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Riemann Liouville Fractional-Like Integral Operators, Convex-

--Manuscript Draft--

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Abstract:	<p>It is well-known fact that fuzzy number theory is based on the characteristic function but in the fuzzy region, the characteristic function is a membership function that depends upon the interval . It means that real numbers and intervals are exceptional cases of fuzzy numbers. By using this approach, another type of Riemann Liouville fractional integral operator has been introduced over a fuzzy domain which is known as Riemann Liouville fractional-like integral operator. These integrals look like Riemann Liouville fractional integral operators but classical integrals are special cases of newly defined integral operators. Some novel versions of these integrals are also obtained and explained by taking the domain’s triangular and trapezoidal fuzzy numbers. We also consider the problem of computing scalar-valued. To discuss the application of new integrals, a new class of convex real-valued functions has been introduced, known as convex-like real-valued mappings. With the help of these new concepts, some applications are taken into account. By using convex-like real-valued mappings and newly proposed fractional-like integral operators, the well-known Hermite-Hadamard <span lang="EN-US" style="font-style: normal; font-size: 10pt; font-family:

	"Times New Roman", serif; color: black;"> type and related inequalities are taken into account in this work. Then by defining new differentiable real-valued functions over fuzzy regions, an identity has been obtained and with the support of this identity, certain inequalities are also acquired. Also, we clearly show the important connections of the derived outcomes with those classical integrals. Finally, some nontrivial numerical examples are also provided to verify the correctness of the presented inequalities that occur with the variation of the parameters and .</p>
Suggested Reviewers:	Aleksandr Rakhmangulov ran@magtu.ru Department of Logistics and Transportation Systems Management, Nosov Magnitogorsk State Technical University, Magnitogorsk, Russia Expert in fractional calculus.
	Michael Voskoglou mvoskoglou@gmail.com Mathematical Sciences, Graduate TEI of Western Greece Expert in fractional calculus.
Response to Reviewers:	

The Editors, Scientific reposts
Subject: Submission of Manuscript.

Dear Editors,
I am submitting my research article "**Riemann Liouville Fractional-like Integral Operators, Convex-Like Real-Valued Mappings and their Applications over Fuzzy Domain**", for possible publication as an original in Scientific Reposts.

I confirm that this manuscript is my original work and has not been published nor has it been submitted simultaneously elsewhere. I have checked the manuscript and have agreed to the submission.

Sincerely,
Juan L. G. Guirao
Departamento de Matema ́tica Aplicada y Estad ́ıstica, Universidad Polit ́ecnica de Cartagena, Hospital de Marina, 30203 Cartagena, Spain

Response to Ref. 1

Dear referee,

We would like to thank you for the valuable comments and replies which were taken into account within the revised version.

Reviewers' comments:

Referee 1

Reviewer #1: This manuscript tries to evaluate the newly defined Riemann - Liouville fractional-like integrals for scalar valued functions over a fuzzy spatial domain specifically over triangular fuzzy intervals and trapezoidal fuzzy intervals. The following are my comments and critics:

General:

Point 1. Abstract is too wordy.

Response: Thanks for your comment. We have tried our best to short the abstract section according to your suggestion but here we want to tell the reason why this abstract is too wordy. Because, in this article almost each and every thing is new and to discuss the main results in abstract section, it is necessary to present results so that the readers can easily know about the main concepts of the article. Moreover, we have not discussed some trivial results.

2- Introduction: it is not necessarily important to fully recall the trivial Hermite -Hadamard inequality for convex mapping.

Response: Thanks for your comment. We have done.

3- Conclusion is less illustrative and captivating for this high quality of work.

Response: We have done.

Methods:

1- The manuscript must clearly present the problem.

Response: Thanks for your comment. Again we have read the main results of the article and improved the presentation of the article. We have done.

2- What is the need to perform this operation (integration) over triangular and trapezoidal fuzzy intervals?

Response: Thanks for your comment. That is the reason, just we have recall the basic concept of fuzzy in abstract section comparing with crisp theory and abstract section is looks too wordy. In literature, we have notice that fuzzy number theory better handle uncertainty as compare to interval theory. We hope that this concept will also be helpful to overcome the uncertainty where intervals have uncertainty because ϑ-cuts characterize the fuzzy numbers to handle easily. Another reason behind to perform this operation over triangular and trapezoidal fuzzy intervals is that interval space is special case of these fuzzy number or interval space and, triangular and trapezoidal fuzzy intervals are types of fuzzy intervals.

Results:

What impact does this work have in improving fuzzy logic?

Response: Yes, obviously. Please see the response of comments 2 (Methods).

Discussion:

So, based on these findings, this evaluation remains valid for another kind of fractional integral operator? Why?

Response: Now we can say yes because during revision of this article (almost six months), by using this approach we are working on many concepts and proved it successfully.

Thank you for your comments on our article. However, we would like to rebut or clarify the answer to the reviewer's comments. We tried to include a better explanation in the revised paper in order to cover the aim and scope of the journal.

Furthermore, we would like to say that we have explicitly presented the conditions and results.
Finally, we express our special thanks to the Editor and referees for his/her excellent comments.

With Regards,

Muhammad Bilal Khan, Juan L. G. Guirao

Highlights:

Riemann Liouville Fractional-Like Integral Operators, Convex-Like Real-Valued Mappings and their Applications over Fuzzy Domain

Muhammad Bilal Khan

• In this paper, we have introduced new fractional operators over the fuzzy environment which
are known as Riemann Liouville Fractional-Like Integral Operators over the fuzzy domain.
• Most of the classical fractional integrals are exceptional cases of this new one.
• We know that, in the fields of applied mathematics and engineering, the theory of convex
mapping has several uses. Therefore, we have defined a new class of convex mappings which
is known as convex-Like real-valued mappings.
• With the help of this class and the newly proposed fractional integral operator over the fuzzy domain, some applications have been presented.
• We have presented the definition of the derivative of a real-valued function over the fuzzy domain.
• We have introduced an identity.
• By applying these definitions, we have amassed some novel and classical exceptional cases
that serve as implementations of the key findings.
• For the purpose of proving the viability of the main results, some nontrivial examples of fuzzy numbered valued convexity are also provided.

Declaration of interests

☒ The author declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

☐ The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

Riemann Liouville Fractional-Like Integral Operators, Convex-Like Real-Valued Mappings and their Applications over Fuzzy Domain

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Abstract: It is well-known fact that fuzzy number theory is based on the characteristic function but in the fuzzy region, the characteristic function is a membership function that depends upon the interval [0,1]. It means that real numbers and intervals are exceptional cases of fuzzy numbers. By using this approach, another type of Riemann Liouville fractional integral operator has been introduced over a fuzzy domain which is known as Riemann Liouville fractional-like integral operator. These integrals look like Riemann Liouville fractional integral operators but classical integrals are special cases of newly defined integral operators. Some novel versions of these integrals are also obtained and explained by taking the domain's triangular and trapezoidal fuzzy numbers. We also consider the problem of computing scalar-valued. To discuss the application of new integrals, a new class of convex real-valued functions has been introduced, known as convex-like real-valued mappings. With the help of these new concepts, some applications are taken into account. By using convex-like real-valued mappings and newly proposed fractional-like integral operators, the well-known Hermite-Hadamard ($\mathcal{H}.\mathcal{H}$) type and related inequalities are taken into account in this work. Then by defining new differentiable real-valued functions over fuzzy regions, an identity has been obtained and with the support of this identity, certain inequalities are also acquired. Also, we clearly show the important connections of the derived outcomes with those classical integrals. Finally, some nontrivial numerical examples are also provided to verify the correctness of the presented inequalities that occur with the variation of the parameters $\mathfrak{v}$ and β.

Keywords: Riemann Liouville fractional-like integral operators over the fuzzy domain, convex-like real-valued mappings, derivative over the fuzzy domain.

I. INTRODUCTION

A class of objects whose members are not clearly defined is referred to as a fuzzy set [1]. In comparison to the traditional mathematical binary representation, fuzzy sets offer a more accurate depiction of reality. The progressive nature of membership in fuzzy sets makes the theory essential for illustrating the finite degree of accuracy in mental representations [2].

Lotfi Asker Zadeh, the theory's originator, released the first study on the fuzzy set theory in 1965. Between 1965 and 1975, Zadeh strengthened the foundation of fuzzy set theory by creating fuzzy similarity links, linguistic hedges, and fuzzy decision-making. In the 1970s, some Japanese research teams began researching fuzzy set theory. In 1970, Mamdani invented the first fuzzy logic controller. Fuzzy logic saw its first commercial use in Europe and Japan in 1977. Fuzzy logic experienced a resurgence in the US at the end of the 1980s as a result of the success it had in Japan at the start of the decade. Numerous criticisms of the theory have surfaced concurrently with the development of the fuzzy set theory [3]. Those esteemed scientists rejected Zadeh's approach to dealing with ambiguity. They complained about it so harshly that the theory's logic was characterized as "Fuzzy logic is the cocaine of science." However, the use of fuzzy logic in actual technological applications demonstrated its effectiveness.

