PROBLEMS PRESENTED AT THE CONFERENCE INTEGRATION, VECTOR MEASURES AND RELATED TOPICS IV

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Haar negligible comeager sets

I have heard the following problem from various people. I am not sure who was the first one to raise it but it is a rather natural question.

In a locally compact group G we have a Haar measure and as on $\mathbb{R}$ one can show that G can be written as the union of a meager set and a set which has measure zero under the Haar measure.

In a nonlocally compact Polish group, Haar measures do not exist. However, one can define Haar null sets. According to Christensen [1], a universally measurable subset M of a Polish group G is **Haar null** if there exists a Borel probability measure μ on G such that $\mu(xMy) = 0$ for all $x, y \in G$. (If G is a locally compact Polish group, a subset of G is Haar null in this sense iff it has measure zero under a Haar measure on G.)

Problem. *Can every Polish group can be written as the union of two sets, one meager and the other Haar null?*

Here is a concrete example where this holds.

Example S_∞, the group of all bijections of $\mathbb{N}$ with the operation of composition and the topology induced by the product topology of $\mathbb{N}^{\mathbb{N}}$, can be written as the union of two such sets. This follows from the following theorems.

Theorem 1. (Truss) [3] *Let M be the set of all $\sigma \in S_\infty$ such that σ has no infinite orbit and for every $n \in \mathbb{N}$, σ has infinitely many orbits of length n. Then, M is the complement of a meager subset of S_∞.*

Theorem 2. (Dougherty-Mycielski) [2] *Let N be the set of all $\sigma \in S_\infty$ such that σ has at most finitely many finite orbits and infinitely many infinite orbits. Then, N is the complement of a Haar null subset of S_∞.*

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Weakly compactly generated Banach lattices

Recall that a Banach space X is said to be *weakly compactly generated* if there is a weakly compact set K in X such that the linear span of K is dense in X; similarly, a Banach lattice E is *weakly compactly generated* if there is a weakly compact subset C of E such that the vector sublattice generated by C is dense in E.

Problem. *Is every Banach lattice that's weakly compactly generated as a Banach lattice a weakly compactly generated Banach space?*

For many classical Banach lattices the positive response follows from known characterizations of weakly compact sets in the spaces under question. One obstacle in general is the fact that it need not be that the solid hull of a weakly compact set be weakly compact.

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Weakly Fatou norms

If E is a vector lattice, a norm $\| \|$ on E is a *Riesz norm* if $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$; it is *Fatou* if in addition $\|\sup A\| = \sup_{x\in A} \|x\|$ whenever $A \subseteq E^+$ is upwards-directed and has a least upper bound in E; and it is *weakly Fatou* if it is a Riesz norm and there is a constant $\alpha \geq 0$ such that $\|\sup A\| \leq \alpha \sup_{x\in A} \|x\|$ whenever $A \subseteq E^+$ is upwards-directed and has a least upper bound. For the basic theory of Fatou norms, see [2] and [1]. Many of these theorems are easy to generalize to the case of weakly Fatou norms. But this could be because there is no real difference.

Problem. *If E is a vector lattice with a weakly Fatou norm, is there an equivalent Fatou norm on E?*

(This is Problem AB of [3].) It is easy to check that any weakly Fatou norm on any ℓ^p space (for $0 < p \leq \infty$) is equivalent to a Fatou norm. Going a little deeper, if E is a weakly (σ,∞)-distributive vector lattice with the countable sup property, then any weakly Fatou norm on E is equivalent to a Fatou norm ([4]). But even the special case $E = C([0,1])$ seems to be challenging.

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$\mathcal{F}$-bases

Definition. Let X be a real Banach space, and $\mathcal{F}$ be a free filter on $\mathbb{N}$. A sequence $(e_n)_{n\in\mathbb{N}} \subset X$ is said to be an $\mathcal{F}$-basis, if for every $x \in X$ there is a unique sequence $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ such that the series $\sum_n a_n e_n$ is $\mathcal{F}$-convergent to x, i.e.

$$\mathcal{F}\text{-}\lim_n \sum_{k=1}^{n} a_k e_k = x.$$

It is easy to show that for every n the $a_n = a_n(x)$ in the above definition depends on x linearly.

