Zassenhaus conjecture for cyclic-by-abelian groups

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Abstract

Zassenhaus Conjecture for torsion units states that every augmentation one torsion unit of the integral group ring of a finite group $G$ is conjugate to an element of $G$ in the units of the rational group algebra $\mathbb{Q}G$. This conjecture has been proved for nilpotent groups, metacyclic groups and some other families of groups. It has been also proved for some special groups. We prove the conjecture for cyclic-by-abelian groups.

In this paper $G$ is a finite group and $RG$ denotes the group ring of $G$ with coefficients in a ring $R$. The units of $RG$ of augmentation one are usually called normalized units. In the 1960’s Hans Zassenhaus established a series of conjectures about the finite subgroups of normalized units of $\mathbb{Z}G$. Namely, he conjectured that every finite group of normalized units of $\mathbb{Z}G$ is conjugate to a subgroup of $G$ in the units of $\mathbb{Q}G$. This conjecture is usually denoted (ZC3), while the version of (ZC3) for the particular case of subgroups of normalized units of the same order as $G$ is usually denoted (ZC2). These conjectures have important consequences. For example, a positive solution of (ZC2) implies a positive solution for the Isomorphism and Automorphism Problems (see [Seh93] for details). The most celebrated positive result for Zassenhaus Conjectures is due to Weiss [Wei91] who proved (ZC3) for nilpotent groups. However, Roggenkamp and Scott discovered a counterexample to the Automorphism Problem, and henceforth to (ZC2) (see [Rog91] and [Kli91]). Later, Hertweck [Her01] provided a counterexample to the Isomorphism Problem.

The only conjecture of Zassenhaus regarding torsion units of group rings that is still open is the version for cyclic subgroups namely:

**Zassenhaus Conjecture for Torsion Units (ZC1).** If $G$ is a finite group then every normalized torsion unit of $\mathbb{Z}G$ is conjugate in $\mathbb{Q}G$ to an element of $G$.

Besides the family of nilpotent groups, (ZC1) has been proved for some concrete groups [BH08, BHK04, HK06, LP89, LT91, Her08b], for groups having a normal Sylow subgroup with an abelian complement [Her06], for some families of cyclic-by-abelian groups [LB83, LT90, LS98, MRSW87, PMS84, PMRS86, dRS06, RS83] and some classes of metabelian groups not necessarily cyclic-by-abelian [MRSW87, SW86]. Other results on Zassenhaus Conjectures can be found in [Seh93, Seh01] and [Seh03, Section 8].

The latest and most general result for (ZC1) on the class of cyclic-by-abelian groups is due to Hertweck [Her08a]. This paper has been our main source of ideas and inspiration. Hertweck proves (ZC1) for finite groups of the form $G = AX$ with $A$ a cyclic normal subgroup of $G$ and $X$ an abelian subgroup of $G$. This includes the class of metacyclic groups, which was not covered in previous results. It is well known that (ZC1) holds for $G$ if and only if the partial augmentations of torsion units are non-negative. In the hypothesis of [Her08a], $C_G(A) = AC_X(A) = AZ(G)$. In the words of Hertweck this is “the main reason for assuming that $A$ is covered by an abelian
Theorem. Let $G$ be a finite cyclic-by-abelian group. Then every normalized torsion unit of $ZG$ is conjugate in $QG$ to an element of $G$.

We argue by contradiction and our strategy uses induction on the order of the group $G$ and on the order of the torsion unit. More precisely, we consider a finite cyclic-by-abelian group $G$, which is a minimal counterexample to (ZC1), and a torsion unit $u$ in $ZG$, which is a minimal counterexample to (ZC1). Here minimal means “of minimal order”. In particular, we assume that (ZC1) holds for proper subgroups and quotients of $G$ and for proper powers of $u$, i.e. for powers $u^k$, with $k$ a divisor of the order of $u$ other than 1.

The main lines of the proof go along the path of [Her08a]. We explain here its structure and the differences with [Her08a]. Let $G$ be a cyclic-by-abelian group and let $u$ be a torsion unit of $ZG$ and assume that $G$ and $u$ are minimal counterexamples to Zassenhaus Conjecture in the sense of the previous paragraph. By a well known result we may assume that one of the partial augmentations of $u$ is negative. This will ultimately yield a contradiction. Let $A$ be a normal cyclic subgroup of $G$ with $G/A$ abelian. Let $C = C_G(A)$ and $D = Z(C)$. Using an idea from [dRS06] we prove that the negative partial augmentations of $u$ only can occur at elements of $D$. Hertweck first considers the case when $\omega_C(u) = 1$ and, as mentioned above, he may even assume in this case that $\omega_A(u) = 1$. In that case Hertweck uses a Theorem of Cliff and Weiss [CW00] to deduce that partial augmentations of $u$ at the elements of $A$ are non-negative. To deal with partial augmentation at elements in $G \setminus A$ Hertweck proves a key result [Her08a, Theorem 5.1] which implies that the $p$-parts of $u$ are $p$-adically conjugate to an element of $A$. We first deal with the case when $\omega_D(u) = 1$ (which, under Hertweck’s assumptions coincides with his first case). However, in general, $AZ(G) \subseteq D \subseteq C$ and both inclusions can be proper. This is the main obstacle to apply directly the arguments of Hertweck in this part of the proof. Instead, we use results from [CW00] and some ideas from proofs contained therein to prove that certain integral linear combinations of these partial augmentations are non-negative (Lemma 2.4). Then we prove a version of [Her08a, Theorem 5.1] suitable for our situation (Theorem 2.5). The combination of these two results finishes the proof of this case. Then we deal with the case when $\omega_D(u) \neq 1$. Here we use the Luthar–Passi Method following the lines of the proof of [Her08a, Theorem 7.3]. Again we encounter difficulties here. Hertweck obtains information on partial augmentations at elements of $A$ and we need to obtain information on partial augmentations at elements of $D$. Hertweck uses the induction argument to reduce the main formula of the Luthar–Passi Method to a formula which relates the partial augmentation of $u$ at an element of $A$ with the multiplicities of the eigenvalues of a representation of $G$ induced by a faithful linear character of $A$. When this method is applied in our case one obtains a more complicated formula where the faithful character of $A$ is replaced by a family of suitable linear characters of $D$ (see Lemma 3.1). It requires some additional work to extract information from this formula. This ultimately leads to the desired contradiction.

1. Notation, preliminaries and some tools

In this section we establish the general notation and collect some elementary observations and known results, which will be used throughout the paper.
The cardinality of a set $X$ is denoted by $|X|$.

We use the standard group theoretical notation. In particular, if $H$ is a group, then $Z(H)$ denotes the center of $H$, $H'$ the commutator subgroup of $H$ and $\exp(H)$ the exponent of $H$. If $g, h \in H$ then $|g|$ denotes the order of $g$, $g^h = h^{-1}gh$, $(g, h) = g^{-1}g^h = g^{-1}h^{-1}gh$ and $g^H$ denotes the conjugacy class of $g$ in $H$. If $X \subseteq H$ then $\langle X \rangle$ denotes the subgroup generated by $X$, $C_H(X) = \{ h \in H : (x, g) = 1 \text{ for every } x \in X \}$, the centralizer of $X$ in $H$, and $N_H(X) = \{ h \in H : X^g \subseteq X \}$, the normalizer of $X$ in $H$. If $Y$ is another subset of $H$ then $(X, Y)$ denotes the group generated by the commutators $(x, y)$ with $x \in X$ and $y \in Y$.