The Sendai Subway system, which opened in 1988 in Sendai, Japan, is one of the best-known uses of fuzzy logic. The train was operated continuously throughout the day using a fuzzy controller for line control. The line now has one of the smoothest operating subway systems in the entire world thanks to the fuzzy controller. Fuzzy logic thermostats are used in commercial HVAC (heating, ventilation, and air conditioning) systems to regulate the heating and cooling, which saves energy by improving system performance and maintaining a more constant temperature than a conventional thermostat. On the basis of the detected barrier, fuzzy logic control systems have also been employed to regulate the speed of autos. A sensor in the front panel of the device detects the presence of impediments. Programmable logic controllers (PLCs) based on fuzzy logic have been developed by businesses like

Moeller, one of the oldest and most reputable brands in the electrical industry. Due of its significant use during the production of The Lord of the Rings, fuzzy logic has also been used in 3D animation systems for creating crowds, see [4]-[9] and the references therein.

On the other hand, every area of scientific research has made significant progress with fuzzy sets. It has a wide range of applications in both theoretical and practical research, from engineering to the arts and humanities, computer science to health sciences, and life sciences to physical sciences. The classification issue in social sciences or geographic information systems is frequently reliant on imprecise language ideas. A better understanding of geographic objects with ill-defined bounds that correspond to various graded categories is facilitated by the depiction of fuzzy boundaries [10]-[12].

The reference [13] is an example of how fuzzy set theory is used in forest planning models and [14] offers a method for calculating forest area using uncertainty levels. See also [15], which presents a method for applying a fuzzy representation of a geographic border to a soil loss model. Reference [16], on the other hand, deals with creating fuzzy category maps from remotely sensed images.

Regarding these applications, it's intriguing to consider how the mathematical principles of integration on fuzzy domains cope with the traits of ambiguous regions. The theory of fuzzy measures and fuzzy integrals was first presented by Sugeno in [17]. References [18]-[24] show that various integral inequalities also hold for the fuzzy context, and some of its features were explored in [25, 26]. The idea of an unknown integral [27] and some of its features are discussed in [28] to provide a further generalization. In contrast, the idea of Gould integrability for interval-valued multifunction with respect to interval-valued set multifunction is offered in [29]. In [30], the Sugeno fuzzy integral for concave functions and non-linear integrals based on decomposition integrals are discussed. Reference [30] discusses the Sugeno fuzzy integral for concave functions, and [31] provides non-linear integrals based on decomposition integrals that take greediness into account. By reducing the issue to specific calculations in the Euclidean space, our method enables the evaluation of the integral over a fuzzy set for the case of real-valued as well as fuzzy-valued functions.

Hermite-Hadamard inequalities in [32] and [33], which is widely employed in many other areas of practical mathematics, particularly optimization and probability, is one of the most important mathematical inequalities pertinent to convex maps. Let's elicit it as shown below:

The $\mathcal{H}.\mathcal{H}$-inequality for convex mapping $\Upsilon: K \to \mathbb{R}$ on an interval $K = [\tau, \varsigma]$ is

$$\Upsilon\left(\frac{\tau+\varsigma}{2}\right) \leq \frac{1}{\varsigma-\tau}\int_{\tau}^{\varsigma}\Upsilon(\varkappa)d\varkappa \leq \frac{\Upsilon(\tau)+\Upsilon(\varsigma)}{2}, \tag{1}$$

for all $\tau, \varsigma \in K$.

Only affine mappings are guaranteed to be equal on both sides. This Hermite and Hadamard finding is really straightforward but extremely potent. It's interesting to note that convex mappings fall on both sides of the aforementioned integral inequality. We recommend readers to read [34]-[36] for a few fascinating facts and uses of the $\mathcal{H}.\mathcal{H}$-disparity.

It has been shown that fractional calculus [37], as a rather resilient technique, is an essential basic ingredient not only in the mathematical sciences but also in the applied sciences. Many researchers have become interested in the area to answer the important topic. As a consequence, many authors have obtained some significant integral inequalities through the efficient interaction of various methods of fractional integrals, including Ahmad et al. [38] in the study of four types of inequalities for convex mappings concerning fractional integrals with exponential kernels, Set et al. [39] in the $\mathcal{H}.\mathcal{H}$-Fejer-type inequality for Atangana-Baleanu fractional integral operators, Khan et al. [40] and Meftah et al. [41] in the $\mathcal{H}.\mathcal{H}$-type inequalities for conformable fractional integral operators, and Dragomir [42] in the $\mathcal{H}.\mathcal{H}$-type inequalities for generalized Riemann Liouville fractional integrals. For more information, see [43]-[45] and the references therein.

Many studies have been conducted in recent years to investigate the relationship between integral inequalities and interval-valued mappings, giving various noteworthy discoveries. The Minkowski-type inequalities and the Beckenbach-type inequalities were established by Roman-Flores [46], the Ostrowski-type inequalities were examined by Chalco-Cano [47] using the extended Hukuhara derivative, and the Opial-type inequalities were first introduced by Costa [48]. Zhao et al. [49] recently built on this idea by introducing interval-valued coordinated convex mappings and related $\mathcal{H}.\mathcal{H}$-type inequalities. Additionally, it served as support for the n-polynomial convex interval-valued mapping's $\mathcal{H}.\mathcal{H}$- and $\mathcal{H}.\mathcal{H}$-Fejér-type inequalities [50].

Recently, Khan and his coworkers recently expanded the idea of convex interval-valued mappings (convex I.V.Ms) and fuzzy-interval-valued mappings (convex F.I.V.Ms) by using fuzzy-order relations. As a result, convex F.I.V.Ms, an apparently new concept, now include (h1, h2)-convex F.I.V.Ms [51] and harmonic convex F-I-Vs, see [52]. We advise interested readers to study certain fundamental ideas connected to fuzzy calculus, see [53]-[70] and the references therein, in order to learn about various recent breakthroughs related to the idea of fuzzy interval-valued analysis of several well-known integral inequalities. For other related concepts, see [71-81] and the references therein.

Inspiring and motivated by the ongoing research work, after some preliminary notions in section 2, we have presented, in section 3, the notion of Riemann-Liouville fractional-like integrals and we have defined the Riemann-Liouville fractional-like integral of a scalar-valued function on a fuzzy domain, showing some examples of integration over triangular fuzzy intervals and trapezoidal fuzzy intervals. Then, in section 4, we present the new class of convex real-valued mapping which is known as convex-like real-valued mapping of the integral of a fuzzy-valued function on a fuzzy domain; and illustrated its calculation with an example. Finally, in section 5, we have presented some applications for the integration over fuzzy domains with higher dimensions. In particular, we have discussed the validation of the main results with the help of nontrivial examples.

II. PRELIMINARIES

First, we provide the concepts and notions required for the sequel. We give our research of the paper to ensure its completion in Section 3. We start with the definition of a fuzzy set such that:

Definition 1: [57, 58] A fuzzy subset T of $\mathbb{R}$ is distinguished by a mapping $\widetilde{\omega}: \mathbb{R} \to [0,1]$ called the membership mapping of T. That is, a fuzzy subset T of $\mathbb{R}$ is a mapping $\widetilde{\omega}: \mathbb{R} \to [0,1]$. So, for further study, we have chosen this notation. We appoint $\mathbb{E}$ to denote the set of all fuzzy subsets of $\mathbb{R}$.

In [70], Goetschel and Voxman initiated to introduce the concept of fuzzy numbers as follows:

Let $\widetilde{\omega} \in \mathbb{E}$. Then, $\widetilde{\omega}$ is known as a fuzzy number or fuzzy interval if the following properties are satisfied by $\widetilde{\omega}$:

(1) $\widetilde{\omega}$ should be normal if there exists $\varkappa \in \mathbb{R}$ and $\widetilde{\omega}(\varkappa) = 1$;

(2) $\widetilde{\omega}$ should be upper semi-continuous on $\mathbb{R}$ if for given $\varkappa \in \mathbb{R}$, there exist $\varepsilon > 0$ there exist $\delta > 0$ such that $\widetilde{\omega}(\varkappa) - \widetilde{\omega}(y) < \varepsilon$ for all $y \in \mathbb{R}$ with $|\varkappa - y| < \delta$;

(3)$\widetilde{\omega}$ should be fuzzy convex that is $\widetilde{\omega}\big((1-\varphi)\varkappa + \varphi y\big) \geq min\left(\widetilde{\omega}(\varkappa), \widetilde{\omega}(y)\right)$, for all $\varkappa, y \in \mathbb{R}$, and $\varphi \in [0, 1]$

(4) $\widetilde{\omega}$ should be compactly supported that is $cl\{\varkappa \in \mathbb{R} |\ \widetilde{\omega}(\varkappa) > 0\}$ is compact.

