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*Research article*

## **Rishi Transform method for solving multi-high order fractional differential equations with constant coefficients**

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**Abstract:** In this paper, we suggest the Rishi transform, which may be used to find the analytic (exact) solution to multi-high-order linear fractional differential equations. where the Riemann-Liouville and Caputo fractional derivatives are used. We first developed the Rishi transform of foundational mathematical functions for this purpose and then described the important characteristics of the Rishi transform, which may be applied to solve ordinary differential equations and fractional differential equations. Following that, the authors found an exact solution to a particular example of fractional differential equations. They looked at four numerical problems and solved them all step by step to demonstrate the value of the Rishi transform. The results show that the suggested novel transform, "Rishi Transform," yields exact solutions to multi-higher-order fractional differential equations without doing complicated calculation work.

**Keywords:** Linear fractional differential equations; Riemann-Liouville's fractional derivative;

Liouville -Caputo's fractional derivative; Rishi transform; Inverse Rishi transform.

**Mathematics Subject Classification:** 30C45, 30C50.

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## 1. Introduction

A strong tool for explaining the memory and inherited characteristics of diverse components and methods is fractional calculus [1-4]. It may be used in a variety of scientific and technical domains, including biology, chemistry, acoustics, fluid mechanics, anomalous diffusion, viscoelasticity, and others. In these applications, a family of integro-differential equations with singularities involved fractional differential equations [5,6]. Several analytic or numerical approaches to solving fractional differential equations have already been published, including [7-10].

Due to their three key characteristics of simplicity, accuracy, and providing results without the need for time-consuming calculation work integral transforms are currently researchers' first choice among other mathematical techniques for figuring out the answers to problems in science, social science, and engineering [11]. The fractional calculus is also useful in engineering and mathematical modeling [12]. And fractional approach many useful in chemistry and using a meshless technique, a numerical solution of fractional reaction-convection-diffusion is shown for modeling PEM fuel cells [13,14]. Some new integral transformations are developed to solve differential equations and fractional differential equations for examples Kamal transformation [15], Aboodh transformation [16], Sumudu transformation [17], Rishi transformation [18], are a few examples of recent developments in integral transforms and fractional differential equations. The renowned issues were examined by Aggarwal and other scholars [19] using a variety of integral transforms.

The goal of this article is to create the "Rishi Transform", a novel integral transform with essential qualities for fractional and fractional differential equations, and to ascertain the solution of the fractional differential equations. The "Rishi transform" that has been suggested is superior to the other transforms that have already been developed since it offers the precise solutions to the issues without requiring time-consuming computations. Laplace transform, a well-known and often used integral transform, and the Rishi transform are dualistic.

We want to generate fractional formulas for another transformation, just as most transformations have them, we have derived the fractional formula of the transformation and use it to analyze some problems. Some of other transformation for fractional integral and derivative was derived, we studied on Rishi transform and derived fractional formula and it is applications.

## 2. Basic definitions and properties

In this section, provides some fundamental definitions and properties of fractional calculus theory, which will be used in this study.

**Definition 2.1** (see [7,20]). The Riemann-Liouville integral operator of fractional order  $\alpha > 0$  is defined as:

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \quad -\infty \leq a < t < \infty, \quad (1)$$

where  $\Gamma(\alpha)$  is Gamma function defined by  $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$ .

**Definition 2.2** (see [7,20]). The Riemann-Liouville differential operator of fractional order  $\alpha > 0$ , and  $m = [\alpha]$  is defined as:

$${}^{RL}D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-s)^{m-\alpha-1} f(s) ds. \quad (2)$$

**Definition 2.3** (see [7,20]). The Liouville -Caputo differential operator of fractional order  $\alpha > 0$ , and  $m = [\alpha]$ , and  $f(t)$  be  $m$  -times differentiable function,  $t > a$  is defined as:

$${}^{LC}D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} \left(\frac{d}{ds}\right)^m f(s) ds. \quad (3)$$

**Remark** (see [21–25]). Some basic properties of fractional calculus are as follows:

1. The fractional (integral and differential) Operator is linear operator.
2. Composition between two Riemann-Liouville integration of orders  $\alpha$  and  $\beta$ , is defined as follows:

$${}_a I_t^\alpha {}_a I_t^\beta f(t) = {}_a I_t^\beta {}_a I_t^\alpha f(t) = I_t^{\alpha+\beta} f(t). \quad (4)$$

3. If  $k \geq \alpha$ , for  $f(t) \in C[a, b]$ , and at every point  $t \in [a, b]$ , then

$${}^{RL}D_t^k ({}_a I_t^\alpha f(t)) = {}^{RL}D_t^{k-\alpha} f(t). \quad (5)$$

The relation is done.