Let $p$ be a prime integer. The ring of $p$-adic integers is denoted by $\mathbb{Z}_p$. If $g$ has finite order then $g_p$ and $g_{p'}$ denote the $p$-part and $p'$-part of $g$, respectively. $O_p(H)$ (respectively, $O_{p'}(H)$) denotes the unique maximal normal $p$-subgroup (respectively, $p'$-subgroup) of $H$. If $H$ has a unique $p$-Sylow subgroup (respectively a unique Hall $p'$-subgroup) then it is denoted $H_p$ (respectively $H_{p'}$).

**Remark 1.1.** Let $N$ be a finite abelian group, $A$ a cyclic subgroup of $N$ and $H$ a subgroup of $N$ such that $H \cap A = 1$ and $H$ is maximal among the subgroups of $N$ with this property. Then $N/H$ is cyclic. Indeed, if $N/H$ is not cyclic there are $x_1, x_2 \in N$ such that $\langle x_1, x_2, H \rangle/H$ is a non-cyclic elementary abelian $p$-subgroup for some prime $p$. By the maximality of $H$, $\langle x_1, H \rangle \cap A$ and $\langle x_2, H \rangle \cap A$ contain the unique subgroup of $A$ of order $p$ and hence $H \cap A = \langle x_1, H \rangle \cap \langle x_2, H \rangle \cap A \neq 1$, a contradiction.

In the remainder of this section $R$ stands for a commutative ring. If $N$ is a normal subgroup of $G$ then the $N$-augmentation map of $RG$ is the unique ring homomorphism $\omega_N : RG \to R(G/N)$ extending the natural map $G \to G/N$ and acting on $R$ as the identity. In particular $\omega = \omega_G$ is the augmentation map of $RG$. Let $r = \sum_{g \in G} r_g g \in RG$ with $r_g \in R$ for every $g$. For every $g \in G$, let $\varepsilon^G_g(r)$ denote the partial augmentation of $r$ at $g$ in $G$, that is

$$\varepsilon^G_g(r) = \sum_{h \in gG} r_h.$$ 

If the group $G$ is clear from the context we simply write $\varepsilon_g(r)$. Conjugacy classes of torsion elements in the units of $RG$ and partial augmentations are strongly related. We collect in the following remark some easy facts about this relation.

**Remark 1.2.** If $x \in G$ then $\varepsilon^G_x : RG \to R$ is an $R$-linear map which satisfies $\varepsilon^G_x(uv) = \varepsilon^G_x(vu)$ for every $u, v \in RG$. Using this it is easy to prove that if $u \in RG$ and $g \in G$ are conjugate in the units of $RG$, then $\varepsilon^G_g(u) = 1$ and $\varepsilon^G_g(u) = 0$ for every $x \in G$ with $x \not\in g^G$. Hence, for such $u$ and $g$ and a normal subgroup $N$ of $G$ we have $g \in N$ if and only if $\omega_N(u) = 1$.

One of the main tools to study (ZC1) is the following well known result which is somehow a converse of Remark 1.2 (see e.g. [Seh93, Lemma 41.5]).

**Proposition 1.3.** Let $u$ be a normalized torsion unit of $ZG$. Then $u$ is conjugate in $QG$ to an element of $G$ if and only if $\varepsilon^G_g(u) \geq 0$ for every $v \in \langle u \rangle$ and every $g \in G$.

Proposition 1.3 is commonly presented in the following equivalent form: A normalized torsion unit $u$ of $ZG$ is conjugate in $QG$ to an element of $G$ if and only if for every $v \in \langle u \rangle$, there is $g \in G$ such that for every $x \in G$ we have $\varepsilon^G_x(v) \neq 0$ if and only if $x \in g^G$. 


The following proposition collects some results from [Her06] and [Her08a] which will be very useful in our arguments.

**Proposition 1.4.** Let $G$ be a finite group and $p$ a prime integer.

(i) Let $R$ be a $p$-adic ring with quotient field $K$ and $u$ a normalized torsion unit of $RG$. Suppose that the $p$-part of $u$ is conjugate to an element $x$ of $G$ in the units of $RG$ and $y$ is an element of $G$ such that the $p$-parts of $x$ and $y$ are not conjugate in $G$. Then $\varepsilon_g(u) = 0$.

(ii) Let $u$ be a normalized torsion unit of $ZG$.

(a) If $\omega_P(u) = 1$ for $P$ a normal $p$-subgroup of $G$, then $u$ is conjugate in $QG$ to an element of $P$.

(b) If $\varepsilon_g(u) \neq 0$ with $g \in G$ then the order of $g$ divides the order of $u$.

A useful technique to deal with Zassenhaus conjecture is the so-called double action formalism introduced by Weiss [Wei88]. Let $\alpha : H \to \text{GL}_k(RG)$ be a group homomorphism. Then $M^\alpha$ denotes the right $R(G \times H)$ module $(RG)^k$ with multiplication defined by $x \cdot r(g, h) = ry^{-1}x\alpha(h)$ for $x \in (RG)^k$, $g \in G$, $h \in H$ and $r \in R$. It is easy to see that if $\beta : H \to \text{GL}_k(RG)$ is another group homomorphism then $M^\alpha \cong M^\beta$ if and only if $\alpha$ and $\beta$ are conjugate in $\text{GL}_k(RG)$, i.e. if and only if there exists $u \in \text{GL}_k(RG)$ such that $\beta(h) = \alpha(h)^u$ for every $h \in H$.

For example, if $C_m = (c)$, the cyclic group of order $m$ generated by $c$, and $u$ and $v$ are torsion units of order $m$ in $\text{GL}_k(RG)$ then $u$ and $v$ are conjugate in $\text{GL}_k(RG)$ if and only if $M^\alpha_u$ and $M^\alpha_v$ are isomorphic, where $\alpha_u$ and $\alpha_v$ denote the homomorphisms $C_m \to \text{GL}_k(RG)$ mapping $c$ to $u$ and $v$ respectively.

Set $\Gamma = G \times C_m$ and let $G[m]$ denote a set of representatives of $G$-conjugacy classes of elements $g \in G$ with $g^m = 1$. For every $g \in G$ let $[g] = \langle (g, c) \rangle \leq \Gamma$. For $H$ a subgroup of a group $K$ we let $\text{ind}_H^K$ denote the ordinary character of $K$ induced from the trivial character of $H$.

Following [CW00], SGL$_k(ZG)$ denotes the kernel of the group homomorphism $\text{GL}_k(ZG) \to \text{GL}_k(Z)$ that acts componentwise as the augmentation map $\omega$. The following lemma can be extracted from Lemma 4.2 and Proposition 4.3 of [CW00] and the expression of $(\text{ind}_H^K)(g, c)$ given in the proof of [CW00, Lemma 4.2]).

**Lemma 1.5.** Let $G$ be a finite nilpotent group, $u \in \text{SGL}_k(ZG)$ with $u^m = 1$ and let $\chi$ denote the character of $M^\alpha_u$. Then for every $g \in G[m]$ we have $\chi(g, c) \in |C_G(g)|Z$ and $\chi = \sum_{g \in G[m]} \chi(g, c) |C_G(g)|^{-1} \text{ind}_H^K$.  

2. Torsion units with $D$-augmentation $1$

The title of this section refers to $D = Z(C_G(A))$ for a cyclic subgroup $A$ of $G$ containing $G'$ (see the introduction). So the aim of this section is to prove (ZC1) for torsion units $u$ with $\omega_D(u) = 1$.

We start with a lemma which is basically contained in [CW00, Lemma 2.6]. We include a proof for the sake of completeness and clarity.

**Lemma 2.1.** Let $N$ be a nilpotent normal subgroup of $G$ and $u$ a torsion unit of $ZG$ such that $\omega_N(u) = 1$. Then

(i) every prime divisor of the order of $u$ divides the order of $N$ and

(ii) if the order of $u$ is a power of a prime $p$, then $\omega_{N_p}(u) = 1$. 