We appoint $\mathbb{E}_C$ to denote the set of all fuzzy numbers of $\mathbb{R}$.

Definition 2: [57, 58] Given $\widetilde{\omega} \in \mathbb{E}_C$, the level sets or cut sets are given by $\left[\widetilde{\omega}\right]^{\mathfrak{v}} = \{\varkappa \in \mathbb{R} |\ \widetilde{\omega}(\varkappa) > \mathfrak{v}\}$ for all $\mathfrak{v} \in [0, 1]$ and by $\left[\widetilde{\omega}\right]^{0} = \{\varkappa \in \mathbb{R} |\ \widetilde{\omega}(\varkappa) > 0\}$. These sets are known as $\mathfrak{v}$-level sets or $\mathfrak{v}$-cut sets of $\widetilde{\omega}$.

From these definitions, we have

$$\left[\widetilde{\omega}\right]^{\mathfrak{v}} = [l(\mathfrak{v}), \curlyvee(\mathfrak{v})], \tag{2}$$

where

$$l(\mathfrak{v}) = inf\{\mathfrak{x} \in \mathbb{R} |\ \omega(\mathfrak{x}) \geq \mathfrak{v}\},$$
$$\curlyvee(\mathfrak{v}) = sup\{\mathfrak{x} \in \mathbb{R} |\ \omega(\mathfrak{x}) \geq \mathfrak{v}\}.$$

Remark 1: [66, 67] For each interval $[\tau, \varsigma] \in \mathcal{X}_C$, there characteristic function $\widetilde{[\tau, \varsigma]}: \mathbb{R} \to [0,1]$ defined by

$$\widetilde{[\tau, \varsigma]}(\varkappa) = \begin{cases} 1 & \varkappa \in [\tau, \varsigma] \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

So, in some sense, we can think that fuzzy numbers generalize the set of closed intervals of real numbers, i.e. that $\mathcal{X}_C \subseteq \mathbb{E}_C$ and therefore $\mathbb{R} \subseteq \mathbb{E}_C$ too, once degenerated intervals can be seen as real numbers and instead of writing $\widetilde{[\varsigma, \varsigma]}$, we just use $\tilde{\varsigma}$. A fuzzy number $\tilde{\varsigma}$ is called crisp number or fuzzy singleton, see [67].

Proposition 1: [62] Let $\widetilde{\omega}, \widetilde{\pi} \in \mathbb{E}_C$. Then relation " $\leq_{\mathbb{F}}$ " given on $\mathbb{E}_C$ by

$$\widetilde{\omega} \leq_{\mathbb{F}} \widetilde{\pi} \text{ when and only when, } \left[\widetilde{\omega}\right]^{\mathfrak{v}} \leq_I \left[\widetilde{\pi}\right]^{\mathfrak{v}}, \tag{4}$$

for every $\mathfrak{v} \in [0, 1]$, it is a partial-order relation.

Proposition 2: [54] Let $\widetilde{\omega}, \widetilde{\pi} \in \mathbb{E}_C$. Then relation " $\supseteq_{\mathbb{F}}$ " given on $\mathbb{E}_C$ by

$$\widetilde{\omega} \supseteq_{\mathbb{F}} \widetilde{\pi} \text{ when and only when, } \left[\widetilde{\omega}\right]^{\mathfrak{v}} \supseteq_I \left[\widetilde{\pi}\right]^{\mathfrak{v}}, \tag{5}$$

for every $\mathfrak{v} \in [0, 1]$, it is up and down-order relation on $\mathbb{E}_C$.

Recalling the approaching notions, which are offered in the literature. If $\widetilde{\omega}, \widetilde{\Pi} \in \mathbb{E}_C$ and $\mathfrak{v} \in \mathbb{R}$, then, for every $\mathfrak{v} \in [0, 1]$, the arithmetic operations are defined by

$$[\widetilde{\omega} \oplus \widetilde{\Pi}]^{\mathfrak{v}} = [\widetilde{\omega}]^{\mathfrak{v}} + [\widetilde{\Pi}]^{\mathfrak{v}}, \tag{6}$$

$$[\widetilde{\omega} \otimes \widetilde{\Pi}]^{\mathfrak{v}} = [\widetilde{\omega}]^{\mathfrak{v}} \times [\widetilde{\Pi}]^{\mathfrak{v}}, \tag{7}$$

$$[\varphi \odot \widetilde{\omega}]^{\mathfrak{v}} = \varphi . [\widetilde{\omega}]^{\mathfrak{v}}. \tag{8}$$

Theorem 1: [57] The space $\mathbb{E}_C$ dealing with a supremum metric, i.e., for $\widetilde{\omega}, \widetilde{\Pi} \in \mathbb{E}_C$

$$d_{\infty}(\widetilde{\omega}, \widetilde{\Pi}) = \sup_{0 \le \mathfrak{v} \le 1} d_H([\widetilde{\omega}]^{\mathfrak{v}}, [\widetilde{\Pi}]^{\mathfrak{v}}), \tag{9}$$

is a complete metric space, where H denotes the well-known Hausdorff metric on space of intervals.
The following is the definition of classical of Riemann-Liouville fractional integral operators over $[\tau, \varsigma]$, where integrable functions are real-valued functions, see [37].
Definition 3: Let $\beta > 0$ and $L([\tau, \varsigma])$ be the collection of all Lebesgue measurable real-valued mapping on $[\tau, \varsigma]$. Then the left and right Riemann-Liouville fractional integral of $\Upsilon \in L([\tau, \varsigma])$ with order $\beta > 0$ are defined by

$$\mathcal{J}^{\beta}_{\tau^+} \Upsilon(\varkappa) = \frac{1}{\Gamma(\beta)} \int_{\tau}^{\varkappa} (\varkappa - \varphi)^{\beta - 1} \Upsilon(\varphi)\, d\varphi, \quad (\varkappa > \tau), \tag{10}$$

and

$$\mathcal{J}^{\beta}_{\varsigma^-} \Upsilon(\varkappa) = \frac{1}{\Gamma(\beta)} \int_{\varkappa}^{\varsigma} (\varphi - \varkappa)^{\beta - 1} \Upsilon(\varphi)\, d\varphi, \quad (\varkappa < \varsigma), \tag{11}$$

respectively, where $\Gamma(\varkappa) = \int_0^{\infty} \varphi^{\varkappa - 1} e^{-\varphi}\, d\varphi$ is the Euler gamma mapping.
Recently Khastan and Rodríguez-López [66] have introduced the real-valued functions over the fuzzy domain and also discussed some of the properties of real-valued functions over fuzzy using the Lebesgue measures. Moreover, they have presented the following definition of integral:
Definition 4: If $\widetilde{\omega} \in \mathbb{E}_C$, and $\Upsilon : [\tau, \varsigma] \subseteq \mathbb{R} \to \mathbb{R}$ is measurable on $[\widetilde{\omega}]^0 \subseteq [\tau, \varsigma]$ (and hence for each $[\widetilde{\omega}]^{\mathfrak{v}}$, for all $\mathfrak{v} \in [0,1]$), then we define

$$\left(\int_{\widetilde{\omega}} \Upsilon\right)(\mathfrak{v}) = \int_{[\widetilde{\omega}]^{\mathfrak{v}}} \Upsilon(\varphi)\, d\varphi, \tag{12}$$

where the integral on the right-hand side is calculated in the sense of Lebesgue. We say the Υ is integrable over the fuzzy domain; if the integral $\int_{[\widetilde{\omega}]^0} \Upsilon(\varphi)\, d\varphi$ is finite. In that case, mapping is defined as

$$\int_{\widetilde{\omega}} \Upsilon : [0, 1] \to \mathbb{R}$$

$$\mathfrak{v} \to \left(\int_{\widetilde{\omega}} \Upsilon\right)(\mathfrak{v}) = \int_{[\widetilde{\omega}]^{\mathfrak{v}}} \Upsilon(\varphi)\, d\varphi.$$

All the above preliminary notions are very useful to discuss the upcoming main results because the problem of interest is to calculate the Riemann-Liouville fractional-like integral of Υ over the fuzzy region $\widetilde{\omega} \in \mathbb{E}_C$.

