[1]Moment Lyapunov exponent and stochastic stability of a vibro-impact system driven by non-Gaussian colored noise

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Abstract Vibro-impact phenomena are prevalent in practical engineering, making research on their stochastic dynamic characteristic of great practical significance. However, the research on stochastic stability and bifurcation of vibro-impact systems is still limited, especially the moment stability. In this paper, based on the *p*th moment Lyapunov exponent, the stochastic stability of a vibro-impact system driven by non-Gaussian colored noise is investigated. Firstly, the smooth stochastic dynamic system is obtained making use of a non-smooth transformation and the non-Gaussian colored noise is simplified to an Ornstein-Uhlenbeck process by utilizing the path-integral method. Thereafter, through applying the L.Arnold perturbation method, the second-order approximate solution of the *p*th moment Lyapunov exponent is calculated, which agree well with the simulation results given by the Monte Carlo method. Finally, the effects of the noise parameters, natural frequency, coefficient of restitution, and damping coefficient on the stochastic stability of the vibro-impact are studied. Due to the existence of impact factor, the natural frequency has a direct and significant effect on the stochastic stability of the system.

Keywords *Stochastic stability*; *Moment Lyapunov exponent*; *Largest Lyapunov exponent*; *Perturbation method*; *Monte Carlo simulation*; *Vibro-impact system.*

1. Introduction

The vibro-impact system, as a typical example of a non-smooth system, is widely prevalent in our daily lives and engineering fields. It can be observed in various applications such as woodpecker toys, car braking systems, vibrating pile drivers, and impact shock absorbers. The presence of vibro-impact significantly influences the dynamic performance, reliability, and lifespan of structures. Due to the universality of vibro-impact in practical engineering field and its importance in aerospace, machinery manufacturing, transportation and energy fields, the study of dynamic characteristics of the vibro-impact systems has

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become increasingly urgent and attracted the attention of many scholars. Furthermore, it is important to note that the loads acting on structures often exhibit random characteristics. These include forces arising from strong winds, waves, atmospheric turbulence, explosions, and earthquakes. Such random forces can induce the system to exhibit complex dynamic behaviors that are fundamentally different from those observed in deterministic systems. Therefore, clarifying the dynamic characteristics of vibro-impact systems under random excitation is more conducive to engineering applications, which also has attracted the research interest of many scholars [1-6].

In the previous decades, there have been significant advancements in the research on the dynamic characteristics of vibro-impact systems under random perturbations [7-9]. Notably, Nayak [10] conducted pioneering work in 1972, exploring the dynamic characteristics of random vibro-impact systems. For a vibro-impact system driven by Gaussian white noise, Jing [11] obtained the exact stationary solution based on the Hertz contact theory. Huang [12] employed the stochastic averaging method to investigate the stationary response of a multi-degree-of-freedom vibro-impact system excited by Gaussian white noise. By applying an improved stochastic averaging method, Namachchivaya [13] examined the stochastic dynamics of a random vibro-impact system. Based on the average energy loss, Gu [14] proposed a new method to study the response of random vibro-impact systems excited by Gaussian white noise. For a vibro-impact system under real noise excitations, Liu [15] applied a similar method to investigate the stationary probabilistic response. Recently, Wang [16, 17] introduced a new path integration method to explore the stochastic response and stochastic bifurcation problems of vibro-impact systems. The study of stochastic bifurcation problems in vibro-impact systems has captured the attention of numerous scholars, leading to significant achievements [18-21]. Xu [22] provided a comprehensive review on the research of stochastic non-smooth systems.

The largest Lyapunov exponent serves as a crucial indicator for studying the local stability and bifurcation behavior of stochastic dynamical systems. For a vibro-impact system with Gaussian white noise perturbation, Feng [23] proposed a method to calculate the largest Lyapunov exponent and subsequently investigated the stochastic bifurcation based on the largest Lyapunov exponent. By calculating the largest Lyapunov exponent, Kumar [24, 25] examined the stochastic bifurcation of a Duffing–Van der Pol vibro-impact system driven by Gaussian white noise. Moreover, Wang [26] recently explored the almost sure stability of a vibro-impact system excited by bounded random perturbations through the computation of the largest Lyapunov exponent. According to the large deviation theory [27], it is important to recognize that even if the stochastic dynamical system is stable with probability 1, the *p*th moment of response may still be unstable and grow exponentially. Therefore, it becomes essential to investigate the moment stability of a stochastic dynamical system. The moment stability is provided by the *p*th moment Lyapunov exponent defined as

$$\Lambda\left(p, \mathbf{x}_0\right) = \lim_{t\to+\infty} \frac{1}{t} \log E\left[\left\|\mathbf{x}(t;\mathbf{x}_0)\right\|^p\right], \tag{1}$$

where $\mathbf{x}\left(t;\mathbf{x}_0\right)$ is the state vector of a stochastic dynamical system. If $\Lambda\left(p,\mathbf{x}_0\right)<0$, then the *p*th moment of the solution $\mathbf{x}\left(t;\mathbf{x}_0\right)$ is stable; otherwise, it is unstable. According to the results of Arnold [28], the limit of Eq. (1) exists and is independent of $\mathbf{x}_0$ under the specified conditions (i.e., $\Lambda\left(p,\mathbf{x}_0\right)=\Lambda\left(p\right)$). And the derivative of *p*th moment Lyapunov exponent at $p=0$ is equal to the largest Lyapunov exponent λ [29], i.e.,

$$\lambda = \frac{\mathrm{d}}{\mathrm{d}p}\Lambda\left(p\right)\bigg|_{p=0} = \lim_{t\to\infty}\frac{1}{t}\log\left\|\mathbf{x}\left(t,\mathbf{x}_0\right)\right\|. \tag{2}$$

The stochastic stability, D-bifurcation and P-bifurcation [27, 30] of a stochastic dynamical system can be determined by the *p*th moment Lyapunov exponent. Consequently, the *p*th moment Lyapunov exponent offers a more comprehensive and profound description of the stochastic dynamical characteristics exhibited by nonlinear dynamical systems.