Proof. (1) We argue by induction on the number of primes dividing \( |N| \). If this number is 0 then \( N = 1 \) and hence \( u = \omega_N(u) = 1 \). Assume that \( |N| \) is divisible by \( p \). By hypothesis \( \omega_{N/N_p}(\omega_{N_p}(u)) = \omega_N(u) = 1 \). Let \( n \) be the order of \( \omega_{N_p}(u) \). By the induction hypothesis every prime divisor of \( n \) divides \( |N : N_p| \). Moreover the order of \( u^n \) is a power of \( p \) by \cite{Seh93}, Lemma 7.5. Let \( q \) be a prime divisor of \( |u| \). Then \( q \) is either \( p \) or a divisor of \( n \). We conclude that \( q \) divides \( |N| \).

(2) Assume that the order of \( u \) is a power of \( p \) and set \( u_1 = \omega_{N_p}(u) \). Then \( u_1 \) is a \( p \)-element of \( \mathbb{Z}(G/N_p) \) such that \( \omega_{N/N_p}(u_1) = \omega_N(u) = 1 \). Since \( p \) is coprime with \( |N : N_p| \), the order of \( u_1 \) is coprime with \( p \), by (1). Hence \( u_1 = 1 \), as desired. \( \square \)

The following two lemmas were improved by the referee with respect to the first version. The first one extends \cite[Claim 5.2]{Her08a}. The argument of the proof of the second one was already used in \cite{dRS06} and \cite{Her08a} to prove that partial augmentations at elements in \( G \setminus C_G(A) \) are non-negative for minimal counterexamples. We need this also for elements in \( G \setminus Z(C_G(A)) \).

**Lemma 2.2.** Let \( A \) be a cyclic subgroup of \( G \) containing \( G' \), let \( N \) be a non-trivial \( p \)-subgroup of \( A \) for some prime \( p \) and let \( x \) be a \( p \)-element of \( C_G(A) \setminus Z(G) \). Then \( O_p(G) \) is a normal Hall \( p' \)-subgroup of both \( C_G(N) \) and \( C_G(x) \).

Proof. Let \( Q = O_p(G) \). Then \( Q \subseteq C_G(O_p(G)) \subseteq C_G(N) \). If \( N_1 \) is the unique minimal non-trivial subgroup of \( N \) and \( \alpha : \text{Aut}(A_p) \rightarrow \text{Aut}(N_1) \) is the restriction map, then the kernel of \( \alpha \) is a \( p \)-group. Therefore the \( p' \)-elements of \( C_G(N) \) commute with the \( p \)-elements of \( A \). \( H \) be a Hall \( p' \)-subgroup of \( C_G(N) \). As \( Q \) are normal in \( G \), \( QH \) is a \( p' \)-subgroup of \( G \) and hence \( A_p \subseteq Q \subseteq H \). We will prove that \( H \) is normal in \( G \). Indeed, let \( h \in H \) and \( g \in G \). As \( G/A \) is abelian, \( h^g = ah \) for some \( a \in A \). As \( a_p \in H \), the order of \( a_p \) is coprime with \( p \). Thus \( (a_p, a_p)h = 1 \) and hence the order of \( h^g \) is divisible by the order of \( a_p \). Thus \( a_p = 1 \) and we conclude that \( h^g = a_p \in H \). This finishes the proof of the claim. Thus \( H = Q \) and so \( Q \) is a normal Hall \( p' \)-subgroup of \( C_G(N) \), as desired.

Let \( M = \langle x^q : g \in G \rangle \). For every \( g \in G \), we have \( x^g = x(x, g) \in xA \) and therefore \( M \) is an abelian normal \( p \)-subgroup of \( G \) contained in \( C_G(A) \). In particular \( M \subseteq O_p(G) \) and hence \( Q \subseteq C_G(M) \). Moreover, if \( y \in C_G(x) \) then \( (x, y^q) = (x, y(y, g)) = 1 \). Thus \( C_G(x) = C_G(M) \). Let \( a = (x, g) \neq 1 \) with \( g \in G \) and \( M_1 = \langle a \rangle \). Then \( M_1 \) is a non-trivial \( p' \)-subgroup of \( A \) contained in \( M \). By the previous paragraph \( Q \) is a normal Hall \( p' \)-subgroup of \( C_G(M_1) \). Since \( Q \subseteq C_G(x) = C_G(M) \subseteq C_G(M_1) \), we deduce that \( Q \) is also a normal Hall \( p' \)-subgroup of \( C_G(x) \), as desired. \( \square \)

**Lemma 2.3.** Let \( A \) be an abelian subgroup of \( G \) containing \( G' \) and assume that \( G/N \) satisfies (ZC1) for every non-trivial subgroup \( N \) of \( A \). Let \( u \) be a normalized torsion unit in \( \mathbb{Z}G \) and let \( x \in G \setminus Z(C_G(A)) \). Then \( \varepsilon_x(u) \geq 0 \).

Proof. By hypothesis and Proposition 1.2, we have \( \varepsilon_{x/N}^{G/N}(\omega_N(u)) \geq 0 \) for every non-trivial normal subgroup \( N \) of \( G \). Let \( C = C_G(A) \) and \( D = Z(C) \). We consider separately the cases where \( x \notin C \) and \( x \in C \).

Assume that \( x \notin C \) and take \( N = \langle (a, x^g) : a \in A, g \in G \rangle \). Then \( N \) is a non-trivial normal subgroup of \( G \) contained in \( A \). If \( a \in A \) and \( g, h \in G \) then \( (a, x^h)x^g = a^{-1}x^{h^{-1}g, (x^h)^{-1}}x^h = a^{x^{h^{-1}g, (x^h)^{-1}a^{-1}x^h}} = x^g(a^x)^{-1} \in x^G \). This proves that \( N \langle x \rangle = x^G \) and hence \( \varepsilon_x(u) = \varepsilon_{x/N}(\omega_N(u)) \geq 0 \).
Suppose that $x \in C \setminus D$ and let $c \in C \setminus C_G(x)$ and $N = \langle (x,c) \rangle$. As $[G', C] = 1$, the Three Subgroup Lemma yields $C' \subseteq Z(G)$ and therefore $N$ is a non-trivial central subgroup of $G$. Moreover, for every $g \in G$ we have $(x,c)x^g = (x,c)x(x,g) = x(x, cg) = x^g$. Therefore $N x^G = x^G$ and again $\varepsilon_{x}^G(u) = \varepsilon_{x N}^G(\omega_N(u)) \geq 0$. }

Next lemma follows the idea of the proof of Theorem 6.3 in [CW00]. We take scalar products of the character obtained through Weiss’ double action formalism and suitable ordinary characters to obtain new information about partial augmentation.