III. RIEMANN-LIOUVILLE FRACTIONAL-LIKE INTEGRALS OVER THE FUZZY DOMAIN

In this section, we have proposed the new concept of Riemann-Liouville fractional integrals operators which is known as Riemann-Liouville fractional-like integral operators.
Definition 5: Let $\beta > 0$ and $L(\widetilde{\omega})$ be the collection of all Lebesgue measurable real-valued mapping over fuzzy domain $\widetilde{\omega}$, where $\widetilde{\omega}$ has the following parametric representation $[\widetilde{\omega}]^{\mathfrak{v}} = [l(\mathfrak{v}), \mathfrak{r}(\mathfrak{v})]$, for all $\mathfrak{v} \in [0,1]$. Must add about $[\widetilde{\omega}]^0$. Then the left and right Riemann-Liouville fractional-like integral of $\Upsilon \in L(\widetilde{\omega})$ with order $\beta > 0$ are defined by

$$\mathcal{J}^{\beta}_{l(\mathfrak{v})^+} \Upsilon(\varkappa) = \frac{1}{\Gamma(\beta)} \int_{l(\mathfrak{v})}^{\varkappa} (\varkappa - \varphi)^{\beta - 1} \Upsilon(\varphi)\, d\varphi, \quad (\varkappa > l(\mathfrak{v})), \tag{13}$$

and

$$\mathcal{J}^{\beta}_{\mathfrak{r}(\mathfrak{v})^-} \Upsilon(\varkappa) = \frac{1}{\Gamma(\beta)} \int_{\varkappa}^{\mathfrak{r}(\mathfrak{v})} (\varphi - \varkappa)^{\beta - 1} \Upsilon(\varphi)\, d\varphi, \quad (\varkappa < \mathfrak{r}(\mathfrak{v})), \tag{14}$$

respectively, where $\Gamma(\varkappa) = \int_0^{\infty} \varphi^{\varkappa - 1} e^{-\varphi}\, d\varphi$ is the Euler gamma mapping.

Particular Cases

Consider the triangular fuzzy numbers $\widetilde{\omega} = (r;\ \lambda, \gamma)$, with $r \in \mathbb{R}$, and $\lambda, \gamma \in \mathbb{R}$, that is

$$\widetilde{\omega}(\varkappa) = \begin{cases} \frac{\omega - r + \lambda}{\lambda}, & \omega \in [r-\lambda, r] \\ \frac{r+\gamma-\omega}{\gamma}, & \omega \in (r, r+\gamma] \\ 0, & \text{otherwise}, \end{cases} \tag{15}$$

whose parametrized form is $[\widetilde{\omega}]^{\mathfrak{v}} = [r - \lambda(1-\mathfrak{v}), r + \gamma(1-\mathfrak{v})]$, for all $\mathfrak{v} \in [0,1]$.

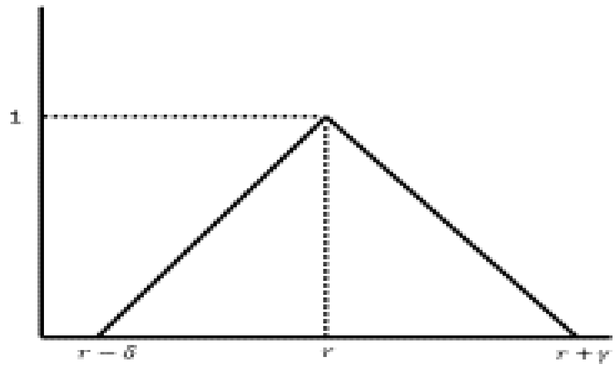

Fig. 1. Triangular fuzzy number

To integrate a real-valued mapping $\Upsilon: [\tau, \varsigma] \to \mathbb{R}$, with $[\widetilde{\omega}]^0 = [r-\lambda, r+\gamma] \subseteq [\tau, \varsigma]$, over the fuzzy set $\widetilde{\omega}$, we have to calculate,

$$\begin{aligned} \left(\mathcal{J}^{\beta}_{(\widetilde{\omega})^{+}}\Upsilon\right)(\mathfrak{v}) &= \mathcal{J}^{\beta}_{\left([\widetilde{\omega}]^{\mathfrak{v}}\right)^{+}}\Upsilon\big(r+\gamma(1-\mathfrak{v})\big) \\ &= \mathcal{J}^{\beta}_{(r-\lambda(1-\mathfrak{v}))^{+}}\Upsilon\big(r+\gamma(1-\mathfrak{v})\big) \\ &= \frac{1}{\Gamma(\beta)}\int_{r-\lambda(1-\mathfrak{v})}^{r+\gamma(1-\mathfrak{v})} (r+\gamma(1-\mathfrak{v})-\varphi)^{\beta-1}\Upsilon(\varphi)\,d\varphi, \end{aligned} \tag{16}$$

given that $\big(r+\gamma(1-\mathfrak{v}) > r-\lambda(1-\mathfrak{v})\big)$,
and

$$\begin{aligned} \left(\mathcal{J}^{\beta}_{(\widetilde{\omega})^{-}}\Upsilon\right)(\mathfrak{v}) &= \mathcal{J}^{\beta}_{\left([\widetilde{\omega}]^{\mathfrak{v}}\right)^{-}}\Upsilon\big(r-\lambda(1-\mathfrak{v})\big) \\ &= \mathcal{J}^{\beta}_{(r+\gamma(1-\mathfrak{v}))^{+}}\Upsilon\big(r-\lambda(1-\mathfrak{v})\big) \\ &= \frac{1}{\Gamma(\beta)}\int_{r-\lambda(1-\mathfrak{v})}^{r+\gamma(1-\mathfrak{v})} \Big(\varphi-\big(r-\gamma(1-\mathfrak{v})\big)\Big)^{\beta-1}\Upsilon(\varphi)\,d\varphi, \end{aligned} \tag{17}$$

given $\big(r-\lambda(1-\mathfrak{v}) < r+\gamma(1-\mathfrak{v})\big)$.

Not that, if Υ is a real constant function, (equal to k), then from (16), we have

$$\left(\mathcal{J}^{\beta}_{(\widetilde{\omega})^{+}}\Upsilon\right)(\mathfrak{v}) = \frac{k\big(-\mathfrak{v}\lambda+\lambda+\gamma(1-\mathfrak{v})\big)^{\beta}}{\beta\Gamma(\beta)} \text{ for } \beta > 0, \tag{18}$$

$$\left(\mathcal{J}^{\beta}_{(\widetilde{\omega})^{-}}\Upsilon\right)(\mathfrak{v}) = \frac{k\big(\lambda(1-\mathfrak{v})+\gamma(1-\mathfrak{v})\big)^{\beta}}{\beta\Gamma(\beta)} \text{ for } \beta > 0, \tag{19}$$

and the function is linear on $\mathfrak{v}$, attaining the value 0 at $\mathfrak{v} = 1$ and $\frac{k(\lambda+\gamma)^{\beta}}{\beta\Gamma(\beta)}$ at $\mathfrak{v} = 0$.

When $\beta = \frac{3}{2}, 1, \frac{1}{2}$, then left and right Riemann-Liouville fractional-like integrals of the constant function $\Upsilon(\varkappa) = 3$ on the triangular fuzzy number $\widetilde{\omega} = \left(1; \frac{1}{2}, 2\right)$ have the following geometrical representation such that

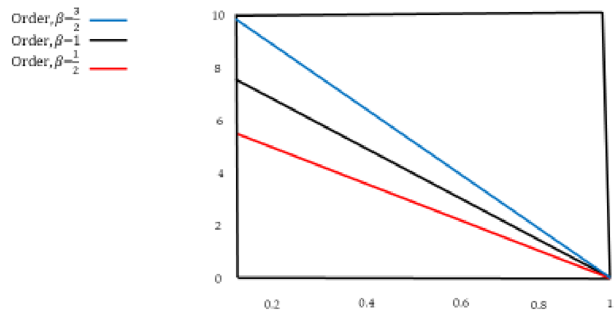

Fig. 2. Riemann-Liouville fractional-type integral of constant function over triangular fuzzy number $\widetilde{\omega} = \left(1; \frac{1}{2}, 2\right)$.

From Figure 2, it can be easily seen that for $\beta = 1$, we obtain the Riemann-like integral of constant function over fuzzy domain.
It is well-known fact that triangular fuzzy number is said to be symmetric, when $\lambda = \gamma$. Here, we are taking Υ as a contact function equal to k, then from (18) and (19), we have

$$\left(\mathcal{J}^{\beta}_{(\widetilde{\omega})^{+}}\Upsilon\right)(\mathfrak{v}) = \frac{k\left(2\lambda(1-\mathfrak{v})\right)^{\beta}}{\beta\Gamma(\beta)} \text{ for } \beta > 0, \tag{20}$$

$$\left(\mathcal{J}^{\beta}_{(\widetilde{\omega})^{-}}\Upsilon\right)(\mathfrak{v}) = \frac{k\left(2\lambda(1-\mathfrak{v})\right)^{\beta}}{\beta\Gamma(\beta)} \text{ for } \beta > 0, \tag{21}$$

From above both expression we have noted that both left and right Riemann-Liouville fractional-like integrals of a constant function attaining the same value. It means that at $\mathfrak{v} = 0$, the error is multiple of k for both $\left(\mathcal{J}^{\beta}_{(\widetilde{\omega})^{+}}\Upsilon\right)(\mathfrak{v}) = \frac{k(2\lambda)^{\beta}}{\beta\Gamma(\beta)} = \left(\mathcal{J}^{\beta}_{(\widetilde{\omega})^{-}}\Upsilon\right)(\mathfrak{v})$, when the triangular number is symmetric.
On the other hand, taking the trapezoidal fuzzy numbers $\widetilde{\omega} = (r, s;\ \lambda, \gamma)$, with $r, s \in \mathbb{R}$, and $\lambda, \gamma \in \mathbb{R}$, that is

$$\widetilde{\omega}(\varkappa) = \begin{cases} 1, & \omega \in [r, s] \\ \frac{\omega - r + \lambda}{\lambda}, & \omega \in [r-\lambda, r] \\ \frac{s+\gamma-\omega}{\gamma}, & \omega \in [s, s+\gamma] \\ 0, & \text{otherwise}, \end{cases} \tag{22}$$

whose parametrized form is $[\widetilde{\omega}]^{\mathfrak{v}} = [r - \lambda(1-\mathfrak{v}), s + \gamma(1-\mathfrak{v})]$, for all $\mathfrak{v} \in [0,1]$.