Although the *p*th moment Lyapunov exponent plays a crucial role in the study of stochastic dynamical systems, its calculation can be particularly challenging. In recent decades, researchers have made significant progress in investigating the *p*th moment Lyapunov exponents for smooth stochastic dynamical systems, yielding fruitful results[31-36]. However, for a non-smooth stochastic dynamical systems, the research on the *p*th moment Lyapunov exponent is still limited. Furthermore, the random excitations often assumed to be Gaussian white noises for the sake of mathematical simplicity, but the Gaussian white noise is only an ideal model and does not exist in practical engineering. The objective of this paper is to explore the *p*th moment Lyapunov exponent and stochastic stability of a vibro-impact system driven by non-Gaussian colored noise. A brief outline of this paper is as follows. In Section 2, the dynamic model of a Rayleigh-Van der Pol stochastic vibro-impact system is introduced. In Section 3, the non-Gaussian colored noise is simplified to an Ornstein-Uhlenbeck random process through applying the path-integral method. In Section 4, the second-order asymptotic analytic solution of the *p*th moment Lyapunov exponent is calculated by employing the L.Arnold perturbation method. And then in Section 5, based on the *p*th moment Lyapunov exponent, stability index and largest Lyapunov exponent, the effects of noise and system parameters on the stochastic dynamics of the vibro-impact system are studied and discussed in detail. Conclusions are drawn in Section 6.

2. Formulation

Consider a Rayleigh-Van der Pol stochastic vibro-impact system, whose governing equation is presented as

$$\begin{aligned} &\ddot{x}+\omega^2 x+\left(-c_1+c_2 x^2+c_3\dot{x}^2\right)\dot{x}=\xi\left(t\right)\dot{x}, \quad x>0 \\ &\dot{x}_+=-r\dot{x}_-, \quad x=0. \end{aligned} \tag{3}$$

Where ω is the natural frequency of the system, c_1 is the linear damping coefficient, c_2 and c_3 denote the nonlinear damping coefficients, $\xi(t)$ is a non-Gaussian colored noise. r ($0<r\le 1$) denotes the coefficient of restitution, which is used to describe the impact energy loss. $\dot{x}_-$, $\dot{x}_+$ represent the impact velocity and rebound velocity respectively. It is very difficult to deal with the non-smooth stochastic dynamical system (3) directly because of the discontinuity of its motion state. For the convenience of analysis, the system (3) is transformed into a smooth system by applying a non-smooth coordinate transformation [37]. The Zhuravlev transformation is given as

$$x=|y|, \quad \dot{x}=\dot{y}\,\mathrm{sgn}(y), \quad \ddot{x}=\ddot{y}\,\mathrm{sgn}(y), \tag{4}$$

where

$$\mathrm{sgn}(y)=\begin{cases}1, & y>0,\\ 0, & y=0,\\ -1, & y<0.\end{cases}$$

By introducing the Dirac delta function[7, 18], and substituting the transformation (4) into Eq. (3) , we can get

$$\ddot{y}+\omega^2 y-\left(c_1-c_2 y^2-c_3\dot{y}^2\right)\dot{y}+(1-r)\dot{y}|\dot{y}|\delta(y)=\dot{y}\xi(t). \tag{5}$$

where $\delta(\bullet)$ is a Dirac delta function. In order to investigate the stochastic stability of the vibro-impact system analytically, we assume the random term is a small quantity of parameter ε ($0<\varepsilon<<1$) and the impact energy loss and damping term are both small quantity of parameter ε^2 . Then, the system (5) becomes

$$\ddot{y}+\omega^2 y+\varepsilon^2\left(-c_1+c_2 y^2+c_3\dot{y}^2+(1-r)|\dot{y}|\delta(y)\right)\dot{y}=\varepsilon\dot{y}\xi(t). \tag{6}$$

3. Approximation to the Markov process

The non-Gaussian colored noise $\xi(t)$ is the solution of the following differential equation

$$\frac{\mathrm{d}\xi(t)}{\mathrm{d}t}=-\frac{1}{\tau_0}\frac{\mathrm{d}}{\mathrm{d}\xi}V_{r_0}(\xi)+\frac{1}{\tau_0}\eta(t), \tag{7}$$

$$V_{r_0}(\xi)=\frac{D_0}{\tau_0(r_0-1)}\ln\left[1+\frac{\tau_0}{D_0}(r_0-1)\frac{\xi^2}{2}\right], \tag{8}$$

where $\eta(t)$ is a Gaussian white noise with intensity $2D_0$; τ_0 denotes the noise correlation time, r_0 is the noise departure coefficient which represents the departure from the Gaussian noise. It can be seen that, when $r_0\to 1$, the non-Gaussian color noise $\xi(t)$ is reduced to an exponential Gaussian color noise with the correlation function $\langle\xi(t)\xi(s)\rangle=(D_0/\tau_0)e^{-|t-s|/\tau_0}$, i.e., an Ornstein-Uhlenbeck(O-U) process with the correlation time τ_0. Moreover $\xi(t)$ will be further simplified to a Gaussian white noise if $\tau_0\to 0$.