**Lemma 2.4.** Let $N$ be an abelian normal subgroup of $G$ and $u$ a torsion unit in $ZG$ with $\omega_N(u) = 1$. Let $\eta$ be an irreducible character of $N$ and $n \in N$. If the order of $n$ divides the order of $u$ then

$$\sum_{h \in \ker \eta} |C_G(h n) : N| \varepsilon_{h n}^G(u) \geq 0.$$

**Proof.** Let $m$ be the order of $u$ and $k = [G : N]$. Consider $ZG$ as a $(ZG,ZN)$-bimodule and fix a transversal of $N$ in $G$ which we consider as a basis of $ZG$ as right $ZN$-module. This defines a ring homomorphism $ZG \rightarrow M_k(ZN)$ mapping an element $x$ of $ZG$ to the matrix associated to the left multiplication by $x$ in the selected basis. Let $v$ be the image of $u$ under this homomorphism. Then $v \in SGL_k(ZN)$. Let $\Gamma = N \times C_m$, with $C_m = \langle c \rangle$, a cyclic group of order $m$, let $\chi$ be the character of $M^{\text{es}}$ and let $N|m| = \{ n \in N : n^m = 1 \}$. By [Wei91, Lemma 1] we know $|C_G(n)|\varepsilon_{n}^G(u) = \chi(n, c)$ for every $n \in N$. (Caution: The sides in the action of the double action formalism is interchanged with respect to [Wei91]. This causes a difference in the formula just used.) Moreover, $\varepsilon_{n}^G(u) = 0$ if $h \in N \setminus N|m|$, by statement (ii)(b) of Proposition 1.4. Combining this with Lemma 1.5 we have

$$\chi = \sum_{h \in N|m|} \frac{\chi(h,c)}{|N|} \text{ind}_{[h]}^1 = \sum_{h \in N|m|} |C_G(h) : N| \varepsilon_{h}^G(u) \text{ind}_{[h]}^1$$

$$= \sum_{h \in N} |C_G(h) : N| \varepsilon_{h}^G(u) \text{ind}_{[h]}^1$$

and for every ordinary irreducible character $\psi$ of $\Gamma$ we get

$$0 \leq \langle \chi, \psi \rangle_{\Gamma} = \sum_{h \in N} |C_G(h) : N| \varepsilon_{h}^G(u) \text{ind}_{[h]}^1 \psi(\varepsilon_{h}^G(u)) = \sum_{h \in N \psi(h,c) = 1} |C_G(h) : N| \varepsilon_{h}^G(u). \quad (2.1)$$

Now let $\psi$ be the character of $\Gamma$ given by $\psi|_{N} = \eta$ and $\psi((1,c)) = \eta(n)^{-1}$. Then $\psi(h,c) = 1$ if and only if $h \in n \ker(\eta)$. So using $\psi$ in (2.1) gives

$$0 \leq \sum_{h \in \ker(\eta)} |C_G(h n) : N| \varepsilon_{h n}^G(u).$$

The following theorem is an adjustment of [Her08a, Theorem 5.1] to our situation.

**Theorem 2.5.** Let $G$ be a finite group and $A$ a cyclic normal subgroup of $G$ containing $G'$. Set $D = Z(C_G(A))$ and let $u$ be a torsion unit of $ZG$ with $\omega_D(u) = 1$. If the order of $u$ is a power of a prime $p$, then $u$ is conjugate in $Z_p G$ to an element of $D_p$. 


Proof. Let $G$, $A$, $D$ and $u$ as in the statement of the theorem. By Lemma 2.1, $\omega_{D_p}(u) = 1$. Then, by statement (ii)(a) of Proposition 1.4, $u$ is conjugate to an $x \in D_p$. Let $R$ be a $p$-adic ring with quotient field $K$ containing a root of unity of order the exponent of $G$. We will prove that $u$ is conjugate to $x$ in $RG$. Then, by [CR81, 30.25], the conjugation already takes place in $\mathbb{Z}_pG$. We assume that $x \in \mathbb{Z}(G)$, for otherwise the result follows at once.

Set $Q = Q_p(G)$ and $E = CG(x)$. By Lemma 2.2, $Q$ is a normal Hall $p'$-subgroup of $E$.

The primitive central idempotents of $KQ$ belong to $RQ$, because the order of $Q$ is invertible in $R$. Moreover $G$ acts on these primitive central idempotents by conjugation. Let $\epsilon_1, \ldots, \epsilon_\beta$ be the sums of the $G$-orbits of this action. Then $RG = \prod_{i=1}^{\beta} \epsilon_iRG$ and therefore it is enough to show that $\epsilon_iu$ is conjugate to $\epsilon_ix$ in $\epsilon_iRG$ for every $i$.

So fix one primitive central idempotent $f$ of $KQ$ and let $\epsilon$ be the sum of the $G$-conjugates of $f$. Note that $\epsilon u \epsilon$ is conjugate to $ex$ in $\epsilon KG$ and a primitive idempotent of $KQ$ stays primitive in $KE$ by Green’s Indecomposability Theorem, since $E/Q$ is a $p$-group [CR81, 19.23]. We have to prove that $\epsilon u$ and $ex$ are conjugate in $\epsilon RG$. Let $\epsilon$ be the sum of the different $E$-conjugates of $f$ and write $e = e_1 + \ldots + e_m$, a sum of orthogonal primitive idempotents of $RQ$. Let $T = CG(e)$ and let $\{1 = s_1, \ldots, s_n\}$ be a transversal of $T$ in $G$. As $e$ is central in $RE$ we have $E \subseteq T$. Then

$$
\epsilon = \sum_{i=1}^{m} \sum_{j=1}^{n} \epsilon_i e_j^{-1}
$$

is a decomposition into primitive orthogonal idempotents of $\epsilon RE$. So by [Her08a, Lemma 4.6] there exist $g_{ij} \in G$ such that

$$
v = \sum_{i=1}^{m} \sum_{j=1}^{n} \epsilon_i e_j^{-1} x^{g_{ij}}
$$

is conjugate to $\epsilon u$ in $RG$. In particular $|v| = |\epsilon u| = |ex|$. Let $C = (c)$ a cyclic group with the same order as $v$ and consider the right $R(G \times C)$-modules $M = M^\alpha$ and $N = M^\beta$, with $\alpha = \alpha_v$ and $\beta = \alpha_ex$ (by slight abuse of notation). We have to show that $v$ is also conjugate to $ex$ in $\epsilon RG$ or equivalently that $M$ and $N$ are isomorphic as $R(G \times C)$-modules.

Every $G$-conjugate of $x$ commutes with every $G$-conjugate of $e_i$ because $x \in O_p(G)$ and $e_i \in RQ$. Using this it easily follows that each $RGe_i x^{s_j^{-1}}$ is a direct summand of both $M$ and $N$. We set $M_{ij} = M e_i x^{s_j^{-1}}$ and $N_{ij} = N e_i x^{s_j^{-1}}$, i.e. both $M_{ij}$ and $N_{ij}$ are $RGe_i x^{s_j^{-1}}$-modules. The strategy of the proof consists in pairing isomorphic $M_{ij}$’s and $N_{ij}$’s.

By the construction, $e_i$ and $e_1$ are conjugate in $REe$ for every $i = 1, \ldots, m$ and for every $t \in T$. Thus for every $i = 1, \ldots, m$ and every $j = 1, \ldots, n$ there is a unit $u$ of $REe x^{s_j^{-1}}$ such that $e_i x^{s_j^{-1}} = (e_1 x^{s_j^{-1}})^u$. Then $a \mapsto au$ is an isomorphism $N_{ij} \to N_{ij}$. Thus

$$
N = \bigoplus_{i,j} N_{ij} \cong \bigoplus_{j=1}^{n} N_{1j}^m. \tag{2.2}
$$

For $j_0 = 1, \ldots, n$, set $X_{j_0} = \{(i,j) : g_{ij} s_j \equiv s_{j_0} \mod T\}$. If $(i,j) \in X_{j_0}$ then $N_{ij_0} \cong M_{ij}$ via $a \mapsto ag_{ij}$ and therefore $N_{1j_0} \cong M_{ij}$. Thus

$$
M = \bigoplus_{i,j} M_{ij} \cong \bigoplus_{j=1}^{n} N_{1j}^{[X_{j_0}]}. \tag{2.3}
$$