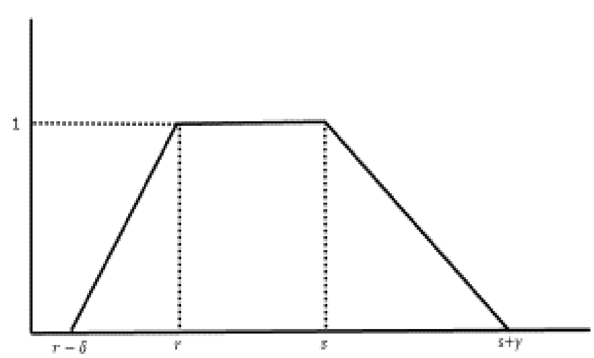

Fig. 3. Trapezoidal fuzzy number

The left and right Riemann-Liouville fractional-like integrals over the fuzzy number $\tilde{\omega}$, we have to calculate:

$$\begin{aligned}\left(\mathcal{J}^{\beta}_{(\tilde{\omega})^{+}}\Upsilon\right)(\mathfrak{v}) &= \mathcal{J}^{\beta}_{\left([\tilde{\omega}]^{\mathfrak{v}}\right)^{+}}\Upsilon\left(s+\gamma(1-\mathfrak{v})\right)\\ &= \mathcal{J}^{\beta}_{\left(r-\lambda(1-\mathfrak{v})\right)^{+}}\Upsilon\left(s+\gamma(1-\mathfrak{v})\right)\\ &= \frac{1}{\Gamma(\beta)}\int_{r-\lambda(1-\mathfrak{v})}^{s+\gamma(1-\mathfrak{v})}\left(s+\gamma(1-\mathfrak{v})-\varphi\right)^{\beta-1}\Upsilon(\varphi)\,d\varphi, \qquad (23)\end{aligned}$$

where $\left(s+\gamma(1-\mathfrak{v})>r-\lambda(1-\mathfrak{v})\right)$.
and

$$\begin{aligned}\left(\mathcal{J}^{\beta}_{(\tilde{\omega})^{-}}\Upsilon\right)(\mathfrak{v}) &= \mathcal{J}^{\beta}_{\left([\tilde{\omega}]^{\mathfrak{v}}\right)^{-}}\Upsilon\left(r-\lambda(1-\mathfrak{v})\right)\\ &= \mathcal{J}^{\beta}_{\left(s+\gamma(1-\mathfrak{v})\right)^{+}}\Upsilon\left(r-\lambda(1-\mathfrak{v})\right)\\ &= \frac{1}{\Gamma(\beta)}\int_{r-\lambda(1-\mathfrak{v})}^{s+\gamma(1-\mathfrak{v})}\left(\varphi-\left(r-\lambda(1-\mathfrak{v})\right)\right)^{\beta-1}\Upsilon(\varphi)\,d\varphi, \qquad (24)\end{aligned}$$

where, $\left(r-\lambda(1-\mathfrak{v})<s+\gamma(1-\mathfrak{v})\right)$.
Not that, if Υ is real constant function, (equal to k), then from (23) and (24), we have

$$\left(\mathcal{J}^{\beta}_{(\tilde{\omega})^{+}}\Upsilon\right)(\mathfrak{v}) = \frac{k\left(s-r-\mathfrak{v}\lambda+\lambda+\gamma(1-\mathfrak{v})\right)^{\beta}}{\beta\Gamma(\beta)} \text{ for } \beta>0, \qquad (25)$$

$$\left(\mathcal{J}^{\beta}_{(\tilde{\omega})^{-}}\Upsilon\right)(\mathfrak{v}) = \frac{k\left(s-r+\lambda(1-\mathfrak{v})+\gamma(1-\mathfrak{v})\right)^{\beta}}{\beta\Gamma(\beta)} \text{ for } \beta>0, \qquad (26)$$

and the function is linear on $\mathfrak{v}$, attaining the value $k(s-r)$ at $\mathfrak{v}=1$ and $\frac{k(s-r+\lambda+\gamma)^{\beta}}{\beta\Gamma(\beta)}$ at $\mathfrak{v}=0$.

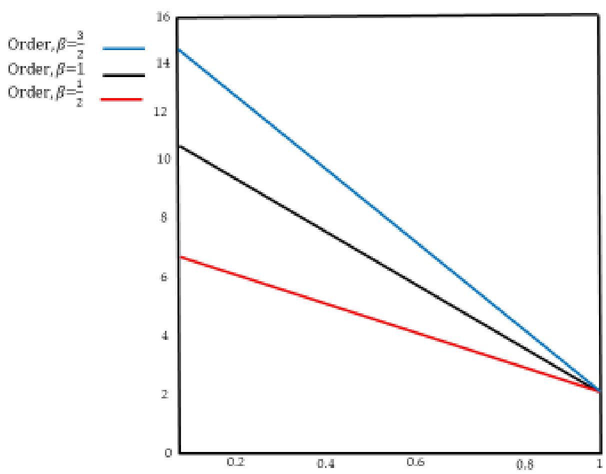

Fig. 4. Riemann-Liouville fractional-type integral of constant function over trapezoidal fuzzy number $\tilde{\omega}=\left(1,2;\frac{1}{2},2\right)$.

If $\beta=\frac{3}{2},1,\frac{1}{2}$, then left and right Riemann-Liouville fractional-like integrals of the constant function $\Upsilon(\varkappa)=3$ on the trapezoidal fuzzy number $\tilde{\omega}=\left(1,2;\frac{1}{2},2\right)$ have the following geometrical representation such that
When $\lambda=\gamma$, then trapezoidal fuzzy number is reduces to be symmetric trapezoidal fuzzy number, Here, we are taking Υ as a constant function equal to k, then from (25) and (26), we have

$$\left(\mathcal{J}^{\beta}_{(\tilde{\omega})^{+}}\Upsilon\right)(\mathfrak{v}) = \frac{k\left(s-r+2\lambda(1-\mathfrak{v})\right)^{\beta}}{\beta\Gamma(\beta)} \text{ for } \beta>0, \qquad (27)$$

$$\left(\mathcal{J}^{\beta}_{(\tilde{\omega})^{-}}\Upsilon\right)(\mathfrak{v}) = \frac{k\left(s-r+2\lambda(1-\mathfrak{v})\right)^{\beta}}{\beta\Gamma(\beta)} \text{ for } \beta>0, \qquad (28)$$

It is familiar fact that trapezoidal fuzzy number becomes triangular number when $s=r$. It means that this discussion coincides with the above discussion related to the fuzzy numbers.

If $\widetilde{\omega} = \widetilde{[\tau, \varsigma]}$, then from Remark 1 and Figure 4, this new Riemann-Liouville fractional-like integrals over fuzzy number reduces to classical Riemann-Liouville fractional integrals $\widetilde{[\tau, \varsigma]}$.
If $\widetilde{\omega} = \widetilde{[\tau, \varsigma]}$ and $\beta = 1$, then By Remark 1, we achieve classical definition of Riemann integrals over real-valued interval.
From Figure 2 and Figure 4, it can be easily seen that for $\beta = 1$, we obtain the Riemann-like integrals as a special case of Riemann-Liouville fractional-like integrals over fuzzy domain.

Note that, further exceptional cases can also be discussed by taking different types of fuzzy numbers by using same approaches.