Based on the results of Fuentes [38], we find that the stationary probability density function $P_s(\xi)$ of Eq. (7) can be normalized if and only if when $r\in(-\infty,3)$, and it is represented as

$$P_s(\xi)=\frac{1}{Z}\left[1+\frac{\tau_0}{D_0}(r_0-1)\frac{\xi^2}{2}\right]^{-1/r_0-1}, \tag{9}$$

where Z denotes a normalization constant. According to Eq. (9), one easily obtains the statistical properties of the process $\xi(t)$:

$$\begin{aligned}&\langle\xi(t)\rangle=0,\\&\langle\xi^2(t)\rangle=\begin{cases}2D_0/\left[\tau_0(5-3r_0)\right], & r_0\in(-\infty,5/3),\\ \infty, & r_0\in(5/3,3).\end{cases}\end{aligned} \tag{10}$$

For $|r-1|<<1$, the following equation is obtained by applying the path-integral method [38-40].

$$\frac{1}{\tau_0}\frac{\mathrm{d}}{\mathrm{d}\xi}V_{r_0}(\xi)=\frac{\xi}{\tau_0}\left[1+\frac{\tau_0}{D_0}(r_0-1)\frac{\xi^2}{2}\right]^{-1}\approx\frac{\xi}{\tau_0}\left[1+\frac{\tau_0}{D_0}(r_0-1)\frac{\langle\xi^2\rangle}{2}\right]^{-1}=\frac{\xi}{\tau_1}, \tag{11}$$

with the associated noise intensity D_1 and the noise correlation time τ_1 respectively are

$$D_1=\left(\frac{2(2-r_0)}{5-3r_0}\right)^2 D_0, \qquad \tau_1=\frac{2(2-r_0)}{5-3r_0}\tau_0. \tag{12}$$

Thus, the non-Gaussian colored noise $\xi(t)$ is reduced to an O-U process with the associated noise intensity D_1 and the noise correlation time τ_1

$$\frac{\mathrm{d}\xi(t)}{\mathrm{d}t}=-\frac{1}{\tau_1}\xi(t)+\frac{1}{\tau_1}\eta_1(t),\text{ i.e., }\ \mathrm{d}\xi(t)=-\alpha_0\xi(t)\mathrm{d}t+\sigma_0\circ\mathrm{d}W(t), \tag{13}$$

where

$$
\begin{cases}
\langle \eta_1(t) \rangle = 0, & \langle \eta_1(t)\eta_1(s) \rangle = 2D_1\delta(t-s), \\
\alpha_0 = 1/\tau_1, & \sigma_0 = \sqrt{2D_1}/\tau_1 = \sqrt{2D_0}/\tau_0,
\end{cases}
\tag{14}
$$

$W(t)$ is a normal Wiener process and the symbol " $\circ$ " denotes a Stratonovich stochastic integral. Because the diffusion coefficient of the O-U process (13) is constant, the Wong–Zakai correction term of Eq. (13) is equal to zero. Hence, the Itô stochastic differential equation of Eq. (13) is given as

$$
\mathrm{d}\xi(t) = -\alpha_0\xi(t)\mathrm{d}t + \sigma_0\mathrm{d}W(t), \tag{15}
$$

with the power spectral density

$$
S(\omega) = \frac{\sigma_0^2}{\alpha_0^2 + \omega^2}. \tag{16}
$$

Neglecting the nonlinear terms in system (6) implies

$$
\begin{aligned}
&\mathrm{d}y_1 = \omega y_2 \mathrm{d}t, \\
&\mathrm{d}y_2 = \left(-\omega y_1 + \varepsilon^2\left(c_1 + (r-1)\omega|y_2|\delta(y_1)\right)y_2 + \varepsilon\xi(t)\,y_2\right)\mathrm{d}t, \\
&\mathrm{d}\xi(t) = -\alpha_0\xi(t)\mathrm{d}t + \sigma_0\mathrm{d}W(t),
\end{aligned}
\tag{17}
$$

here $y = y_1, \dot{y} = \omega y_2$.

By using the transformation $y_1 = e^{\rho}\cos\varphi, y_2 = -e^{\rho}\sin\varphi, \varphi \in [0, 2\pi]$, Eq. (17) is converted to

$$
\begin{aligned}
&\mathrm{d}\rho = \left(\varepsilon^2 q_2(\varphi) + \varepsilon q_1(\varphi)\xi(t)\right)\mathrm{d}t, \\
&\mathrm{d}\varphi = \left(\omega + \varepsilon^2 h_2(\varphi) + \varepsilon h_1(\varphi)\xi(t)\right)\mathrm{d}t, \\
&\mathrm{d}\xi(t) = -\alpha_0\xi(t)\mathrm{d}t + \sigma_0\mathrm{d}W(t).
\end{aligned}
\tag{18}
$$

where

$$
q_1(\varphi) = \frac{1}{2}(1 - \cos 2\varphi), \qquad h_1(\varphi) = \frac{1}{2}\sin 2\varphi,
$$

$$
q_2(\varphi) = \frac{1}{2}\left(c_1 + (r-1)\omega|\sin\varphi|\delta(\cos\varphi)\right)(1 - \cos 2\varphi),
$$

$$
h_2(\varphi) = \frac{1}{2}\left(c_1 + (r-1)\omega|\sin\varphi|\delta(\cos\varphi)\right)\sin 2\varphi.
$$