4. Composition between fractional (differentiation and integration) of Liouville-Caputo operator of order  $\alpha$ , is defined as follows:

$${}^{LC}D_t^k ({}_a I_t^\alpha f(t)) = f(t). \quad (6)$$

5. Composition between fractional (integration and differentiation) of Liouville -Caputo operator of order  $\alpha$ , and  $m = [\alpha]$  is defined as follows:

$${}_a I_t^\alpha ({}^{LC}D_t^k f(t)) = f(t) - \sum_{k=0}^{m-1} \frac{(t-a)^k}{k!} f^{(k)}(a). \quad (7)$$

In general,  ${}^{LC}D_t^k ({}_a I_t^\alpha f(t)) \neq {}_a I_t^\alpha ({}^{LC}D_t^k f(t))$ .

6. Apply fractional integral and differential of Liouville-Caputo operator of function  $t^n, n \geq 0$ ,

$$\text{we get: } {}_a I_t^\alpha t^n = \frac{\Gamma(1+n)}{\Gamma(1+n+\alpha)} t^{n+\alpha} \text{ and } {}^{LC}D_t^k t^n = \frac{\Gamma(1+n)}{\Gamma(1+n-\alpha)} t^{n-\alpha}. \quad (8)$$

### 3. Fundamental properties of Rishi transform

In this section, we have defined the Rishi transform and its properties (see [11]). Using several other researches, we will learn an introduction to the fractional formula of the new transform:

**Definition 3.1 [Rishi transforms, 11].** The piecewise continuous function of exponential order using the Rishi transform  $f(t)$  defined in the interval  $[0, \infty)$ , is given by:

$$R\{f(t)\} = \left(\frac{p}{q}\right) \int_0^{\infty} f(t) e^{-\left(\frac{q}{p}\right)t} dt = F(q, p), \quad q > 0, p > 0.$$

**Property 1** (see [24]). Some fundamental functions and apply the Rishi transformation:

$f(t), t > 0$	$R\{f(t)\} = F(q, p)$	$f(t), t > 0$	$R\{f(t)\} = F(q, p)$
1	$\left(\frac{p}{q}\right)^2$	$\sin kt$	$\frac{kp^3}{q(q^2 + k^2p^2)}$
$e^{kt}$	$\frac{p^2}{q(q - kp)}$	$\cos kt$	$\frac{p^2}{(q^2 + p^2k^2)}$
$t^\beta, \beta \in N$	$\beta! \left(\frac{p}{q}\right)^{\beta+2}$	$\sinh kt$	$\frac{kp^3}{q(q^2 - k^2p^2)}$
$t^\beta, \beta > -1, \beta \in R$	$\Gamma(\beta + 1) \left(\frac{p}{q}\right)^{\beta+2}$	$\cosh kt$	$\frac{p^2}{(q^2 - p^2k^2)}$

**Property 2** (see [24]). Some fundamental functions, and apply inverse the Rishi transformation:

$F(q, p)$	$R^{-1}\{F(q, p)\} = f(t)$	$F(q, p)$	$R^{-1}\{F(q, p)\} = f(t)$
$\left(\frac{p}{q}\right)^2$	1	$\frac{kp^3}{q(q^2 + k^2p^2)}$	$\sin kt$
$\frac{p^2}{q(q - kp)}$	$e^{kt}$	$\frac{p^2}{(q^2 + p^2k^2)}$	$\cos kt$
$\beta! \left(\frac{p}{q}\right)^{\beta+2}$	$t^\beta, \beta \in N$	$\frac{kp^3}{q(q^2 - k^2p^2)}$	$\sinh kt$
$\Gamma(\beta + 1) \left(\frac{p}{q}\right)^{\beta+2}$	$t^\beta, \beta > -1, \beta \in R$	$\frac{p^2}{(q^2 - p^2k^2)}$	$\cosh kt$

**Property 3 [Convolution property, 21].** If  $R\{f(t)\} = F(q, p)$  and  $R\{g(t)\} = G(q, p)$ , then

$$R\{f(t) * g(t)\} = \left(\frac{q}{p}\right) F(q, p) G(q, p),$$

where  $*$  denotes convolution of  $f$  and  $g$ , then  $f(t) * g(t) = \int_0^t f(t-s) g(s) ds$ .