Therefore the isomorphism $M \cong N$ will follow from (2.2) and (2.3) provided $|X_{j_0}| = m$. The remainder of the proof is dedicated to prove this equality. For this we will use a representation of $\epsilon KG$ and investigate the multiplicities of eigenvalues of $ex$ and $v$ under this representation. They are the same for $ex$ and $v$ are conjugate in $\epsilon KG$. This is also the strategy in the proof of
This can be chosen satisfying $\psi$ for every $\epsilon \in \mathrm{morphism}$ between matrix rings of size $\chi$ that do not vanish on $e$. So $\psi(\varepsilon) = 1$ for every primitive idempotent $\varepsilon$ of $RQ$ satisfying $ee \neq 0$ and hence $\psi(1) = m$. Let $\rho$ be a representation of $Q \times D_p$ affording $\psi \otimes \pi$. This can be chosen satisfying $\rho(e_i) = E_i$, where $E_i$ denotes the elementary matrix with 1 in the $i$-th diagonal entry and 0 anywhere else. Then $\rho(e_i y) = \pi(y) E_i$ for any $i$ and $y \in D_p$. Let $\chi = \text{ind}_{Q \times D_p}^G(\psi \otimes \pi)$, let $\Delta$ be a representation of $T$ affording $\chi$ and let $\{t_1, \ldots, t_k\}$ be a transversal of $Q \times D_p$ in $T$. After a suitable conjugation one may assume that

$$\Delta \left( \sum_{i=1}^{m} e_i y_i \right) = \text{diag} \left( \pi \left( y_i^{1} \right) : 1 \leq i \leq m, \ 1 \leq j \leq k \right) \in M_{mk}(K)$$

for every $y_1, \ldots, y_m \in D_p$. Denote by $\bar{\Delta} : M_{n}(eKT) \to M_{nmk}(K) = M_{n}(M_{mk}(K))$ the homomorphism between matrix rings of size $n$ which acts on each entry as $\Delta$.

As right $KG$-modules we have $eKG = \sum_{j=1}^{n} e_{j}^{-1} KG \cong (eKG)^{n}$ and so

$$eKG \cong \text{End}_{KG}(eKG) \cong M_{n}(\text{End}_{KG}(eKG)) \cong M_{n}(eKG) \cdot \text{End}_{KG}(eKG). \quad (2.4)$$

Moreover, $eKG = e \sum_{j=1}^{n} s_{j} KTe = e \sum_{j=1}^{n} e_{j}^{-1} s_{j} KTe = eKT$, since $e$ is orthogonal to any different conjugate of itself. So (2.4) gives an isomorphism $\delta : eKG \cong M_{n}(eKT)$, which satisfies

$$\delta \left( \sum_{j=1}^{n} e_{j}^{-1} y_{j} \right) = \text{diag}(e y_{1}, \ldots, e y_{n})$$

for $y_1, \ldots, y_n \in KT$. Set $\hat{\Delta} = \Delta \circ \delta$. Then we have

$$\hat{\Delta}(ex) = \Delta \left( \text{diag}(e x^{s_{1}}, \ldots, e x^{s_{n}}) \right) = \text{diag} \left( \pi \left( x^{s_{j} l_{j}} \right) : 1 \leq i \leq m, \ 1 \leq j \leq n, \ 1 \leq l \leq k \right).$$

Observe that the index $i$ only affects the diagonal entries of $\hat{\Delta}(ex)$ by repeating each entry $m$ times. Furthermore $\{s_{j} l_{j} : 1 \leq j \leq n, 1 \leq l \leq k \}$ is a transversal of $Q \times D_p$ in $G$. Note that if $x \not\equiv x$, then $x^{\alpha} = ax$ with $1 \not\equiv a \in A$ and so $\pi(x^{\alpha}) = \pi(a) \pi(x) \not\equiv \pi(x)$ by the choice of $\pi$. So $\hat{\Delta}(ex)$ is a diagonal matrix with $[G : E]$ different eigenvalues each with multiplicity $m[E : Q \times D_p]$.

On the other hand we have

$$\hat{\Delta}(v) = \Delta \left( \sum_{i=1}^{m} \sum_{j=1}^{n} e_{i} s_{j} x^{g_{i} j} \right) = \Delta \left( \text{diag} \left( \sum_{i=1}^{m} e_{i} x^{g_{i_{1}} s_{1}}, \ldots, \sum_{i=1}^{m} e_{i} x^{g_{i_{m}} s_{n}} \right) \right)$$

$$= \text{diag} \left( \pi \left( x^{g_{i_{j}} s_{j}} \right) : 1 \leq i \leq m, \ 1 \leq j \leq n, \ 1 \leq l \leq k \right)$$

Let $\{v_1, \ldots, v_r\}$ be a transversal of $E$ in $T$ and set $Y_{j_0} = \{(i, j, l) : g_{i j} s_{j} l_{i} \equiv s_{j_0} \mod E\}$ and $Y_{j_0} = \{(i, j, l) : g_{i j} s_{j} v_{l} \equiv s_{j_0} \mod E\}$. Arguing as in the previous paragraph we deduce that the multiplicity of $\pi(x^{g_{i_{j}}})$ as eigenvalue of $\hat{\Delta}(v)$ is $|Y_{j_0}| = |Y_{j_0}|[E : Q \times D_p]$. As $v$ and $ex$ are conjugate in $eKG$ the multiplicities of the eigenvalues of $\hat{\Delta}(ex)$ and $\hat{\Delta}(v)$ coincide. Thus $m = |Y_{j_0}|$. Finally, $(i, j, l) \mapsto (i, j)$ defines a bijective map $Y_{j_0} \to X_{j_0}$. Therefore $|X_{j_0}| = m$ as desired.

**Corollary 2.6.** Let $G$ be a finite group and $A$ a normal cyclic subgroup of $G$ containing $G'$. Let $D = Z(C_G(A))$ and $u$ a torsion unit in $ZG$ satisfying $\omega_D(u) = 1$. Then $u$ is conjugate in $QG$ to an element of $D$. 

\[\square\]
Proof. Since $G/A$ is abelian there exists some $b \in G$ such that $\omega_A(u) = bA$. We claim that \{\{n \in D : \varepsilon_n(u) \neq 0\}\} is contained in $bA$. Indeed, let $p$ be some prime dividing $|u|$. By Theorem 2.5, $u_p$ is conjugate in $\mathbb{Z}_pG$ to some $n_{p,0} \in D$. By statement (i) of Proposition 1.4, if $\varepsilon_n(u) \neq 0$ then $n_p$ is conjugate to $n_{p,0}$, so $n_pA = n_{p,0}A$. On the other hand we have that $b_pA = \omega_A(u_p)$ is conjugate to $n_{p,0}A = \omega_A(n_{p,0})$, so $b_pA = n_{p,0}A$ and hence $b_pA = n_{p,0}A$ for all primes $p$ dividing $|u|$ and all $n \in D$ with $\varepsilon_n(u) \neq 0$. As \{\{n\}\} divides \{\{u\}\}, by statement (ii)(b) of Proposition 1.4, we deduce that $nA = bA$ and the claim is proved.

Assume that the statement of the corollary is false. Then, by Proposition 1.3 and Lemma 2.3, there exists an $n \in D$ such that $\varepsilon_n(u) < 0$. So $n \in bA$, by the previous paragraph. Moreover, the order of $n$ divides the order of $u$, by statement (ii)(b) of Proposition 1.4. By Remark 1.1, there exists a subgroup $H$ in $D$ such that $H \cap A = 1$ and $D/H$ is cyclic. So $D$ has a linear character $\eta$ with kernel $H$. Then, by Lemma 2.4, we have $0 \leq \sum_{h \in H} [C_G(hn):D]\varepsilon_{hn}(u)$. But if $\varepsilon_{hn}(u) \neq 0$ with $h \in H$ then $hn \in bA \cap nH = nA \cap nH = n(A \cap H) = \{n\}$. Hence
\[0 \leq \sum_{h \in H} [C_G(hn):D]\varepsilon_{hn}(u) = [C_G(n):D]\varepsilon_n(u) < 0,
\]
a contradiction. \hfill \qed

3. Reduction to torsion units of $D$-augmentation 1

In this section we complete the proof of our main theorem, by using that it has been proved for units with $\omega_B(u) = 1$ (Corollary 2.6), to prove it for the remaining units with an induction argument. (Here $A$ and $D$ are as in Corollary 2.6.) Our arguments follow the lines of the proof of Theorem 7.1 of [Her08a]. Hertweck examined the multiplicities of eigenvalues of the images of the relevant torsion units under a representation of $G$ affording the character induced from a faithful linear character of $A$. In our study we need to replace $A$ by $D = Z(C_G(A))$. We consider linear characters of $D$, whose kernel does not contain any non-trivial normal subgroup of $G$. Observe that if $D$ is cyclic then the characters satisfying this condition are precisely the faithful linear characters of $D$.