Problem. *Is it true that a_n depend on x continuously?*

In particular we don't know the answer in the case of $\mathcal{F}$ being the filter $\mathcal{F}_{st}$ of statistical convergence. Recall that by definition $A \in \mathcal{F}_{st}$ if its natural density equals 1, i.e., if

$$\lim_n \frac{|\{k \leq n : k \in A\}|}{n} = 1.$$

Since this problem is open, in the few papers where we were dealing with statistical bases [2] or with $\mathcal{F}$-bases for general filters on $\mathbb{N}$ [3] we were urged to include the continuity of coordinate functionals into the definition of $\mathcal{F}$-basis. See also [1] for some related results.

After the problem was presented at the conference, T.Kochanek found an affirmative answer for countably generated filters ([4]).

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Weak*-exposed points

Let X be a Banach space and X^* be its conjugate space. If $\emptyset \neq K \subset X^*$ is a weak*-compact convex set, then $x_0^* \in K$ is weak* exposed if there is $x_0 \in X$ with $x_0(x_0^*) > x_0(x^*)$ for every $x^* \in K \setminus \{x_0^*\}$. In such a case x_0 weak* exposes x_0^*. If moreover, the convergence $x_0(x_n^*) \to x_0(x_0^*)$ yields $\|x_n^* - x_0^*\| \to 0$, then x_0^* is called weak* strongly exposed ($x_0^* \in (w^* - str\, exp(K))$).

It is known (cf. [2, Theorem 4.2.13]) that X^* has the RNP if and only if each weak*-compact convex $\emptyset \neq K \subset X^*$ is of the form $K = \overline{conv(w^* - str\ exp(K))}^*$ ($\overline{W}^*$ is the weak*-closure of W).

Problem. *Which Banach spaces have the property that every non-empty weak*-compact convex set $K \subset X^*$ admits a weak*-exposed point?*

Henstock-Kurzweil integrability

Problem. *Let $\{E_n : n \in \mathbb{N}\}$ be a collection of pairwise disjoint Lebesgue measurable subsets of $[0,1]$. When is a real valued function $f = \sum_{n=1}^{\infty} a_n \chi_{E_n}$ Henstock-Kurzweil integrable?*

A similar question can be formulated for Henstock and Henstock-Kurzweil-Pettis integrable functions with values in a Banach space.

It is known (cf. [3]) that if X is a Banach space, then $f = \sum_{n=1}^{\infty} a_n \chi_{E_n}$ is Bochner integrable if and only if $\sum_{n=1}^{\infty} a_n |E_n|$ is absolutely convergent and, $f = \sum_{n=1}^{\infty} a_n \chi_{E_n}$ is Pettis integrable if and only if $\sum_{n=1}^{\infty} a_n |E_n|$ is unconditionally convergent.

Required definitions and partial solutions to this problem are contained in [1].

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Pettis and McShane integrals

The comparison between different generalizations of the Lebesgue integral is an important part of the research on vector-valued integration. Here we focus on the Pettis and McShane integrals for functions $f : [0,1] \to X$, where $[0,1]$ is equipped with the Lebesgue measure λ and X is a Banach space. Recall that f is said to be McShane integrable, with integral $x \in X$, if for every $\epsilon > 0$ there is a positive function δ on $[0,1]$ such that the inequality $\| \sum_{i=1}^{p} \lambda(I_i) f(t_i) - x\| \leq \epsilon$ holds for every finite collection $I_1, \dots, I_p$ of non-overlapping closed intervals covering $[0,1]$ and every choice of points $t_i \in [0,1]$ with $I_i \subset (t_i - \delta(t_i), t_i + \delta(t_i))$. In general, McShane integrability lies strictly between Bochner and Pettis integrability, but McShane and Pettis integrability are equivalent for functions taking values in *separable* Banach spaces; see [6, 7, 8]. Hence every separable Banach space belongs to the class $\mathcal{C}$ defined by:

$$X \in \mathcal{C} \text{ iff every Pettis integrable function } f : [0,1] \to X \text{ is McShane integrable.}$$