**Property 4** (see [11]). The Rishi transform operator and inverse Rishi transform is linear operator.

1. Rishi Transform is linear operator

$$R\{\sum_{i=0}^n k_i f_i(t)\} = \sum_{i=0}^n k_i R\{f_i(t)\}. \quad \text{Where } k_i \text{ are arbitrary constant.}$$

2. Inverse Rishi Transform is linear operator

If  $f_i(t) = R^{-1}\{F_i(q, p)\}$ , then  $R^{-1}\{\sum_{i=0}^n k_i F_i(q, p)\} = \sum_{i=0}^n k_i R^{-1}\{F_i(q, p)\}$ , where  $k_i$  are arbitrary constant.

#### 4. Rishi transform of fractional integrals and derivatives

In this section, we derived fractional formula of Rishi transform for fractional integral and fractional derivative by using those properties.

**Property 5:** (see [11]). The Rishi transformation for integer order derivative of  $f(t)$  is:

$$R\{f^{(m)}(t)\} = \left(\frac{q}{p}\right)^m F(q, p) - \sum_{k=0}^{m-1} \left(\frac{q}{p}\right)^{k-1} f^{(m-1-k)}(0),$$

It is easy prove by Mathematical Induction.

**Theorem 4.1:** If fractional order  $\alpha \in [m - 1, m)$ , and apply the Rishi transform of the fractional

integral is 
$$R\{I_t^\alpha f(t)\} = \left(\frac{p}{q}\right)^\alpha F(q, p) .$$

**Proof:** From definition of Riemann-Liouville integral of equation (1), and apply Rishi transform

$$R\{I_t^\alpha f(t)\} = R\{{}^{RL}D_t^{-\alpha} f(t)\} = R\left\{\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds\right\} = \frac{1}{\Gamma(\alpha)} R\left\{\int_0^t (t-s)^{\alpha-1} f(s) ds\right\} .$$

By using properties (3), and (1), we get:

$$R\{I_t^\alpha f(t)\} = \frac{1}{\Gamma(\alpha)} \frac{q}{p} R\{t^{\alpha-1}\} F(q, p) = \frac{1}{\Gamma(\alpha)} \frac{q}{p} \left(\frac{p}{q}\right)^{\alpha+1} \Gamma(\alpha) F(q, p).$$

Hence 
$$R\{I_t^\alpha f(t)\} = \left(\frac{p}{q}\right)^\alpha F(q, p).$$

**Theorem 4.2:** If  $f(t)$  is a function and  $F(q, p)$  is a Rishi transform for Reimann-Liouville fractional derivative of order  $\alpha > 0$  is:

$$R\{{}^{RL}D_t^\alpha f(t)\} = \left(\frac{q}{p}\right)^\alpha F(q, p) - \sum_{k=0}^{m-1} \left(\frac{q}{p}\right)^{k-1} [{}^{RL}D_t^{\alpha-k-1} f(t)]_{t=0}.$$

**Proof:** From definition of Riemann-Liouville fractional derivative, we have:

$$R\{{}^{RL}D_t^\alpha f(t)\} = R\{D_t^m I_t^{m-\alpha} f(t)\},$$

Now, from property (5), and apply Rishi transform of above relation, we get:

$$R\{D_t^m I_t^{m-\alpha} f(t)\} = \left(\frac{q}{p}\right)^m R\{I_t^{m-\alpha} f(t)\} - \sum_{k=0}^{m-1} \left(\frac{q}{p}\right)^{k-1} D_t^{m-k-1} [I_t^{m-\alpha} f(t)]_{t=0},$$

By using Theorem (4.1), and equation (5), we get:

$$= \left(\frac{q}{p}\right)^m \left(\frac{p}{q}\right)^{m-\alpha} F(q, p) - \sum_{k=0}^{m-1} \left(\frac{q}{p}\right)^{k-1} [{}^{RL}D_t^{\alpha-k-1} f(t)]_{t=0}.$$

Hence,

$$R\{{}^{RL}D_t^\alpha f(t)\} = \left(\frac{q}{p}\right)^\alpha F(q, p) - \sum_{k=0}^{m-1} \left(\frac{q}{p}\right)^{k-1} [{}^{RL}D_t^{\alpha-k-1} f(t)]_{t=0}.$$

**Theorem 4.3:** If  $f(t)$  is a function and  $F(q, p)$  is a Rishi transform then the Rishi transform for Liouville-Caputo fractional derivative of order  $\alpha > 0$  is:

$$R\{{}^{LC}D_t^\alpha f(t)\} = \left(\frac{q}{p}\right)^\alpha F(q, p) - \sum_{k=0}^{m-1} \left(\frac{p}{q}\right)^{m-k-\alpha+1} f^{(m-k-1)}(0).$$