As it is customary $\varphi$ denotes Euler’s totient function. For every integer $n$ we let $\zeta_n$ denote a fixed complex primitive root of unity of order $n$.

Let $N$ be an abelian normal subgroup of $G$ and let $\psi_1, \ldots, \psi_m$ be the linear characters of $N$. The following formula is easy to check for $u \in G$ by the orthogonality relations and by linearity it holds in general
\[|C_G(x)|\varepsilon_x^G(u) = \sum_{i=1}^m \psi_i(x)\psi_i^G(u) \quad \text{for every } x \in N \text{ and } u \in CG. \quad (3.1)\]

Let
\[\mathbb{K} = \mathbb{K}_N = \{K \leq N : N/K \text{ is cyclic and } K \text{ does not contain any non-trivial normal subgroup of } G\}.\]

For every $K \in \mathbb{K}$ we select a linear character $\psi_K$ of $N$ with kernel $K$ and we fix a representation $\rho_K$ of $G$ affording the induced character $\psi_K^G$. Observe that if $K_1$ and $K_2$ are conjugate in $G$ then $\psi_{K_1}^G = \psi_{K_2}^G$ and therefore we may assume that $\rho_{K_1} = \rho_{K_2}$. Let $C_\mathbb{K}$ be the set of $G$-conjugacy classes of elements of $\mathbb{K}$. For every $C \in C_\mathbb{K}$ select a representative $K_C$ of $C$. Let $Q_K = \mathbb{Q}(\zeta_{[N:K]}^\mathbb{K})$.

For a square matrix $U$ with entries in $\mathbb{C}$ and $\alpha \in \mathbb{C}$ let $\mu_U(\alpha)$ denote the multiplicity of $\alpha$ as eigenvalue of $U$. If $U_m = 1$ and $\alpha^m = 1$ then we have the following formula (see [LP89])
\[\mu_U(\alpha) = \frac{1}{m} \sum_{d|m} \frac{\text{tr}(\alpha^d)}{\text{tr}(U^d)\alpha^{-d}}. \quad (3.2)\]
This formula is the bulk of the Luthar–Passi Method.

**Lemma 3.1.** Let $G$ be a finite group such that (ZC1) holds for every proper quotient of $G$. Let $N$ be an abelian normal subgroup of $G$. Let $u$ be a unit of $ZG$ with $\omega_N(u) \neq 1$ and let $x \in N$.

(i) Then

$$|C_G(x)|\varepsilon_x(u) = \sum_{K \in \mathbb{K}} \text{tr}_{Q_K/Q}(\overline{\psi_K(x)}\psi_K^G(u)).$$ \hspace{1cm} (3.3)

(ii) Assume moreover that $m = |u|$, $f = |\omega_N(u)|$, $x^m = 1$ and $u^d$ is conjugate in $QG$ to an element of $G$ for every $1 \neq d | m$. Then for every $h \neq 1$ we have

$$\sum_{K \in \mathbb{K}} \left[ Q_K : Q \right] \mu_{p_K(u)}(\psi_K(x)) = \frac{\varphi(m)}{m} |C_G(x)|\varepsilon_x(u) + \frac{1}{h} \sum_{K \in \mathbb{K}} \left[ Q_K : Q \right] \mu_{p_K(u^h)}(\psi_K(x^h))$$ \hspace{1cm} (3.4)

(iii) Assume moreover that $G'$ is cyclic and $u^f$ is conjugate in $QG$ to an element $y$ of $N$. Let $u_C = \{g \in G : x^f \in y^g K_C^q\}$ for $C \in \mathcal{C}_G$. Then we have

$$\sum_{K \in C} \mu_{p_K(u^f)}(\psi_K(x^f)) = \frac{|C_G(y) : N|}{|N_G(K_C)|} u_C.$$ \hspace{1cm} (3.5)

**Proof.** (i) Let $\psi$ be a linear character of $N$ such that the kernel of $\psi$ contains a non-trivial normal subgroup $U$ of $G$. Then $\psi$ is inflated from a linear character $\phi$ of $N/U$. By the induction hypothesis $\omega_U(u)$ is conjugate in $Q(G/U)$ to an element of $G'/U \setminus N/U$. Then $\psi^G(u) = \phi^{G/U}(\omega_U(u)) = 0$. Hence we can drop in (3.1) all the summands labeled by linear characters of $N$ whose kernel is not in $\mathbb{K}$. The remaining characters are those of the form $\sigma \circ \psi_K$ for a $K \in \mathbb{K}$ and $\sigma \in \text{Gal}(Q_K/Q)$. Hence

$$|C_G(x)|\varepsilon_x(u) = \sum_{K \in \mathbb{K}} \sum_{\sigma \in \text{Gal}(Q_K/Q)} \sigma \circ \psi_K(x)(\sigma \circ \psi_K^G(u)) = \sum_{K \in \mathbb{K}} \text{tr}_{Q_K/Q}(\overline{\psi_K(x)}\psi_K^G(u)),$$

as desired.

(ii) Let $d$ be a divisor of $m$ such that $d \neq 1$ and $\omega_N(u^d) \neq 1$. By hypothesis, $u^d$ is conjugate to an element of $G$ which does not belong to $N$, by Remark 1.2. Therefore $\varepsilon_n(u^d) = 0$ for every $n \in N$. Hence, by (3.3) we have $\sum_{K \in \mathbb{K}} \text{tr}_{Q_K/Q}(\psi_K^G(u^d)\psi_K(n)^{-1}) = 0$. Thus,

if $f \neq 1 | m$, $d \neq 1$ and $n \in N$ then $\sum_{K \in \mathbb{K}} \text{tr}_{Q_K/Q}(\psi_K^G(u^d)\psi_K(n)^{-1}) = 0.$ \hspace{1cm} (3.6)

For every $K \in \mathbb{K}$ and every integer $d$ we use the notation $\mu(K,d) = \mu_{p_K(u^d)}(\psi_K(x^d))$, the multiplicity of $\psi_K(x^d)$ as an eigenvalue of $\rho_K(u^d)$. By (3.2), for every $e | m$ we have

$$\sum_{K \in \mathbb{K}} \left[ Q_K : Q \right] \mu(K,e) = \frac{e}{m} \sum_{K \in \mathbb{K}} \sum_{d|m/e} \left[ Q_K : Q \right] \text{tr}_{Q_K/Q}(\psi_K^G(u^d)\psi_K(x)^{-ed}).$$ \hspace{1cm} (3.7)