IV. APPLICATIONS OF RIEMANN-LIOUVILLE FRACTIONAL-LIKE INTEGRAL OPERATORS

In this section, firstly with the help of real-valued mapping over the fuzzy domain, we propose a new class of convex mapping which is known as convex-like real-valued mapping. Then, some applications of convex-like real-valued mapping and Riemann-Liouville fractional-like integrals over the fuzzy domain will be discussed by introducing new versions of $\mathcal{H}.\mathcal{H}$-type inequalities. Before finding the application of Riemann-Liouville fractional-like integrals over the fuzzy domain, firstly we present the class of classical convex real-valued mapping and then we will define the novel class convex-like real-valued mapping.
Definition 6: The real-valued mapping $\Upsilon: [\tau, \varsigma] \to \mathbb{R}$ is called convex real-valued mapping on $[\tau, \varsigma]$ if

$$\Upsilon(\varphi\varkappa + (1-\varphi)s\,) \leq \varphi\Upsilon(\varkappa) + (1-\varphi)\Upsilon(s), \tag{29}$$

for all $\varkappa, s \in [\tau, \varsigma], \varphi \in [0, 1]$. If (29) is reversed then, Υ is called concave real-valued mapping on $[\tau, \varsigma]$. Υ is affine if and only if it is both convex and concave real-valued mapping.
Definition 7: The real-valued mapping $\Upsilon: \widetilde{\omega} \to \mathbb{R}$ is called convex-like real-valued mapping on $\widetilde{\omega}$ if

$$\Upsilon(\varphi\varkappa + (1-\varphi)s\,) \leq \varphi\Upsilon(\varkappa) + (1-\varphi)\Upsilon(s), \tag{30}$$

for all $\varkappa, s \in \widetilde{\omega}, \varphi \in [0, 1]$. If (30) is reversed then, Υ is called type-2 concave real-valued mapping on $\widetilde{\omega}$. Υ is type 2 affine if and only if it is both type 2 convex and type-2 concave real-valued mapping.
Remark 2: By Remark 1, if $\widetilde{\omega} = \widetilde{[\tau, \varsigma]}$, then from Definition 7, we acquire the Definition 6.

Riemann-Liouville fractional-like integral Inequalities

Here is our first main application of Riemann-Liouville fractional-like integral in relation to the $\mathcal{H}.\mathcal{H}$-type inequalities, which is based on convex-like real-valued mapping on $\widetilde{\omega}$.
Theorem 2: Let $\Upsilon: \widetilde{\omega} \to \mathbb{R}$ be a convex-like real-valued mapping on $\widetilde{\omega}$, whose parametrized form is $\left[\widetilde{\omega}\right]^{\mathfrak{v}} = [l(\mathfrak{v}), \mathfrak{r}(\mathfrak{v})]$, for all $\mathfrak{v} \in [0, 1]$. If $\Upsilon \in L(\widetilde{\omega})$, then

$$\Upsilon\left(\frac{l(\mathfrak{v})+\mathfrak{r}(\mathfrak{v})}{2}\right) \leq \frac{\Gamma(\beta+1)}{2\left(\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})\right)^{\beta}}\left[\mathcal{J}^{\beta}_{l(\mathfrak{v})^{+}}\Upsilon\left(\mathfrak{r}(\mathfrak{v})\right) + \mathcal{J}^{\beta}_{\mathfrak{r}(\mathfrak{v})^{-}}\Upsilon\left(l(\mathfrak{v})\right)\right] \leq \frac{\Upsilon\left(l(\mathfrak{v})\right)+\Upsilon\left(\mathfrak{r}(\mathfrak{v})\right)}{2}. \tag{31}$$

If $\Upsilon(\varkappa)$ is type-2 concave real-valued mapping, then

$$\Upsilon\left(\frac{l(\mathfrak{v})+\mathfrak{r}(\mathfrak{v})}{2}\right) \geq \frac{\Gamma(\beta+1)}{2\left(\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})\right)^{\beta}}\left[\mathcal{J}^{\beta}_{l(\mathfrak{v})^{+}}\Upsilon\left(\mathfrak{r}(\mathfrak{v})\right) + \mathcal{J}^{\beta}_{\mathfrak{r}(\mathfrak{v})^{-}}\Upsilon\left(l(\mathfrak{v})\right)\right] \geq \frac{\Upsilon\left(l(\mathfrak{v})\right)+\Upsilon\left(\mathfrak{r}(\mathfrak{v})\right)}{2}. \tag{32}$$

Proof: Let $\Upsilon: \widetilde{\omega} \to \mathbb{R}$ be a convex-like real-valued mapping, . Then, by hypothesis, we have

$$2\Upsilon\left(\frac{l(\mathfrak{v})+\mathfrak{r}(\mathfrak{v})}{2}\right) \leq \Upsilon\left(\varphi l(\mathfrak{v}) + (1-\varphi)\mathfrak{r}(\mathfrak{v})\right) + \Upsilon\left((1-\varphi)l(\mathfrak{v}) + \varphi\mathfrak{r}(\mathfrak{v})\right).$$

Therefore, for every $\mathfrak{v} \in [0, 1]$, we have

$$2\Upsilon\left(\frac{l(\mathfrak{v})+\mathfrak{r}(\mathfrak{v})}{2}\right) \leq \Upsilon\left(\varphi l(\mathfrak{v}) + (1-\varphi)\mathfrak{r}(\mathfrak{v})\right) + \Upsilon\left((1-\varphi)l(\mathfrak{v}) + \varphi\mathfrak{r}(\mathfrak{v})\right),$$

Multiplying both sides by $\varphi^{\beta-1}$ and integrating the obtained result with respect to φ over (0,1), we have

$$2\int_0^1 \varphi^{\beta-1}\Upsilon\left(\frac{l(\mathfrak{v})+\mathfrak{r}(\mathfrak{v})}{2}\right)d\varphi$$
$$\leq \int_0^1 \varphi^{\beta-1}\Upsilon\big(\varphi l(\mathfrak{v})+(1-\varphi)\mathfrak{r}(\mathfrak{v})\big)d\varphi$$
$$+\int_0^1 \varphi^{\beta-1}\Upsilon\big((1-\varphi)l(\mathfrak{v})+\varphi\mathfrak{r}(\mathfrak{v})\big)d\varphi.$$

Let $\varkappa=\varphi l(\mathfrak{v})+(1-\varphi)\mathfrak{r}(\mathfrak{v})$ and $s=(1-\varphi)l(\mathfrak{v})+\varphi\mathfrak{r}(\mathfrak{v})$. Then we have

$$\frac{2}{\beta}\Upsilon\left(\frac{l(\mathfrak{v})+\mathfrak{r}(\mathfrak{v})}{2}\right) \leq \frac{1}{\big(\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})\big)^{\beta}}\int_{l(\mathfrak{v})}^{\mathfrak{r}(\mathfrak{v})}(\mathfrak{r}(\mathfrak{v})-s)^{\beta-1}\Upsilon(s)\,ds$$
$$+\frac{1}{\big(\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})\big)^{\beta}}\int_{l(\mathfrak{v})}^{\mathfrak{r}(\mathfrak{v})}\big(\varkappa-l(\mathfrak{v})\big)^{\beta-1}\Upsilon(\varkappa)\,d\varkappa$$
$$=\frac{\Gamma(\beta)}{\big(\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})\big)^{\beta}}\left[\mathcal{J}^{\beta}_{l(\mathfrak{v})^+}\Upsilon\big(\mathfrak{r}(\mathfrak{v})\big)+\mathcal{J}^{\beta}_{\mathfrak{r}(\mathfrak{v})^-}\Upsilon\big(l(\mathfrak{v})\big)\right]. \qquad (33)$$

For the right part of double inequality (31), considering convex-like real-valued mapping on $\widetilde{\omega}$, then for $\mathfrak{v},\varphi\in[0,1]$, we have

$$\Upsilon\big(\varphi l(\mathfrak{v})+(1-\varphi)\mathfrak{r}(\mathfrak{v})\big)\leq\varphi\Upsilon\big(l(\mathfrak{v})\big)+(1-\varphi)\Upsilon\big(\mathfrak{r}(\mathfrak{v})\big), \qquad (34)$$

and

$$\Upsilon\big((1-\varphi)l(\mathfrak{v})+\varphi\mathfrak{r}(\mathfrak{v})\big)\leq(1-\varphi)\Upsilon\big(l(\mathfrak{v})\big)+\varphi\Upsilon\big(\mathfrak{r}(\mathfrak{v})\big). \qquad (35)$$

By adding (34) and (35), we obtain the following resultant

$$\Upsilon\big(\varphi l(\mathfrak{v})+(1-\varphi)\mathfrak{r}(\mathfrak{v})\big)+\Upsilon\big((1-\varphi)l(\mathfrak{v})+\varphi\mathfrak{r}(\mathfrak{v})\big)$$
$$\leq\Upsilon\big(l(\mathfrak{v})\big)+\Upsilon\big(\mathfrak{r}(\mathfrak{v})\big). \qquad (36)$$