**Proof:** From definition of Liouville -Caputo fractional derivative, we have:

$$R\{{}^{LC}D_t^\alpha f(t)\} = R\{I_t^{m-\alpha} D_t^m f(t)\}.$$

Now, by using theorem (4.1), we get:

$$R\{I_t^{m-\alpha} D_t^m f(t)\} = \left(\frac{p}{q}\right)^{m-\alpha} R\{D_t^m f(t)\}.$$

Also, by using property (5), we get:

$$\begin{aligned} \left(\frac{p}{q}\right)^{m-\alpha} R\{D_t^m f(t)\} &= \left(\frac{p}{q}\right)^{m-\alpha} \left[ \left(\frac{q}{p}\right)^m F(q, p) - \sum_{k=0}^{m-1} \left(\frac{q}{p}\right)^{k-1} f^{(m-k-1)}(0) \right] \\ &= \left(\frac{q}{p}\right)^\alpha F(q, p) - \sum_{k=0}^{m-1} \left(\frac{p}{q}\right)^{m-k-\alpha+1} f^{(m-k-1)}(0). \end{aligned}$$

Hence

$$R\{{}^{LC}D_t^\alpha f(t)\} = \left(\frac{q}{p}\right)^\alpha F(q, p) - \sum_{k=0}^{m-1} \left(\frac{p}{q}\right)^{m-k-\alpha+1} f^{(m-k-1)}(0).$$

## 5. Illustrative examples

In this section, contains four analytical problems for explaining the utility of Rishi transform for determining the exact (analytic) solution of multi-high order linear fractional differential equations of Riemann-Liouville and Liouville -Caputo Sense.

**Example 5.1:** Consider the linear fractional differential equation for Riemann-Liouville's fractional derivative

$${}^{RL}D_t^{1.3} f(t) + 2 \frac{d}{dt} f(t) = \frac{6}{\Gamma(2.7)} t^{1.7} - \frac{2}{\Gamma(1.7)} t^{0.7} + 6t^2 - 4t,$$

with the initial conditions:  $f(0) = 0$  ,  $[{}^{RL}D_t^{0.3}(f(t))]_{t=0} = 0$  and  $[{}^{RL}D_t^{-0.7}(f(t))]_{t=0} = 0$ .

**Solution:** Apply Rishi transform for both sides, by using property (4), we get:

$$R\{{}^{RL}D_t^{1.3}y(t)\} + R\left\{2\frac{d}{dt}y(t)\right\} = R\left\{\frac{6}{\Gamma(2.7)}t^{1.7} - \frac{2}{\Gamma(1.7)}t^{0.7} + 6t^2 - 4t\right\}.$$

By using theorem (4.2) and property (1), we get:

$$\left(\left(\frac{q}{p}\right)^{1.3} F(q, p) - \sum_{k=0}^1 \left(\frac{q}{p}\right)^{k-1} [{}^{RL}D_t^{1.3-k-1}f(t)]_{t=0}\right) + 2\left(\frac{q}{p}F(q, p) - \sum_{k=0}^0 \left(\frac{q}{p}\right)^{k-1} f^{(1-1-k)}(0)\right) = \frac{6}{\Gamma(2.7)}\left(\frac{p}{q}\right)^{3.7} \Gamma(2.7) - \frac{2}{\Gamma(1.7)}\left(\frac{p}{q}\right)^{2.7} \Gamma(1.7) + 12\left(\frac{p}{q}\right)^4 - 4\left(\frac{p}{q}\right)^3,$$

$$\left(\frac{q}{p}\right)^{1.3} F(q, p) - \left(\frac{q}{p}\right)^{-1} [{}^{RL}D_t^{0.3}f(t)]_{t=0} - [{}^{RL}D_t^{-0.7}f(t)]_{t=0} + 2\frac{q}{p}F(q, p) - 2\left(\frac{q}{p}\right)^{-1} f(0) = 6\left(\frac{p}{q}\right)^{3.7} - 2\left(\frac{p}{q}\right)^{2.7} + 12\left(\frac{p}{q}\right)^4 - 4\left(\frac{p}{q}\right)^3,$$

$$F(q, p) \left(\left(\frac{q}{p}\right)^{0.3} + 2\right) \left(\frac{p}{q}\right)^{-1} = 6\left(\frac{p}{q}\right)^{3.7} + 12\left(\frac{p}{q}\right)^4 - 2\left(\frac{p}{q}\right)^{2.7} - 4\left(\frac{p}{q}\right)^3,$$

$$F(q, p) \left(\left(\frac{q}{p}\right)^{0.3} + 2\right) = 6\left(\frac{p}{q}\right)^{4.7} + 12\left(\frac{p}{q}\right)^5 - 2\left(\frac{p}{q}\right)^{3.7} - 4\left(\frac{p}{q}\right)^4,$$

$$F(q, p) = \frac{1}{\left(\frac{q}{p}\right)^{0.3} + 2} \left(6\left(\frac{p}{q}\right)^5 \left(\left(\frac{p}{q}\right)^{-0.3} + 2\right) - 2\left(\frac{p}{q}\right)^4 \left(\left(\frac{p}{q}\right)^{-0.3} + 2\right)\right),$$

$$F(q, p) = \frac{1}{\left(\frac{q}{p}\right)^{0.3} + 2} \left(6\left(\frac{p}{q}\right)^5 - 2\left(\frac{p}{q}\right)^4\right) \left(\left(\frac{q}{p}\right)^{0.3} + 2\right).$$