4. Moment Lyapunov exponent

According to Eq. (18), one easily finds that the random process $\varphi(t)$ is independent of the variable ρ. Thus, the random process $\varphi(t)$ alone forms a diffusive Markov process with the following generator:

$$L_\varepsilon(p) = L_0(p) + \varepsilon L_1(p) + \varepsilon^2 L_2(p), \tag{19}$$

where

$$L_0(p) = -\alpha_0 \xi \frac{\partial}{\partial \xi} + \frac{1}{2}\sigma_0^2 \frac{\partial^2}{\partial \xi^2} + \omega \frac{\partial}{\partial \varphi},$$

$$L_1(p) = h_1(\varphi)\xi \frac{\partial}{\partial \varphi} + p q_1(\varphi)\xi,$$

$$L_2(p) = h_2(\varphi)\frac{\partial}{\partial \varphi} + p q_2(\varphi).$$

The moment Lyapunov exponent $\Lambda(p)$ of system (18) is the largest eigenvalue of the operator $L_\varepsilon(p)$ [28, 32, 41] , i.e.,

$$L_\varepsilon(p) T(p) = \Lambda(p) T(p). \tag{19}$$

Both the moment Lyapunov exponent $\Lambda(p)$ and eigenfunction $T(p)$ are expressed as the power series of ε, respectively, i.e.,

$$\begin{aligned} \Lambda(p) &= \Lambda_0(p) + \varepsilon\Lambda_1(p) + \varepsilon^2\Lambda_2(p) + \cdots + \varepsilon^n \Lambda_n(p) + \cdots, \\ T(p) &= T_0(p) + \varepsilon T_1(p) + \varepsilon^2 T_2(p) + \cdots + \varepsilon^n T_n(p) + \cdots. \end{aligned} \tag{20}$$

Substituting Eq. (20) into Eq. (19) gives

$$\varepsilon^0 : \left(L_0(p) - \Lambda_0(p)\right) T_0(p) = 0, \tag{21}$$

$$\varepsilon^1 : \left(L_0(p) - \Lambda_0(p)\right) T_1(p) = \left(\Lambda_1(p) - L_1(p)\right) T_0(p), \tag{22}$$

$$\begin{aligned} &\varepsilon^2 : \left(L_0(p) - \Lambda_0(p)\right) T_2(p) = \left(\Lambda_1(p) - L_1(p)\right) T_1(p) + \left(\Lambda_2(p) - L_2(p)\right) T_0(p), \\ &\qquad \vdots \end{aligned} \tag{23}$$

4.1 Zeroth-order perturbation

Based on the definition of pth moment Lyapunov exponent $\Lambda(p)$, we get that $\Lambda_0(p) \equiv 0$ for any p. Thus, Eq. (21) is reduced to

$$-\alpha_0 \xi \frac{\partial T_0(p)}{\partial \xi} + \frac{1}{2}\sigma_0^2 \frac{\partial^2 T_0(p)}{\partial \xi^2} + \omega \frac{\partial T_0(p)}{\partial \varphi} = 0. \tag{24}$$

The separation of variables method can be employed to solve Eq. (24). Let $T_0(p) = \Phi(\varphi)\Xi(\xi)$, one has

$$\frac{\dot{\Phi}}{\Phi} = c, \qquad -\alpha_0 \xi \frac{\dot{\Xi}}{\Xi} + \frac{1}{2}\sigma_0^2 \frac{\ddot{\Xi}}{\Xi} = -\omega c, \tag{25}$$

which along with the periodic boundary condition $\Phi(\varphi+2\pi)=\Phi(\varphi)$ asserts that $c=0$ and $\Phi(\varphi)$ is a constant. Thus, we have

$$-\alpha_0\xi\frac{\dot{\Xi}(\xi)}{\Xi(\xi)}+\frac{1}{2}\sigma_0^2\frac{\ddot{\Xi}(\xi)}{\Xi(\xi)}=0. \tag{26}$$

From Eq. (26), we can get

$$\Xi(\xi)=C_1+C_2 erf\left(j\frac{\sqrt{\alpha_0}}{\sigma_0}\xi\right),$$

where $erf(\bullet)$ is the error function; j stands for the imaginary unit. Because $\Xi(\xi)$ is bounded as $\xi\to\infty$, so we obtain that $C_2=0$ i.e., $\Xi(\xi)$ is a constant. Therefore, $T_0(p)$ is a constant, i.e., $T_0(p)=C$ .wq

The adjoint equation of Eq. (24) is given as

$$L_0^*(p)T_0^*(p)=0. \tag{27}$$

By applying a similar method to solve Eq. (24), we obtain that

$$T_0^*(p)=\frac{P_s(\xi)}{2\pi C}, \tag{28}$$

where $P_s(\xi)$ is the stationary probability density function of the random process $\xi(t)$.