Let $d | m$ and $\alpha = \psi_K^G(u^d)\psi_K(x)^{-d}$. Clearly the image of $\psi_K$ is in $Q_K$ and $\psi_K^G(u^d) \in Q(\zeta_m^d)$. Moreover, $x^m = 1$, by hypothesis. Thus $\alpha \in Q_K \cap Q(\zeta_m^d)$. Let $L = Q_K(\zeta_m^d)$. Then

$$[L : Q_K]\text{tr}_{Q_K/Q}(\alpha) = (\text{tr}_{Q_K/Q} \circ \text{tr}_{L/Q_K})(\alpha) = \text{tr}_{L/Q}(\alpha)$$

$$= (\text{tr}_{Q(\zeta_m^d)/Q} \circ \text{tr}_{L/Q(\zeta_m^d)})(\alpha) = [L : Q(\zeta_m^d)]\text{tr}_{Q(\zeta_m^d)/Q}(\alpha).$$
Moreover, by (3.6), this is 0 provided \( f \mid d \mid m \) and \( d \neq 1 \). Thus, for \( e = 1 \) and \( 1 \neq h \mid f \), (3.7) can be reduced to the following

\[
\sum_{K \in \mathcal{K}} [Q_K : Q] \mu(K, 1) = \frac{|Q(\zeta^d_m) : Q|}{m} \left( \frac{|C_G(x)|}{m} \right) \varepsilon_x(u) + \frac{1}{m} \sum_{K \in \mathcal{K} \Delta|d|m} [Q_K : Q] \sum_{h \in [x, h] \mathcal{G}} \frac{|C_G(x)|}{m} \varepsilon_x(u) + \frac{1}{h} \sum_{K \in \mathcal{K}} [Q_K : Q] \mu(K, h),
\]

where in the last equality we have used (3.7) for \( e = h \). This proves (3.4).

(iii) Finally assume that \( G' \) is cyclic and \( u^f \) is conjugate in \( QG \) to \( y \in N \). Then \( \rho_K(u^f) \) and \( \text{diag}(\psi_K(y^g) : g \in T) \) are conjugate in the matrices over \( K \), where \( T \) is a transversal of \( N \). Observe that \( \psi_K(y^g) = \psi_K(y^h) \) if and only if \( \psi_K((y, g), h) = \psi_K((y, g), h) \) if and only if \( (y, g) \in K \) if and only if \( (y, g) = (y, h) \) (because \( K \cap G' = 1 \)), if and only if \( gh^{-1} \in C_G(y) \). Therefore each eigenvalue of \( \rho_K(u^f) \) has multiplicity \( |C_G(y) : N| \). On the other hand \( \psi_K(x^f) \) is an eigenvalue of \( \rho_K(u^f) \) if and only if \( \psi_K(x^f) = \psi_K(y^g) \) for some \( g \in G \), if and only if \( x^f \in y^gK \). Therefore if \( C \subset K \) and \( K_C \) is a representative of \( C \) then

\[
\sum_{K \in C} \mu_{\rho_K(u^f)}(\psi_K(x^f)) = \frac{1}{|N_G(K_C)|} \sum_{g \in G, x^f \in y^gK_C^y} [C_G(y) : N] = \frac{|C_G(y) : N|}{|N_G(K_C)|} u_C,
\]

as desired.

**Remark 3.2.** Let \( A \) be a cyclic normal subgroup of \( G \) containing \( G' \) and \( N \) an abelian subgroup of \( G \). Clearly every element of \( K_N \) intersects \( A \) and \( Z(G) \) trivially. Conversely, let \( H \) be a subgroup of \( G \) containing a non-trivial normal subgroup \( U \) of \( G \) and such that \( H \cap Z(G) = 1 \). If \( 1 \neq n \in U \) then \( 1 \neq (n, g) \in A \cap U \) for some \( g \in G \) and therefore \( A \cap H \neq 1 \). Thus \( K_N = \{ K \leq N : A \cap K = Z(G) \cap K = 1 \} \) and \( N/K \) is cyclic.

Observe that \( K_N \) can be empty. For example, this is the case if \( N \cap Z(G) \) is not cyclic.

**Lemma 3.3.** Assume that \( A \) is a cyclic subgroup of \( G \) containing \( G' \). Let \( N \) be an abelian subgroup of \( G \) containing \( A \) and \( K = K_N \). Then for every \( K \in K \) we have \( |K| \leq |K| = \frac{|N|}{\exp(N)} \).

**Proof.** Write \( N = C \times H \) with \( C \) cyclic of maximal order in \( N \) and selected in such a way that if \( p \) is prime and \( \exp(N_p) = \exp(A_p) \), then \( C_p = A_p \). We claim that if \( K \in K \), then \( C \cap K = 1 \). Otherwise \( C_p \cap K \neq 1 \) for some prime \( p \) and therefore \( \exp(C_p) = \exp(N_p) > \exp(A_p) \). Let \( x \) be a generator of \( C_p \), \( q = |A_p| \) and \( a = (x, g) \) with \( g \in G \). Then \( a \in A_p \) and therefore \( a^q = 1 \). Thus \( (x^q)^3 = x^3 \). This proves that \( x^3 \) is a non-trivial central element of \( G \). Then \( Z(G) \cap K \neq 1 \), contradicting the fact that \( K \) does not contain any normal subgroup of \( G \). This proves the claim.
Let $\pi_1$ and $\pi_2$ be the projections $N \to C$ and $N \to H$ along the decomposition $N = C \times H$. By the previous paragraph $K \cap \ker \pi_2 = 1$ and therefore $|K| \leq |H| = \frac{|N|}{\exp(N)}$. As $N/K$ is cyclic we have $\exp(N) \leq |N : K| = \exp(N/K) \leq \exp(N)$. Hence $|K| = \frac{|N|}{\exp(N)} = |H|$ and therefore $\pi_2|_K : K \to H$ is an isomorphism for every $K \in \mathbb{K}$. Therefore $K = \{f(h)h : h \in H\}$ for a homomorphism $f : H \to C$. (More precisely $f = \pi_1 \circ \pi_2|^{-1}_K$.) Thus $K$ is completely determined by $f$ and hence $|\mathbb{K}| \leq |\text{Hom}(H,C)| = |H|$. The last equality follows easily from the fact that $C$ is cyclic and $\exp(H)$ divides the order of $C$. \qed

We are ready to prove our main result.

**Theorem 3.4.** If $G$ is a cyclic-by-abelian finite group then every normalized torsion unit of $ZG$ is conjugate in $QG$ to an element of $G$.

**Proof.** By means of contradiction we assume that $G$ is a counterexample of minimal order to the theorem and $u$ is a normalized torsion unit of minimal order of $ZG$ which is not conjugate to an element of $G$ in $QG$. We select a cyclic subgroup $A$ of $G$ with $G/A$ abelian and take $D = Z(C_G(A))$ and $\mathbb{K} = \mathbb{K}_D$. By Proposition 1.3, we may assume without loss of generality that $\varepsilon_x(u) < 0$ for some $x \in G$. This implies that the order of $x$ divides the order of $u$ by statement (ii)(b) of Proposition 1.4. Set $m = |u|$ and $f = |\omega_D(u)|$. By assumption $u^d$ is conjugate in $QG$ to an element of $G$ for every $1 \neq d \mid m$.