Hence

$$F(q, p) = 6\left(\frac{p}{q}\right)^5 - 2\left(\frac{p}{q}\right)^4,$$

Take Inverse Rishi transform for both sides  $R^{-1}\{F(q, p)\} = R^{-1}\left\{6\left(\frac{p}{q}\right)^5 - 2\left(\frac{p}{q}\right)^4\right\}$ .

The exact solution is:  $f(t) = t^2(t - 1)$ .

**Example 5.2** (see [15]). Consider the linear fractional differential equation for Riemann-Liouville's fractional derivative, given by:

$${}^{RL}D_t^{\frac{1}{2}}f(t) + f(t) = \frac{1}{2}t + \frac{\sqrt{t}}{\sqrt{\pi}},$$

with the initial conditions:  $\left[{}^{RL}D_t^{-\frac{1}{2}}f(t)\right]_{t=0} = 0$ .

**Solution:** Apply Rishi transform for both sides of problem above, using property (4), we get:

$$R \left\{ {}^{RL}D_t^{\frac{1}{2}} f(t) \right\} + R\{f(t)\} = R \left\{ \frac{1}{2}t + \frac{\sqrt{t}}{\sqrt{\pi}} \right\},$$

By using theorem (4.2), property (1) and definition (3.1), we can say:

$$R \left\{ {}^{RL}D_t^{\frac{1}{2}} f(t) \right\} = \left( \frac{q}{p} \right)^{\frac{1}{2}} F(q, p) - \sum_{k=0}^0 \left( \frac{q}{p} \right)^{k-1} \left[ {}^{RL}D_t^{\frac{1}{2}-k-1} f(t) \right]_{t=0} : R\{f(t)\} = F(q, p),$$

$$R \left\{ \frac{1}{2}t + \frac{\sqrt{t}}{\sqrt{\pi}} \right\} = \frac{1}{2} \left( \frac{p}{q} \right)^3 + \frac{1}{\sqrt{\pi}} \left( \frac{p}{q} \right)^{\frac{5}{2}} \Gamma\left(\frac{3}{2}\right),$$

So, the above equation as follows:

$$\left( \frac{q}{p} \right)^{\frac{1}{2}} F(q, p) - \sum_{k=0}^0 \left( \frac{q}{p} \right)^{k-1} \left[ {}^{RL}D_t^{\frac{1}{2}-k-1} f(t) \right]_{t=0} + F(q, p) = \frac{1}{2} \left( \frac{p}{q} \right)^3 + \frac{1}{\sqrt{\pi}} \left( \frac{p}{q} \right)^{\frac{5}{2}} \Gamma\left(\frac{3}{2}\right),$$

$$F(q, p) \left( \left( \frac{q}{p} \right)^{\frac{1}{2}} + 1 \right) = \frac{1}{2} \left( \frac{p}{q} \right)^3 + \frac{1}{\sqrt{\pi}} \left( \frac{p}{q} \right)^{\frac{5}{2}} = \frac{1}{2} \left( \frac{p}{q} \right)^3 \left( 1 + \left( \frac{q}{p} \right)^{\frac{1}{2}} \right),$$

$$F(q, p) = \frac{1}{2} \left( \frac{p}{q} \right)^3.$$

Now, take inverse Rishi transform for both sides, we get:

$$R^{-1}\{F(q, p)\} = R^{-1}\left\{ \frac{1}{2} \left( \frac{p}{q} \right)^3 \right\}. \quad \text{Hence, the exact solution is } f(t) = \frac{1}{2}t.$$