4.2 First-order perturbation

In light of the results in section 4.1, Eq. (22) is reduced to

$$\begin{aligned}L_0(p)T_1(p)&=\left(\Lambda_1(p)-L_1(p)\right)T_0(p)\\&=\left(\Lambda_1(p)-pq_1(\varphi)\xi\right)T_0(p).\end{aligned} \tag{29}$$

The solvability condition to Eq. (29) is

$$\left\langle\left(\Lambda_1(p)-L_1(p)\right)T_0(p),\ T_0^*(p)\right\rangle=\int_0^{2\pi}\int_{-\infty}^{+\infty}\left(\Lambda_1(p)-L_1(p)\right)T_0(p)T_0^*(p)\,\mathrm{d}\xi\mathrm{d}\varphi=0. \tag{30}$$

Thus, we can get

$$\Lambda_1(p)=\frac{1}{2\pi}\int_0^{2\pi}pq_1(\varphi)\int_{-\infty}^{+\infty}\xi P_s(\xi)\mathrm{d}\xi\mathrm{d}\varphi=0. \tag{31}$$

Then, Eq. (29) becomes

$$L_0(p)T_1(p)=-pq_1(\varphi)\xi T_0(p),\ \text{ i.e.,}$$

$$\left(-\alpha_0\xi\frac{\partial}{\partial\xi}+\frac{1}{2}\sigma_0^2\frac{\partial^2}{\partial\xi^2}+\omega\frac{\partial}{\partial\varphi}\right)T_1(p)=-pC\xi q_1(\varphi). \tag{32}$$

Based on the results of Refs. [30, 32, 33, 42], the solution of Eq. (32) is given as following

$$T_1(p)=T_1(\varphi,\xi;p)=Cp\int_0^{+\infty}q_1(\varphi+\omega\tau)K(\xi,\tau)\mathrm{d}\tau, \tag{33}$$

where

$K(\xi,\tau)=\int_{-\infty}^{+\infty}\xi_\tau P(\xi_\tau,\tau;\xi,0)\,d\xi_\tau$, $P(\xi_\tau,\tau;\xi,0)$ is the transient density.

4.3 Second-order perturbation

By using the results obtained in sections. 4.1 and 4.2, Eq. (23) is converted to

$$L_0(p)T_2(p)=C\left(\Lambda_2(p)-pq_2(\varphi)\right)-L_1(p)T_1(p). \tag{34}$$

The solvability condition to Eq. (29) is

$$\begin{aligned}&\left\langle C\left(\Lambda_2(p)-pq_2(\varphi)\right)-L_1(p)T_1(p),\ T_0^*(p)\right\rangle\\&=\int_0^{2\pi}\int_{-\infty}^{+\infty}\left(\Lambda_2(p)-pq_2(\varphi)\right)\frac{P_s(\xi)}{2\pi}-L_1(p)T_1(p)\frac{P_s(\xi)}{2\pi C}\mathrm{d}\xi\mathrm{d}\varphi=0.\end{aligned} \tag{35}$$

From the solvability condition (35), one easily gets

$$\begin{aligned}\Lambda_2(p)&=\frac{1}{2\pi}\int_0^{2\pi}pq_2(\varphi)\mathrm{d}\varphi+\int_0^{2\pi}\int_{-\infty}^{+\infty}L_1(p)T_1(p)\frac{P_s(\xi)}{2\pi C}\mathrm{d}\xi\mathrm{d}\varphi\\&=\frac{p}{16}\left(\left(S(2\omega)+2S(0)\right)p+2S(2\omega)+8\left(c_1+\frac{2\omega(r-1)}{\pi}\right)\right),\end{aligned} \tag{36}$$

where $S(\omega)$ is the power spectral density of the random process $\xi(t)$, and

$$S(\omega)=2\int_0^{+\infty}R(\tau)\cos(\omega\tau)\mathrm{d}\tau,\qquad R(\tau)=\int_{-\infty}^{+\infty}\xi P_s(\xi)K(\xi,\tau)\mathrm{d}\xi.$$

Since $\Lambda_0(p)=0$, $\Lambda_1(p)=0$, and ε is a small parameter, the *p*th moment Lyapunov exponent is approximately formulated as

$$\Lambda(p)\cong\varepsilon^2\Lambda_2(p)=\frac{\varepsilon^2}{16}p\left(\left(S(2\omega)+2S(0)\right)p+2S(2\omega)+8\left(c_1+\frac{2\omega(r-1)}{\pi}\right)\right). \tag{37}$$

The stability index δ_p is the non-zero solution of $\Lambda(p)=0$ .Hence, from Eq. (37), one has

$$\delta_p=\frac{1}{S(2\omega)+2S(0)}\left(\frac{16\omega}{\pi}(1-r)-2S(2\omega)-8c_1\right). \tag{38}$$

The stability index $\delta_p>0$ indicates that the system is pth moment-stable as $0<p<\delta_p$. And according to Eq. (2), the largest Lyapunov exponent can be given as

$$\lambda=\frac{\mathrm{d}}{\mathrm{d}p}\Lambda(p)\bigg|_{p=0}\cong\frac{1}{8}\varepsilon^2\left(S(2\omega)+4c_1+\frac{8\omega(r-1)}{\pi}\right). \tag{39}$$

5. Results and discussions

In order to verify the reliability of the approximate analytical solution of the moment

Lyapunov exponent $\Lambda(p)$ given by the L. Arnold perturbation method, the moment Lyapunov exponents $\Lambda(p)$ of the random vibro-impact system (3) are calculated numerically by using the Monte Carlo method [43]. The original system (3) is simulated in MATLAB by using the fourth-order Runge-Kutta method, and there have been fruitful achievements [44] in the research of numerical calculation methods for vibro-impact systems. Fig. 1 plots the analytical results and numerical results of the *p*th moment Lyapunov exponent $\Lambda(p)$ for different noise parameters. It is clear that the differences between analytical and numerical results are very small, which indicates that the approximate analytical solution of $\Lambda(p)$ obtained by the perturbation method is valid. Thus, by using the analytical results of the moment Lyapunov exponent $\Lambda(p)$ and largest Lyapunov exponent λ, the stochastic stability of the vibro-impact system (3) can be further discussed in detail. From Eqs. (37),(38) and (39), it can be seen that the stochastic stability of the system (3) is related to the power spectral density $S(\omega)$, the coefficient of restitution r, the damping coefficient c_1, and the natural frequency ω.

