim \left(\frac{d}{N}\right)^{\frac{d-1}{2d}}$$

Again, in terms of Fourier analysis ...

The Sidon constant of the characters x^S, $|S| = d$ on the group $\{\pm 1\}^N$ up to uniform constants equals $\left(\frac{d}{N}\right)^{\frac{d-1}{2d}}$.

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Summary

	Bohr	Boole
all functions	$\sqrt{\frac{\log N}{N}}$	$\frac{1}{N}$
all degree-d fct.	$\left(\frac{d}{N}\right)^{\frac{d-1}{2d}}$	$\frac{1}{\sqrt{dN}}$
all d-homo. fct.	$\left(\frac{d}{N}\right)^{\frac{d-1}{2d}}$	$\left(\frac{d}{N}\right)^{\frac{d-1}{2d}}$
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For the last two estimates we need BH-inequalities for functions on the Boolean cube – with a good control of the constants.

Theorem, Blei 2003

For each $d \in \mathbb{N}$ there is a (best) constant $\mathrm{BH}^{\leq d}_{\{\pm 1\}}$ such that for every $f : \{\pm 1\}^N \to \mathbb{R}$ of degree d

$$\Big(\sum_{|S| \leq d} |\widehat{f}(S)|^{\frac{2d}{d+1}} \Big)^{\frac{d+1}{2d}} \leq \mathrm{BH}^{\leq d}_{\{\pm 1\}} \, \|f\|_{\{\pm 1\}^N} .$$

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Why is it interesting to improve the constants?

First reason – as mentioned, we need it for proofs of results like

$$\lim_{N\to\infty} \frac{\rho(\mathcal{B}^N_{\text{hom}})}{\sqrt{\frac{\log N}{N}}} = 1$$

Second reason: the Aaronson-Ambainis conjecture, 2011

Aaronson-Ambainis, 2011:

The need for structure in quantum speedups

Informal conjecture

Every quantum query algorithm can be approximated by a classical algorithm on 'most' inputs.

This follows from the AA conjecture – a conjecture in Fourier analysis

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$f : \{\pm 1\}^N \to [-1, 1]$ of degree d
$\mathrm{Var}(f) := \sum_{S \neq \emptyset} \widehat{f}(S)^2$ variance of f
$\mathrm{Inf}_j(f) := \sum_{S : j \in S} \widehat{f}(S)^2$ influence of the variable x_j

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From the book of O'Donnell: Analysis of Boolean functions

If true, this conjecture would have significant consequences regarding the limitations of efficient quantum computation.

What is known so far...

(i) True for Boolean functions $f : \{\pm 1\}^N \to \{\pm 1\}$
O'Donnell, Schramm, Saks and Servedio, 2005

(ii) True replacing in the lower bound d by 2^d
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- BH-inequalities are useful for the study of XOR-games.
- Can one prove that $\mathbf{BH}^{\leq d}_{\{\pm 1\}} \leq \mathbf{poly}(d)$?
- Whould this imply the AA-conjecture ? In certain cases yes!

State of art today...

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Theorem

There exists a constant $C > 0$ such that for all d

$$\mathrm{BH}_{\{\pm 1\}}^{\leq d} \leq C^{\sqrt{d \log d}}$$

In particular,

$$\overline{\lim}_d \sqrt[d]{\mathrm{BH}_{\{\pm 1\}}^{\leq d}} = 1\,.$$

Esto está en contraste con...

- The real BH-inequality: $\overline{\lim}_d \sqrt[d]{\mathrm{BH}^{\leq d}_{[-1,1]}} = 1 + \sqrt{2}$

- The complex BH-inequality: $\overline{\lim}_d \sqrt[d]{\mathrm{BH}^{\leq d}_{\mathbb{T}}} = 1$

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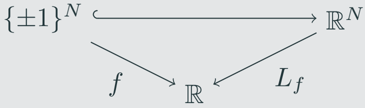

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- ....and the tetrahedral case is somewhat in between!
- The complex BH-inequality: $\overline{\lim}_d \sqrt[d]{\mathrm{BH}^{\leq d}_{\mathbb{T}}} = 1$

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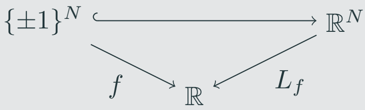

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- Blei's decomposition for such forms–extended version of B-P-S
- Hypercontractivity (of the noise operator)
- a new Harris type polarization formula for real degree-d polynomials