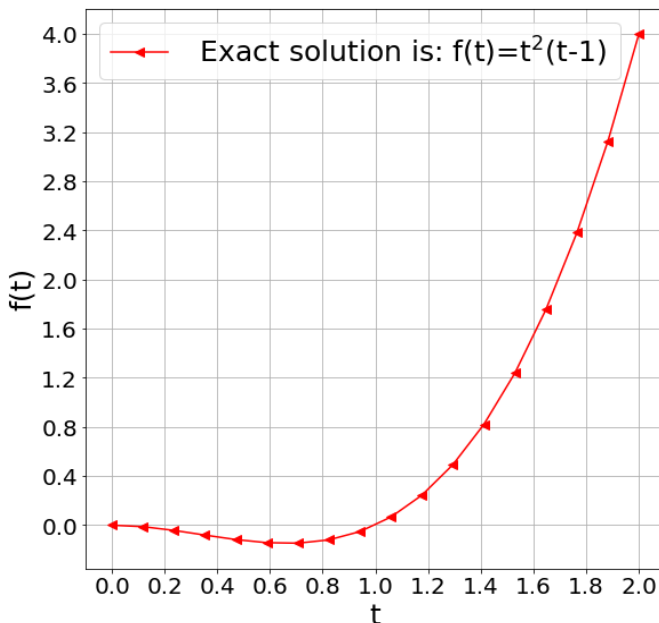


Figure 1. Exact solution of Example 5.1.

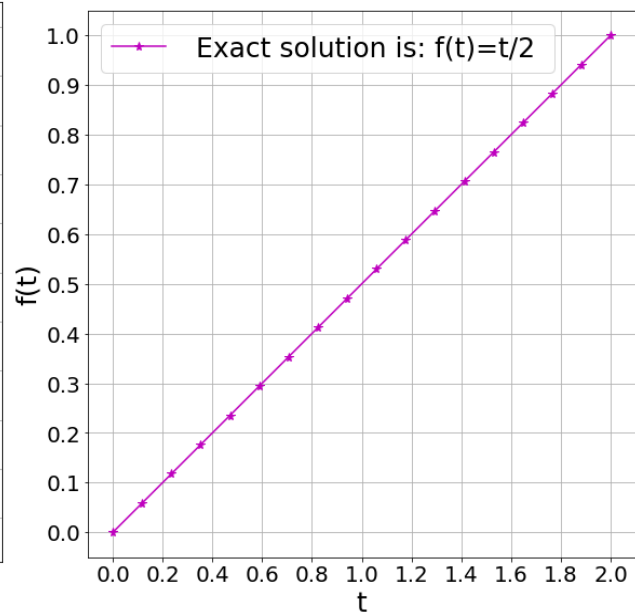


Figure 2. Exact solution of Example 5.2.

**Remark:** The above example was proved by Kamal transform in [12]. Exact solutions were obtained from both methods.



**Example 5.3** Consider the linear fractional differential equation for Liouville -Caputo fractional derivative is given by:

$${}^L C_0 D_t^{1.2} f(t) + {}^L C_0 D_t^{0.2} f(t) + f(t) = \frac{\Gamma(2.5)}{\Gamma(1.3)} t^{0.3} + \frac{\Gamma(2.5)}{\Gamma(2.3)} t^{1.3} + t^{1.5} + 1,$$

with the initial conditions:  $f(0) = 1, f'(0) = 0$ .

**Solution:** Take Rishi transform for both sides, by using property (4), we obtain:

$$R\{{}^L C_0 D_t^{1.2} f(t) + {}^L C_0 D_t^{0.2} f(t)\} + R\{f(t)\} = R\left\{\frac{\Gamma(2.5)}{\Gamma(1.3)} t^{0.3} + \frac{\Gamma(2.5)}{\Gamma(2.3)} t^{1.3} + t^{1.5} + 1\right\}.$$

By using theorem (4.3), property (1) and definition (3.1), we get:

$$\begin{aligned} \left(\frac{q}{p}\right)^{1.2} F(q, p) - \sum_{k=0}^1 \left(\frac{p}{q}\right)^{1.8-k} f^{(1-k)}(0) + \left(\frac{q}{p}\right)^{0.2} F(q, p) - \sum_{k=0}^0 \left(\frac{p}{q}\right)^{0.8-k} f^{(-k)}(0) + F(q, p) \\ = \frac{\Gamma(2.5)}{\Gamma(1.3)} \left(\frac{p}{q}\right)^{2.3} \Gamma(1.3) + \frac{\Gamma(2.5)}{\Gamma(2.3)} \left(\frac{p}{q}\right)^{3.3} \Gamma(2.3) + \Gamma(2.5) \left(\frac{p}{q}\right)^{3.5} + \left(\frac{p}{q}\right)^2. \end{aligned}$$