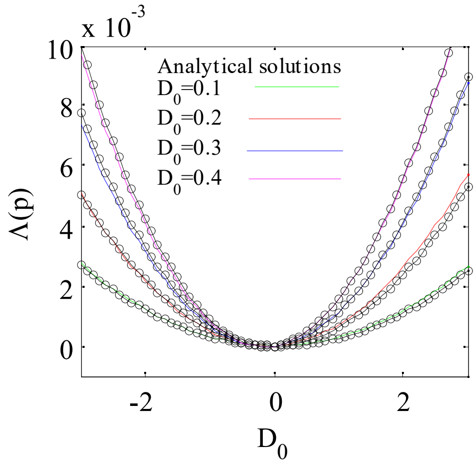

Fig. 1 Analytical and numerical solutions of moment Lyapunov exponents for the case $\varepsilon=0.1, \omega=1, r=0.8, c_1=0.1; r_0=0.95, \tau_0=0.5$.

According to Eq. (39), one easily finds that the largest Lyapunov exponent λ increases with the increase of power spectral density $S(2\omega)$, which implies that the almost-sure stability of the vibro-impact system is reduced with the increase of the power spectral density $S(2\omega)$. When $S(2\omega)<-4c_1+8\omega(1-r)/\pi$, the vibro-impact system is almost-sure stable, and the system is unstable when $S(2\omega)>-4c_1+8\omega(1-r)/\pi$. From Eqs. (12), (14) and (16), it is clear that the power spectral density is determined by the noise parameters D_0, r_0 and τ_0. For different parameters of the non-Gaussian colored noise $\xi(t)$, Figs. 2 and Figs. 3 depict the moment Lyapunov exponents $\Lambda(p)$ and largest Lyapunov exponents λ respectively. It can be observed from Fig. 2 that the stability region and stability index of the vibro-impact system gradually decrease with increasing of noise intensity D_0 and departure coefficient r_0, i.e., the stochastic stability of the system is reduced by the increase of noise intensity D_0 and departure coefficient r_0. Since the effect of departure coefficient r_0 on the stochastic stability is very small as $r_0 \to 1$, it can be neglected. In view of Fig. 2(c), it is seen that the stochastic stability of the system is enhanced by the increase of correlation time τ_0. And for the case $\tau_0 = 0.0001$, $\tau_0 = 0.001$ and $\tau_0 = 0.01$, the *p*th moment Lyapunov exponents $\Lambda(p)$ are almost identical, which means that the stochastic stability of the system is affected by the noise correlation time τ_0 slightly when the parameter τ_0 is very small. From Fig. 3, it can be found that the noise intensity D_0 significantly effects the stochastic stability of the vibro-impact system, especially in the situation of small value of natural frequency. Therefore, more attention should be paid to noise and strong noise should be avoided, especially when the value of natural frequency ω of the vibro-impact system is small.