Taking multiplication of (36) inequality with $\varphi^{\beta-1}$, then integrating the resultant over $[0,1]$, we have

$$\int_0^1\varphi^{\beta-1}\Upsilon\big(\varphi l(\mathfrak{v})+(1-\varphi)\mathfrak{r}(\mathfrak{v})\big)+\int_0^1\varphi^{\beta-1}\Upsilon\big((1-\varphi)l(\mathfrak{v})+\varphi\mathfrak{r}(\mathfrak{v})\big)$$
$$\leq\int_0^1\varphi^{\beta-1}\Upsilon\big(l(\mathfrak{v})\big)+\int_0^1\varphi^{\beta-1}\Upsilon\big(\mathfrak{r}(\mathfrak{v})\big).$$

Again taking substitution, $\varkappa=\varphi l(\mathfrak{v})+(1-\varphi)\mathfrak{r}(\mathfrak{v})$ and $s=(1-\varphi)l(\mathfrak{v})+\varphi\mathfrak{r}(\mathfrak{v})$, then by simple calculation we obtain

$$\frac{\Gamma(\beta)}{\big(\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})\big)^{\beta}}\left[\mathcal{J}^{\beta}_{l(\mathfrak{v})^+}\Upsilon\big(\mathfrak{r}(\mathfrak{v})\big)+\mathcal{J}^{\beta}_{\mathfrak{r}(\mathfrak{v})^-}\Upsilon\big(l(\mathfrak{v})\big)\right]\leq\frac{\Upsilon\big(l(\mathfrak{v})\big)+\Upsilon\big(\mathfrak{r}(\mathfrak{v})\big)}{\beta}. \qquad (37)$$

Combining (33) and (37), we have

$$\Upsilon\left(\frac{l(\mathfrak{v})+\mathfrak{r}(\mathfrak{v})}{2}\right)\leq\frac{\Gamma(\beta+1)}{2\big(\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})\big)^{\beta}}\left[\mathcal{J}^{\beta}_{l(\mathfrak{v})^+}\Upsilon\big(\mathfrak{r}(\mathfrak{v})\big)+\mathcal{J}^{\beta}_{\mathfrak{r}(\mathfrak{v})^-}\Upsilon\big(l(\mathfrak{v})\big)\right]$$
$$\leq\frac{\Upsilon\big(l(\mathfrak{v})\big)+\Upsilon\big(\mathfrak{r}(\mathfrak{v})\big)}{2}.$$

Hence, the required result.

Particular Cases

Here some of the exceptional cases have been discussed which depend upon the triangular fuzzy number and trapezoidal fuzzy number.

Firstly, taking triangular fuzzy number such that

$$\left[\widetilde{\omega}\right]^{\mathfrak{v}}=[r-\lambda(1-\mathfrak{v}),r+\gamma(1-\mathfrak{v})], \qquad (38)$$

then inequality (31) reduces to the Riemann-Liouville fractional-like integral $\mathcal{H}.\mathcal{H}$-type inequalities over triangular fuzzy number $\widetilde{\omega}$ such that

$$\Upsilon\left(\frac{2r+(\gamma-\lambda)(1-\mathfrak{v})}{2}\right)\leq\frac{\Gamma(\beta+1)}{2\big((\gamma+\lambda)(1-\mathfrak{v})\big)^{\beta}}\left[\mathcal{J}^{\beta}_{(r-\lambda(1-\mathfrak{v}))^+}\Upsilon\big(r+\gamma(1-\mathfrak{v})\big)+\mathcal{J}^{\beta}_{(r+\gamma(1-\mathfrak{v}))^-}\Upsilon\big(r-\lambda(1-\mathfrak{v})\big)\right]$$
$$\leq\frac{\Upsilon\big(r-\lambda(1-\mathfrak{v})\big)+\Upsilon\big(r+\gamma(1-\mathfrak{v})\big)}{2}. \qquad (39)$$

Secondly, taking trapezoidal fuzzy number such that

$$\left[\widetilde{\omega}\right]^{\mathfrak{v}}=[r-\lambda(1-\mathfrak{v}),s+\gamma(1-\mathfrak{v})], \qquad (40)$$

then inequality (31) reduces to the Riemann-Liouville fractional-like integral $\mathcal{H}.\mathcal{H}$-type inequalities over trapezoidal fuzzy number $\widetilde{\omega}$ such that

$$\Upsilon\left(\frac{r+s+(\gamma-\lambda)(1-\mathfrak{v})}{2}\right)$$
$$\leq \frac{\Gamma(\beta+1)}{2\left(s-r+(\gamma+\lambda)(1-\mathfrak{v})\right)^{\beta}}\left[\mathcal{J}^{\beta}_{\left(r-\lambda(1-\mathfrak{v})\right)^{+}}\Upsilon\big(s+\gamma(1-\mathfrak{v})\big)+\mathcal{J}^{\beta}_{\left(s+\gamma(1-\mathfrak{v})\right)^{-}}\Upsilon\big(r-\lambda(1-\mathfrak{v})\big)\right]$$
$$\leq \frac{\Upsilon\big(r-\lambda(1-\mathfrak{v})\big)+\Upsilon\big(s+\gamma(1-\mathfrak{v})\big)}{2}. \quad (41)$$

Note that, if $s=r$, then both double inequalities (39) and (41) coincides.
Remark 3: From Theorem 2. we clearly see that
Let $\beta=1$. Then, inequality (31) reduces to the following inequality which is also new one:

$$\Upsilon\left(\frac{l(\mathfrak{v})+\mathfrak{r}(\mathfrak{v})}{2}\right)\leq \frac{1}{\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})}\int_{l(\mathfrak{v})}^{\mathfrak{r}(\mathfrak{v})}\Upsilon(\varkappa)d\varkappa \leq \frac{\Upsilon(l(\mathfrak{v}))\oplus\Upsilon(\mathfrak{r}(\mathfrak{v}))}{2}. \quad (42)$$

If $\widetilde{\omega}=\widetilde{[\tau,\varsigma]}$, then from (31), we get following classical fractional $\mathcal{H}.\mathcal{H}$-inequality.

Let $\widetilde{\omega}=\widetilde{[\boldsymbol{\tau},\boldsymbol{\varsigma}]}$. Then inequality (31) reduces to the classical fractional $\boldsymbol{\mathcal{H}}.\boldsymbol{\mathcal{H}}$-inequality, see [43].

Let $\boldsymbol{\beta}=\mathbf{1}$ and $\widetilde{\omega}=\widetilde{[\boldsymbol{\tau},\boldsymbol{\varsigma}]}$. Then from Theorem 2, we achieve the classical $\boldsymbol{\mathcal{H}}.\boldsymbol{\mathcal{H}}$-inequality (1).

Derivative of real valued function over fuzzy domain

Here we have proposed the notion of derivative of real valued function over fuzzy domain, and discussed its application in next Theorem 2 and Example 1.

Definition 8: Let $\widetilde{\omega}\in\mathbb{E}_C$, and $\Upsilon:[\tau,\varsigma]\subseteq\mathbb{R}\to\mathbb{R}$ is said to be on $[\widetilde{\omega}]^0\subseteq[\tau,\varsigma]$ (and hence for each $[\widetilde{\omega}]^{\mathfrak{v}}$, for all $\mathfrak{v}\in[0,1]$), and $t_0\in[\widetilde{\omega}]^{\mathfrak{v}}$. We define derivative of Υ, $\Upsilon'(t_0)\in\mathbb{R}$ (provided it exists) as

$$\left(\Upsilon'_{t_0\in\widetilde{\omega}}\right)(\mathfrak{v})=\lim_{h\to 0^-}\frac{\Upsilon(t_0+h)-\Upsilon(t_0)}{h}=\Upsilon'_{t_0\in[\widetilde{\omega}]^{\mathfrak{v}}}(t_0), \quad (43)$$

We call $\Upsilon'_{t_0\in[\widetilde{\omega}]^{\mathfrak{v}}}(t_0)$ the derivative of Υ at $t_0\in[\widetilde{\omega}]^{\mathfrak{v}}$. Also, we define the left derivative $^{-}\Upsilon'(t_0)\in\mathbb{R}$ (provided it exists) as

$$^{-}\left(\Upsilon'_{t_0\in\widetilde{\omega}}\right)(\mathfrak{v})=\lim_{h\to 0^-}\frac{\Upsilon(t_0+h)-\Upsilon(t_0)}{h}={}^{-}\Upsilon'_{t_0\in[\widetilde{\omega}]^{\mathfrak{v}}}(t_0). \quad (44)$$

and the right derivative $^{+}\Upsilon'(t_0)\in\mathbb{R}$ (provided it exists) as

$$^{+}\left(\Upsilon'_{t_0\in\widetilde{\omega}}\right)(\mathfrak{v})=\lim_{h\to 0^+}\frac{\Upsilon(t_0+h)-\Upsilon(t_0)}{h}={}^{+}\Upsilon'_{t_0\in[\widetilde{\omega}]^{\mathfrak{v}}}(t_0). \quad (45)$$