Simplify the statement

$$\begin{aligned} \left(\frac{q}{p}\right)^{1.2} F(q, p) - \left(\frac{p}{q}\right)^{1.8} f'(0) - \left(\frac{p}{q}\right)^{0.8} f(0) + \left(\frac{q}{p}\right)^{0.2} F(q, p) - \left(\frac{p}{q}\right)^{1.8} f(0) + F(q, p) = \\ \Gamma(2.5) \left(\frac{p}{q}\right)^{2.3} + \Gamma(2.5) \left(\frac{p}{q}\right)^{3.3} + \Gamma(2.5) \left(\frac{p}{q}\right)^{3.5} + \left(\frac{p}{q}\right)^2. \end{aligned}$$

After input initial conditions, we get:

$$F(q, p) \left( \left(\frac{q}{p}\right)^{1.2} + \left(\frac{q}{p}\right)^{0.2} + 1 \right) = \Gamma(2.5) \left(\frac{p}{q}\right)^{2.3} + \Gamma(2.5) \left(\frac{p}{q}\right)^{3.3} + \Gamma(2.5) \left(\frac{p}{q}\right)^{3.5} + \left(\frac{p}{q}\right)^2 + \left(\frac{p}{q}\right)^{0.8} + \left(\frac{p}{q}\right)^{1.8},$$

$$\begin{aligned} F(q, p) \left( \left(\frac{q}{p}\right)^{1.2} + \left(\frac{q}{p}\right)^{0.2} + 1 \right) &= \Gamma(2.5) \left(\frac{p}{q}\right)^{3.5} \left( \left(\frac{q}{p}\right)^{1.2} + \left(\frac{q}{p}\right)^{0.2} + 1 \right) + \left(\frac{p}{q}\right)^2 \left( \left(\frac{q}{p}\right)^{1.2} + \left(\frac{q}{p}\right)^{0.2} + 1 \right), \\ F(q, p) \left( \left(\frac{q}{p}\right)^{1.2} + \left(\frac{q}{p}\right)^{0.2} + 1 \right) &= \left( \left(\frac{q}{p}\right)^{1.2} + \left(\frac{q}{p}\right)^{0.2} + 1 \right) \left[ \Gamma(2.5) \left(\frac{p}{q}\right)^{3.5} + \left(\frac{p}{q}\right)^2 \right]. \end{aligned}$$

Implies that  $F(q, p) = \Gamma(2.5) \left(\frac{p}{q}\right)^{3.5} + \left(\frac{p}{q}\right)^2$ .

Now, take inverse Rishi transform for both sides, we get:

$$R^{-1}\{F(q, p)\} = R^{-1}\left\{\Gamma(2.5) \left(\frac{p}{q}\right)^{3.5} + \left(\frac{p}{q}\right)^2\right\}.$$

Hence, the exact solution is:  $f(t) = 1 + t\sqrt{t}$ .

**Example 5.4** Consider the linear fractional differential equation given by

$${}^L C_0 D_t^{1.9} f(t) + f(t) = \frac{2}{\Gamma(1.1)} t^{0.1} + t^2 + 1,$$

with the initial conditions:  $f(0) = 1, f'(0) = 0$

**Solution:** Take Rishi transform for both sides, and by property (4), we obtain:

$$R\{{}^LC_0D_t^{1.9}f(t)\} + R\{f(t)\} = R\left\{\frac{2}{\Gamma(1.1)}t^{0.1} + t^2 + 1\right\}.$$

By using theorem (4.3) and property (1), we get:

$$\begin{aligned} \left(\frac{q}{p}\right)^{1.9} F(q, p) - \sum_{k=0}^1 \left(\frac{p}{q}\right)^{1.1-k} f^{(1-k)}(0) + F(q, p) &= \frac{2}{\Gamma(1.1)} \left(\frac{p}{q}\right)^{2.1} \Gamma(1.1) + \left(\frac{p}{q}\right)^4 \Gamma(3) + \left(\frac{p}{q}\right)^2, \\ \left(\frac{q}{p}\right)^{1.9} F(q, p) - \left(\frac{p}{q}\right)^{1.1} f'(0) - \left(\frac{p}{q}\right)^{0.1} f(0) + F(q, p) &= 2 \left(\frac{p}{q}\right)^{2.1} + 2 \left(\frac{p}{q}\right)^4 + \left(\frac{p}{q}\right)^2. \end{aligned}$$