By Lemma 2.3, $x \in D$ and by Corollary 2.6, $\omega_D(u) \neq 1$. Thus $1 \neq f \mid m$ and in particular, $u^f$ is conjugate in $QG$ to some $y \in D$. By the first induction hypothesis (ZC1) holds for every proper quotient of $G$ and hence we can use Lemma 3.1 for $N = D$, $u$ and $x$. Since $|K| = \frac{|D|}{\exp(D)}$ for every $K \in \mathbb{K}$, by Lemma 3.3, and $[Q_K : Q] = \varphi([D : K])$ we can write (3.4) for $h = f$ as

$$
\sum_{K \in \mathbb{K}} \mu_{\pi_K(u)}(\psi_K(x)) = \frac{\varphi(m)}{m} \frac{|C_G(x)|}{\varphi([D : K])} \varepsilon_x(u) + \frac{1}{f} \sum_{K \in \mathbb{K}} \mu_{\pi_K(u^f)}(\psi_K(x^f)). \tag{3.8}
$$

We claim that

$$
\frac{1}{f} \sum_{K \in \mathbb{K}} \mu_{\pi_K(u^f)}(\psi_K(x^f)) \leq \frac{\varphi(m)}{m} \frac{|C_G(x)|}{\varphi([D : K])}. \tag{3.9}
$$

Write $f = f_1 f_2$ with $f_1$ and $f_2$ positive integers such that the prime divisors of $f_1$ divide $|D|$ and $(f_2,|D|) = 1$. Then $m = f_2 m'$ with all prime divisors of $m'$ dividing $|D|$. Note that $(x^f) = (x^{f_1})$ and so $C_G(x^f) = C_G(x^{f_1})$. Consider the map $\alpha : C_G(x^{f_1}) \to A$ given by $g \mapsto (x,g)$. If $a = (x,g)$ then $x^g = ax$ and therefore $x^{f_1} = (x^{f_1})^g = a^{f_1} x^{f_1}$. Hence the image of $\alpha$ is contained in $\{a \in A : a^{f_1} = 1\}$ and this is a subgroup of $A$ of order $\leq f_1$. On the other hand $\alpha(g) = \alpha(h)$ if and only if $gh^{-1} \in C_G(x)$. Therefore

$$
[C_G(x^{f_1}) : C_G(x)] \leq f_1. \tag{3.10}
$$

Assume that $K \in \mathbb{K}$ and $y_1$ and $y_2$ are elements of $G$ in the same conjugacy class such that $y_1 K = y_2 K$. Then $y_2 \in y_1 A \cap y_1 K = \{y_1\}$ because $A \cap K = 1$. Therefore, if $C \in \mathbb{C}_K$, then $\{g \in G : (x^{f_1})^g = y_1 C_G(K)\}$ is the disjoint union of the subsets $X_{C,y_1} = \{g \in G : (x^{f_1})^g = y_1 K\}$ with $y_1 \in y_1^G$. If $g,h \in X_{C,y_1}$ then $(x^{f_1})^{gh} = ((x^{f_1})^h)^{(x^{f_1})^{-1}} = x^{f_1} h^{-1} k_{h^{-1}}$ for some $k \in K_C$. Then $(x^{f_1}, gh^{-1}) \in A \cap K_{h^{-1}} = 1$ and hence $gh^{-1} \in C_G(x^{f_1})$. Conversely, if $gh^{-1} \in C_G(x^{f_1})$ and $g \in X_{C,y_1}$ then $h \in X_{C,y_1}$. This proves that if $X_{C,y_1}$ is not empty, then it is a coset of $C_G(x^{f_1}) = C_G(x^{f_1})$. Therefore for $u_C$ as in Lemma 3.1 we get, using (3.10),

$$
u_C = \sum_{y_1 \in y_1^G} |X_{C,y_1}| \leq |C_G(x^{f_1})| |y_1^G| = |C_G(x^{f_1})| |G : C_G(y)| \tag{3.11}
$$

$$
\leq f_1 |C_G(x)| |G : C_G(y)|,
$$
and hence

\[ \sum_{C \in \mathcal{C}_K} \frac{u_C}{|N_G(K_C)|} \leq \frac{1}{f_1} |C_G(x)| \sum_{C \in \mathcal{C}_K} [G : N_G(K_C)] = \frac{1}{f_1} |C_G(x)| |\mathbb{K}|. \quad (3.12) \]

Thus

\[ \frac{1}{f_1} |C_G(y) : D| \sum_{C \in \mathcal{C}_K} \frac{u_C}{|N_G(K_C)|} \leq \frac{1}{f_2} |\mathbb{K}| |C_G(x) : D|. \quad (3.13) \]

By Lemma 3.3, \(|\mathbb{K}| \leq |K|\). Moreover, every prime divisor of \(m'\) divides \(\exp(D) = |D : K|\) and therefore

\[ \frac{1}{f_2} \leq \frac{\varphi(f_2) \varphi(m')}{f_2 m'} \frac{|D : K|}{\varphi([D : K])} = \frac{\varphi(m)}{m} \frac{|D : K|}{\varphi([D : K])}. \quad (3.14) \]

Thus

\[ \frac{1}{f_2} |\mathbb{K}| |C_G(x) : D| \leq \frac{\varphi(m)}{m} \frac{|D : K|}{\varphi([D : K])} |\mathbb{K}| |C_G(x) : D| = \frac{\varphi(m)}{m} \frac{|C_G(x)|}{\varphi([D : K])}. \quad (3.15) \]

Combining (3.5), (3.13) and (3.15) we conclude that

\[ \frac{1}{f_1} \sum_{K \in \mathbb{K}} \mu_{\rho_K(u)}(\psi_K(x^j)) \leq \frac{1}{f_1} |C_G(y) : D| \sum_{C \in \mathcal{C}_K} \frac{u_C}{|N_G(K_C)|} \leq \frac{\varphi(m)}{m} \frac{|C_G(x)|}{\varphi([D : K])}. \quad (3.16) \]

This proves (3.9).

As the left side of (3.8) is non-negative we have

\[ \frac{1}{f_1} \sum_{K \in \mathbb{K}} \mu_{\rho_K(u)}(\psi_K(x^j)) \geq -\frac{\varphi(m)}{m} \frac{|C_G(x)|}{\varphi([D : K])} \varepsilon_x(u) \geq \frac{\varphi(m)}{m} \frac{|C_G(x)|}{\varphi([D : K])}, \]

because \(\varepsilon_x(u) < 0\). Therefore equality holds in (3.9) and hence equality holds in all the displayed formulas from (3.10) to (3.16). This has the following consequences: \(\varphi(f_2) = 1\), so that \(f_2 \leq 2\); \(|C_G(x^j)| = f_1|C_G(x)|\) and hence the conjugacy class of \(x\) contains all the elements of the form \(a^j x\), where \(a\) is an element of \(A\) of order \(f_1\); \(m\) is divisible by all the primes dividing \(|D|\); \(X_{C_{y_i}} \neq \emptyset\) for every \(y_i \in y^G\); the left hand side of (3.8) is zero and hence \(\psi_K(x)\) is not an eigenvalue of \(\rho_K(u)\) for every \(K \in \mathbb{K}\). Applying this to conjugates of \(x\) we deduce that \(\psi_K(x^g)\) is not an eigenvalue of \(\rho_K(u)\) for every \(g \in G\) and every \(K \in \mathbb{K}\). We claim that \(f_2 = 2\) and all the eigenvalues of \(\rho_K(u)\) have even order. If \(\xi\) is an eigenvalue of \(\rho_K(u)\) then \(\xi^f\) is an eigenvalue of both \(\rho_K(u^f)\) and \(\rho_K(y)\). Thus \(\xi^f = \psi_K(y_1)\) for some \(y_1 \in y^G\). As \(X_{C_{y_1}} \neq \emptyset\), \(\xi^f = \psi_K((x^j)^g)\) for some \(g \in G\) and therefore \(\xi = \zeta_j^f \psi_K(x^g)\) for some \(j\). However, \(a^j x^g\) is conjugate to \(x^g\) for every \(0 \leq i < f_1\). Hence \(\psi_K(a^j x^g) = \zeta_j^f \psi_K(x^g)\) is not an eigenvalue of \(\rho_K(u)\) for every \(0 \leq i < f_1\). This implies that \(f_2 = 2\) and \(j\) is odd. Therefore the order of \(\xi\) is even. This proves the claim.

Then all the eigenvalues of \(\rho_K(u^\Xi)\) are equal to \(-1\) and hence \(\psi_K(u^\Xi) = -|G : D|\). However \(u^\Xi\) is conjugate to an element of \(G \setminus D\) and therefore \(\psi_K(u^\Xi) = 0\), a contradiction.

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