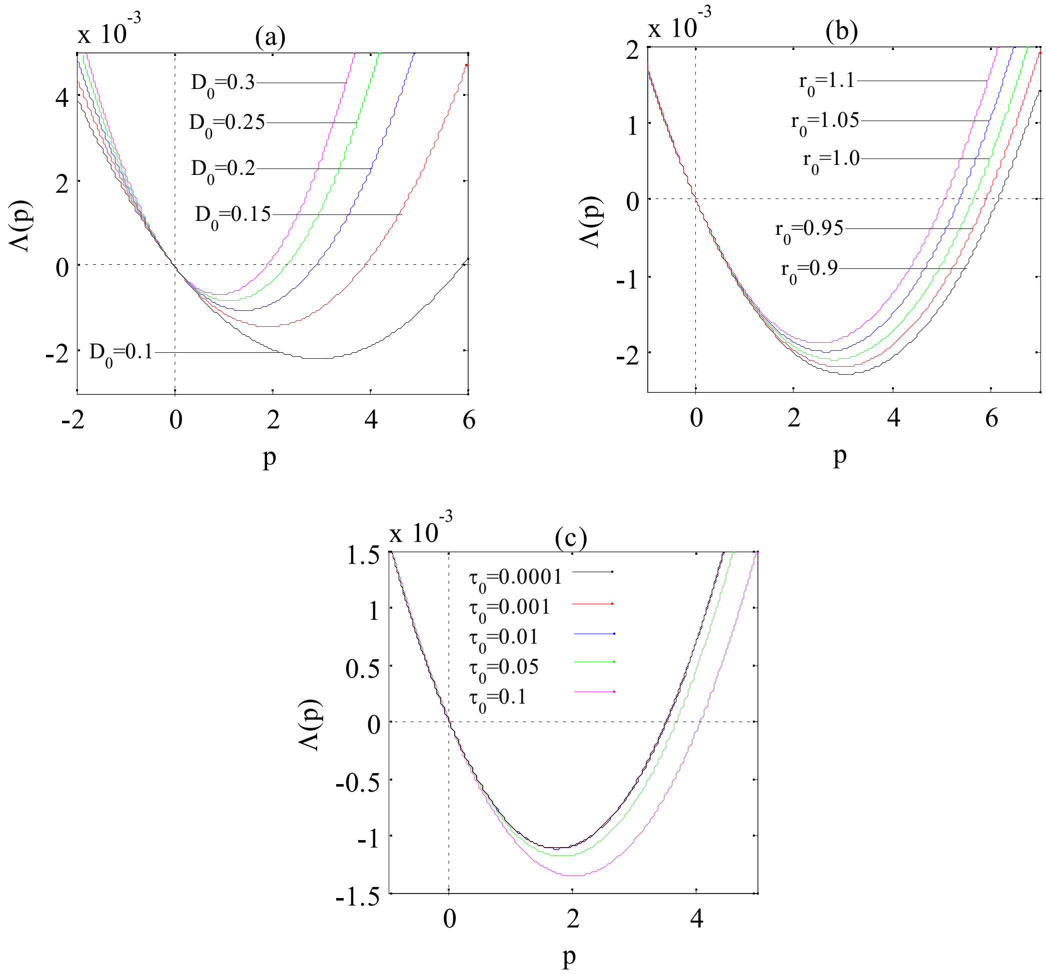

Fig. 2 Effect of noise intensity on the moment Lyapunov exponent for $\varepsilon = 0.1, r = 0.8, \omega = \pi, c_1 = 0.1$; (a) $r_0 = 0.95, \tau_0 = 0.5$; (b) $D_0 = 0.1, \tau_0 = 0.5$; (c) $D_0 = 0.1, r_0 = 0.95$.

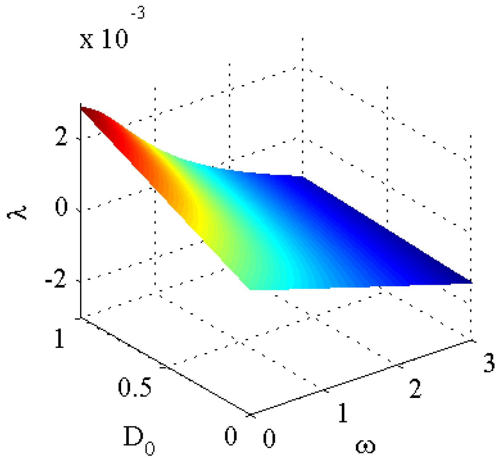

Fig. 3 Largest Lyapunov exponent for $\varepsilon = 0.1, r = 0.8, c = 0.1; r_0 = 0.95, \tau_0 = 0.5$.

For different values of natural frequency ω, the largest Lyapunov exponents λ and pth moment Lyapunov exponents $\Lambda(p)$ of the vibro-impact are described in Fig. 3 and Fig 4. Based on the fact disclosed in Fig. 4, it can be seen that along with increasing the value of parameter ω, the value of stability index δ_p gradually becomes larger. One can also easily find that the largest Lyapunov exponent λ decreases significantly with the increase of parameter ω as shown in Fig. 3. According to Eq. (37-39), it can be found that the natural frequency ω can directly influence on the stochastic stability of the system, moreover increasing the parameter ω lead to the strongly stable of the system.

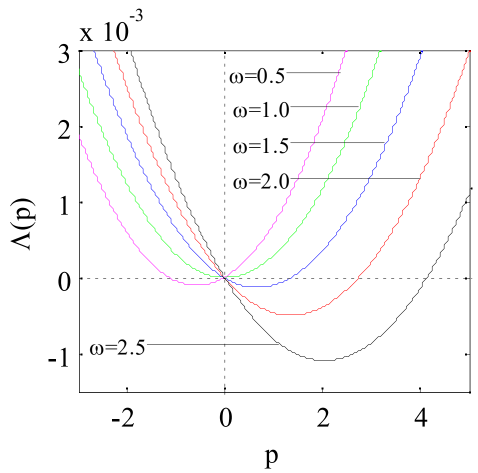

Fig. 4 Effect of natural frequency ω on the moment Lyapunov exponent for $\varepsilon = 0.1, r = 0.8, c_1 = 0.1; D_0 = 0.1, r_0 = 0.95, \tau_0 = 0.5$.

The pth moment Lyapunov exponents $\Lambda(p)$ of the vibro-impact system for different values of parameter r are plotted in Fig. 5. In view of Fig. 5, it is clear that the stability index δ_p gradually decreases with the increase of parameter r, which means the stochastic stability of the system is reduced with increasing the parameter r. From Eq. (39), we find that the largest Lyapunov exponent λ of the vibro-impact system increases with the increase of parameter r. The system is almost-sure stable when $r < 1 - \pi\left(S(2\omega) + 4c_1\right)/8\omega$ and the system is almost-sure unstable when $1 - \pi\left(S(2\omega) + 4c_1\right)/8\omega < r \le 1$. Therefore, the stochastic stability of the vibro-impact system is reduced with increasing the coefficient of restitution r. However, the energy loss of the vibro-impact system is assumed to be a small quantity (i.e., $0 \le 1 - r << 1$) in the above calculation. Thus, we should choose the appropriate coefficient of restitution r to stabilize the random vibro-impact system.