After input initial conditions, we get:

$$F(q, p) \left( \left(\frac{q}{p}\right)^{1.9} + 1 \right) = 2 \left(\frac{p}{q}\right)^{2.1} + 2 \left(\frac{p}{q}\right)^4 + \left(\frac{p}{q}\right)^2 + \left(\frac{p}{q}\right)^{0.1},$$

$$F(q, p) = \frac{1}{\left(\frac{q}{p}\right)^{1.9} + 1} \left( 2 \left(\frac{p}{q}\right)^4 \left(\left(\frac{q}{p}\right)^{1.9} + 1\right) + \left(\frac{p}{q}\right)^2 \left(\left(\frac{q}{p}\right)^{1.9} + 1\right) \right),$$

$$F(q, p) = \left( \frac{1}{\left(\frac{q}{p}\right)^{1.9} + 1} \right) \left( \left(\frac{q}{p}\right)^{1.9} + 1 \right) \left( 2 \left(\frac{p}{q}\right)^4 + \left(\frac{p}{q}\right)^2 \right),$$

$$F(q, p) = \left(\frac{p}{q}\right)^2 + 2 \left(\frac{p}{q}\right)^4.$$

Finally, take inverse Rishi transform for both sides, we get:

$$R^{-1}\{F(q, p)\} = R^{-1}\left\{\left(\frac{p}{q}\right)^2 + 2 \left(\frac{p}{q}\right)^4\right\}.$$

Hence, the exact solution is:  $f(t) = 1 + t^2$ .

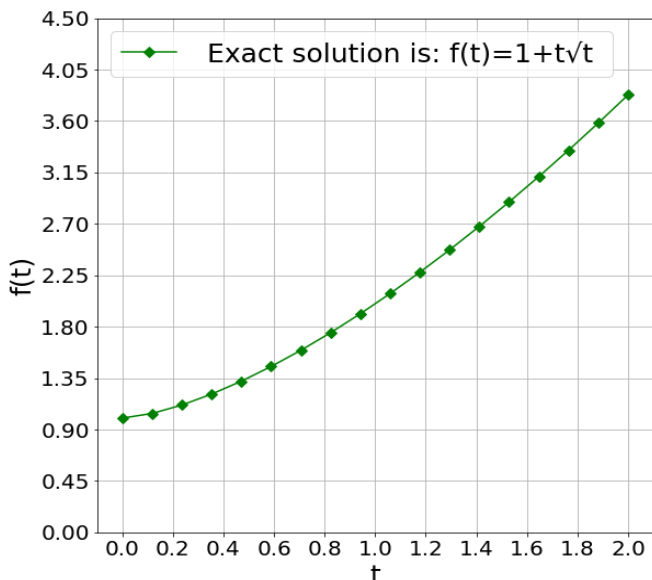


Figure 3. Exact solution of Example 5.3.

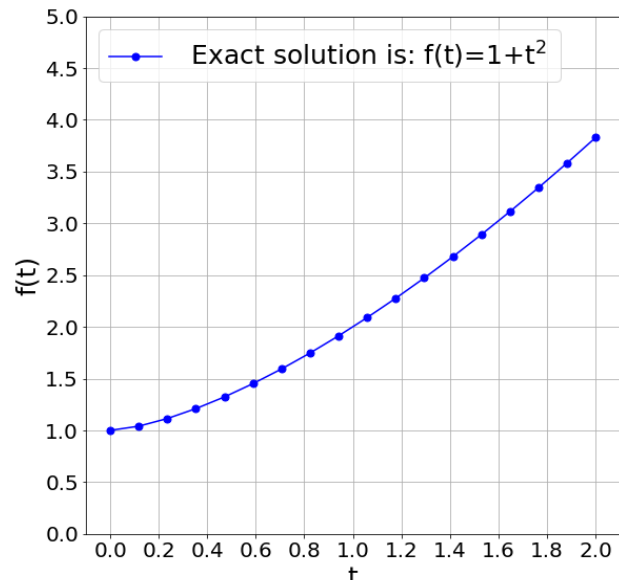


Figure 4. Exact solution of Example 5.4.

In this work, we proposed a transform for solving the fractional differential equations, but more studies will be required for solving the real-world problems modeled as fractional differential equations by using our proposed method.

## 6. Conclusion

After analyzing the literature regarding fractional differential equations (FDEs), it can be concluded that the Rishi transformation method is a feasible approach for solving such equations. This method involves transforming the FDEs into a simpler form. The Rishi transformation method offers several advantages, such as the reduction of the order of the equation, simplification of the form of the equation, and the ability to solve the FDEs with constant coefficients. What we have been able to achieve is that we have derived Rishi transform formulas for fractional derivation that can be used for later research. It is an applicable method for the analysis of many types of fractional differential equations, several studies have shown that the Rishi transformation method is an effective tool for solving different types of the FDEs.

In conclusion, the Rishi transformation method is a promising method for solving the FDEs, and with further research and development, it can become an even more efficient and effective tool for solving increasingly complex FDEs.

Concerning future work, we plan to extend the numerical schemes presented here to apply them on other types of fractional differential equations including exponential and Mittag-Leffler in kernels to establish similar results (visit [21, 26] to see these operators).

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