The coincidence of McShane and Pettis integrability for *non-separable* Banach spaces was studied by Di Piazza and Preiss [4], who showed that $c_0(\Gamma) \in \mathcal{C}$ (for any set Γ) and that all super-reflexive Banach spaces (e.g., Hilbert spaces) also belong to $\mathcal{C}$. It was proved later that $L^1(\mu) \in \mathcal{C}$ (for any probability μ), [9]. All these results were generalized simultaneosly in [3] by showing that every (subspace of a) Hilbert generated Banach space belongs to $\mathcal{C}$. To the best of our knowledge, this is the most general positive answer to the Di Piazza-Preiss question whether every weakly compactly generated (WCG) Banach space belongs to $\mathcal{C}$. It turns out that the answer is negative in general, even for reflexive Banach spaces, see [1]. However, a related question still remains open:

Problem. *Let $\mathcal{K}$ be the class of Eberlein compacta K for which $C(K) \in \mathcal{C}$. Does $\mathcal{K}$ coincide with the class of uniform Eberlein compacta?*

Recall that a compact space K is called Eberlein (resp. uniform Eberlein) if it is homeomorphic to a weakly compact subset of a Banach (resp. Hilbert) space. It is known that a compact space K is Eberlein (resp. uniform Eberlein) if, and only if, $C(K)$ is WCG (resp. Hilbert generated); see [5, Chapter 5] and [2]. Thus, the results of [1, 3] show that $\mathcal{K}$ contains all uniform Eberlein compacta but not all Eberlein compacta.

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Let μ be a Borel probability measure on a standard Borel space X. Let $\mathrm{Aut}(\mu)$ be the group of all equivalence classes of μ-measure preserving transformations, with two transformations declared equivalent if they are equal on a set of measure one. The group operation in $\mathrm{Aut}(\mu)$ is composition and the topology is the weak topology, that is, the weakest topology making all functions of the form

$$[f] \mapsto \mu(A \triangle f(A)) : \mathrm{Aut}(\mu) \to \mathbb{R}, \text{ for Borel } A \subseteq X,$$

continuous. $\mathrm{Aut}(\mu)$ is Polish; see [4, 17.46].

Let G be a Polish group. A continuous homomorphism $\phi\colon G \to \mathrm{Aut}(\mu)$ is said to have a *point realization* if there is a Borel action $\alpha\colon G \times X \to X$ such that for each $g \in G$ the transformation $\alpha(g, \cdot)$ is in the equivalence class $\phi(g)$.

Let $C(2^{\mathbb{N}}, \mathbb{T})$ be the group of all continuous functions from the Cantor set to the circle with pointwise multiplication and with the uniform convergence topology.

Problem. *Is there a continuous homomorphism from $C(2^{\mathbb{N}}, \mathbb{T})$ to $\mathrm{Aut}(\mu)$ without a point realization?*

The classical theorem of Mackey ([6]) asserts that for a Polish locally compact group G each continuous homomorphism $G \to \mathrm{Aut}(\mu)$ has a point realization. Glasner and Weiss ([3]) showed that this also holds for all closed subgroups of the group of all permutations of $\mathbb{N}$ (taken with the pointwise convergence topology), and Kwiatkowska and Solecki, giving a common generalization of the above two results, proved in [5] that this is true for all Polish groups of isometries of locally compact, separable metric spaces. Examples of Polish groups failing this condition were given by Becker ([1]) and, independently, by Glasner, Tsirelson, and Weiss ([2]). The group $C(2^{\mathbb{N}}, \mathbb{T})$ is situated between the groups from the positive results of [3,5,6] and the examples of [1,2].

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