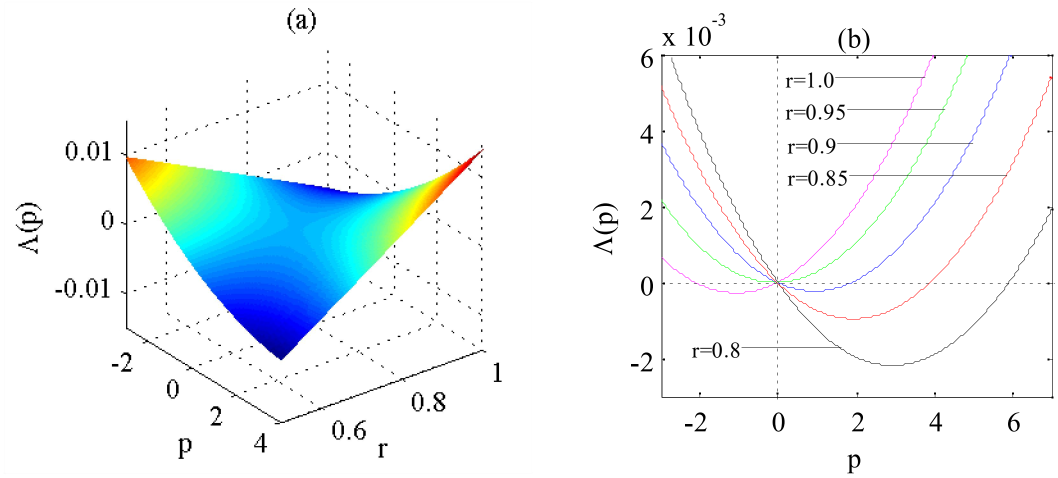

Fig. 5 Effect of parameter r on the moment Lyapunov exponent for $\varepsilon = 0.1, \omega = \pi, c_1 = 0.1; D_0 = 0.1, r_0 = 0.95, \tau_0 = 0.5$.

The pth moment Lyapunov exponents $\Lambda(p)$ of the vibro-impact system for different values of the damping coefficient c_1 are shown by Fig. 6, it can be seen that the moment stability of the system is reduced with increasing the parameter c_1. According to Eq. (39), we assert the largest Lyapunov exponent λ of the vibro-impact system increases with the increase of parameter c_1, i.e., the almost-sure stability of the system is reduced with the parameter c_1. If $c_1 < -S(2\omega)/4 + 2\omega(1-r)/\pi$, the system will be almost-sure stable, otherwise, the system is almost-sure unstable. Therefore, the stochastic stability of the vibro-impact system is reduced with the increase of linear damping coefficient c_1.

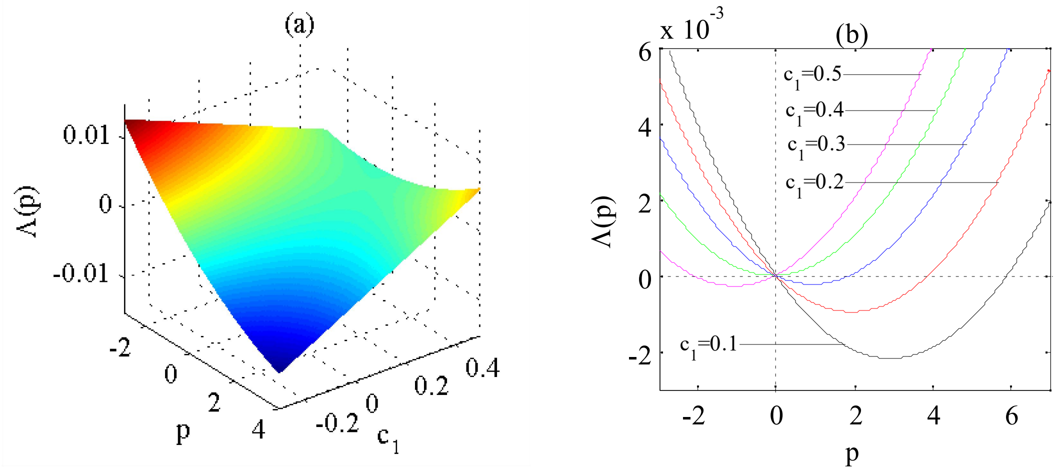

Fig. 6 Effect of damping coefficient c_1 on the moment Lyapunov exponent for $\varepsilon = 0.01, r = 0.8, \omega = \pi; D_0 = 0.1, r_0 = 0.95, \tau_0 = 0.5$.

5. Conclusion

By calculating the *p*th moment Lyapunov exponent $\Lambda(p)$, the stochastic stability of the vibro-impact system driven by non-Gaussian colored noise excitation is studied. The random vibro-impact system is transformed into a smooth random dynamical system through using the Zhuravlev transformation, and the non-Gaussian colored noise $\xi(t)$ is simplified to be an O-U random process by the path-integral method. For weak noise excitation and finite values of *p*, the *p*th moment Lyapunov exponent $\Lambda(p)$ is calculated by applying the L.Arnold perturbation method, and the second-order approximate solution of $\Lambda(p)$ is derived. Thereafter, in term of the value of $\Lambda(p)$, the stability index δ_p and largest Lyapunov exponent λ are obtained. Furthermore, the numerical results of the *p*th moment Lyapunov exponent $\Lambda(p)$ are provided by utilizing Monte Carlo simulation, which are well consistent with the approximate analytical solutions. Finally, based on the *p*th moment Lyapunov exponent $\Lambda(p)$ and largest Lyapunov exponent λ, the effects of the noise, the coefficient of restitution r, the linear damping coefficient c_1, and the natural frequency ω on the stochastic stability are discussed.

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