We say that Υ is differentiable on $[\widetilde{\omega}]$ if it is differentiable at each fuzzy point on $[\widetilde{\omega}]$. At the end points of $[\widetilde{\omega}]$, we only consider the one sided derivative.
where the integral on the right-hand side is calculated in the sense of Lebesgue. We say the Υ is integrable over fuzzy domain , if the derivative $\Upsilon'_{t_0\in[\widetilde{\omega}]^{\mathfrak{v}}}(t_0)$ is finite. In that case, mapping is defined as

$$\Upsilon'_{(\widetilde{\omega})}:[0,1]\to\mathbb{R}$$

$$\boldsymbol{\mathfrak{v}}\to\left(\boldsymbol{\Upsilon}'_{(\widetilde{\omega})}\right)(\boldsymbol{\mathfrak{v}})=\boldsymbol{\Upsilon}'_{[\widetilde{\omega}]^{\mathfrak{v}}}(\boldsymbol{\Upsilon}(\boldsymbol{\varphi})).$$

Now we need a following Lemma which will be helpful to prove the upcoming result.
Lemma 1: Let $\Upsilon:\widetilde{\omega}\to\mathbb{R}$ be a real-valued mapping on $\widetilde{\omega}$, whose parametrized form is $[\widetilde{\omega}]^{\mathfrak{v}}=[l(\mathfrak{v}),\mathfrak{r}(\mathfrak{v})]$, for all $\mathfrak{v}\in[0,1]$. If Υ is differentiable on $\big(l(\mathfrak{v}),\mathfrak{r}(\mathfrak{v})\big)$ and $\Upsilon\in L(\widetilde{\omega})$, then the following inequality hold for Riemann-Liouville fractional-like integrals:

$$\frac{\boldsymbol{\Upsilon}(\boldsymbol{l}(\boldsymbol{\mathfrak{v}}))+\boldsymbol{\Upsilon}(\mathfrak{r}(\boldsymbol{\mathfrak{v}}))}{2}-\frac{\boldsymbol{\Gamma}(\boldsymbol{\beta}+\mathbf{1})}{2\big(\mathfrak{r}(\boldsymbol{\mathfrak{v}})-\boldsymbol{l}(\boldsymbol{\mathfrak{v}})\big)^{\boldsymbol{\beta}}}\left[\boldsymbol{\mathcal{J}}^{\boldsymbol{\beta}}_{\boldsymbol{l}(\boldsymbol{\mathfrak{v}})^{+}}\boldsymbol{\Upsilon}\big(\mathfrak{r}(\boldsymbol{\mathfrak{v}})\big)+\boldsymbol{\mathcal{J}}^{\boldsymbol{\beta}}_{\mathfrak{r}(\boldsymbol{\mathfrak{v}})^{-}}\boldsymbol{\Upsilon}\big(\boldsymbol{l}(\boldsymbol{\mathfrak{v}})\big)\right]$$

$$=\frac{\mathfrak{r}(\boldsymbol{\mathfrak{v}})-\boldsymbol{l}(\boldsymbol{\mathfrak{v}})}{2}\int_{\mathbf{0}}^{\mathbf{1}}\left[(\mathbf{1}-\boldsymbol{\varphi})^{\boldsymbol{\beta}}-\boldsymbol{\varphi}^{\boldsymbol{\beta}}\right]\boldsymbol{\Upsilon}'\big(\boldsymbol{\varphi}\boldsymbol{l}(\boldsymbol{\mathfrak{v}})+(\mathbf{1}-\boldsymbol{\varphi})\mathfrak{r}(\boldsymbol{\mathfrak{v}})\big)\boldsymbol{d}\boldsymbol{\varphi}. \quad (46)$$

Proof: From the right part of (43), just we have taken,

$$L = \int_0^1 [(1-\varphi)^\beta - \varphi^\beta] \Upsilon'(\varphi l(\mathfrak{v}) + (1-\varphi)\mathfrak{r}(\mathfrak{v}))$$
$$= \frac{\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})}{2} \int_0^1 (1-\varphi)^\beta \Upsilon'(\varphi l(\mathfrak{v}) + (1-\varphi)\mathfrak{r}(\mathfrak{v}))d\varphi - \frac{\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})}{2} \int_0^1 \varphi^\beta \Upsilon'(\varphi l(\mathfrak{v}) + (1-\varphi)\mathfrak{r}(\mathfrak{v}))d\varphi$$
$$= L_1 + L_2. \tag{47}$$

Using the rule integration by parts, we have

$$L_1 = \frac{\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})}{2} \int_0^1 (1-\varphi)^\beta \Upsilon'(\varphi l(\mathfrak{v}) + (1-\varphi)\mathfrak{r}(\mathfrak{v}))d\varphi$$
$$= \frac{\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})}{2} \left[(1-\varphi)^\beta \frac{\Upsilon(\varphi l(\mathfrak{v})+(1-\varphi)\mathfrak{r}(\mathfrak{v}))}{l(\mathfrak{v})-\mathfrak{r}(\mathfrak{v})} \Big|_0^1 + \frac{\beta}{l(\mathfrak{v})-\mathfrak{r}(\mathfrak{v})} \int_0^1 (1-\varphi)^{\beta-1} \Upsilon(\varphi l(\mathfrak{v}) + (1-\varphi)\mathfrak{r}(\mathfrak{v}))d\varphi\right]$$
$$= \frac{\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})}{2} \left[\frac{\Upsilon(\mathfrak{r}(\mathfrak{v}))}{l(\mathfrak{v})-\mathfrak{r}(\mathfrak{v})} + \frac{\beta}{l(\mathfrak{v})-\mathfrak{r}(\mathfrak{v})} \int_0^1 (1-\varphi)^{\beta-1} \Upsilon(\varphi l(\mathfrak{v}) + (1-\varphi)\mathfrak{r}(\mathfrak{v}))d\varphi\right]$$

taking $\varkappa = \varphi l(\mathfrak{v}) + (1-\varphi)\mathfrak{r}(\mathfrak{v})$, we have

$$L_1 = \frac{\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})}{2} \left[\frac{\Upsilon(\mathfrak{r}(\mathfrak{v}))}{l(\mathfrak{v})-\mathfrak{r}(\mathfrak{v})} - \frac{\Gamma(\beta+1)}{(\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v}))^2} \frac{1}{\Gamma(\beta)} \int_{\mathfrak{r}(\mathfrak{v})}^{l(\mathfrak{v})} \left(\frac{l(\mathfrak{v})-\varkappa}{l(\mathfrak{v})-\mathfrak{r}(\mathfrak{v})}\right)^{\beta-1} \Upsilon(\varkappa)d\varkappa\right]$$
$$= \frac{\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})}{2} \left[\frac{\Upsilon(\mathfrak{r}(\mathfrak{v}))}{l(\mathfrak{v})-\mathfrak{r}(\mathfrak{v})} - \frac{\Gamma(\beta+1)}{(\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v}))^{\beta+1}} \mathcal{J}^\beta_{\mathfrak{r}(\mathfrak{v})^-} \Upsilon(l(\mathfrak{v}))\right]. \tag{48}$$

Now calculating L_2, we have

$$L_2 = -\frac{\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})}{2} \int_0^1 \varphi^\beta \Upsilon'(\varphi l(\mathfrak{v}) + (1-\varphi)\mathfrak{r}(\mathfrak{v}))d\varphi$$
$$= -\frac{\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})}{2} \left[\varphi^\beta \frac{\Upsilon(\varphi l(\mathfrak{v})+(1-\varphi)\mathfrak{r}(\mathfrak{v}))}{l(\mathfrak{v})-\mathfrak{r}(\mathfrak{v})} \Big|_0^1 - \frac{\beta}{l(\mathfrak{v})-\mathfrak{r}(\mathfrak{v})} \int_0^1 \varphi^{\beta-1} \Upsilon(\varphi l(\mathfrak{v}) + (1-\varphi)\mathfrak{r}(\mathfrak{v}))d\varphi\right]$$
$$= \frac{\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})}{2} \left[\frac{\Upsilon(l(\mathfrak{v}))}{l(\mathfrak{v})-\mathfrak{r}(\mathfrak{v})} + \frac{\beta}{l(\mathfrak{v})-\mathfrak{r}(\mathfrak{v})} \int_0^1 \varphi^{\beta-1} \Upsilon(\varphi l(\mathfrak{v}) + (1-\varphi)\mathfrak{r}(\mathfrak{v}))d\varphi\right]$$

again taking $\varkappa = \varphi l(\mathfrak{v}) + (1-\varphi)\mathfrak{r}(\mathfrak{v})$, we have

$$L_2 = \frac{\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})}{2} \left[\frac{\Upsilon(l(\mathfrak{v}))}{l(\mathfrak{v})-\mathfrak{r}(\mathfrak{v})} - \frac{\Gamma(\beta+1)}{(\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v}))^2} \frac{1}{\Gamma(\beta)} \int_{\mathfrak{r}(\mathfrak{v})}^{l(\mathfrak{v})} \left(\frac{\mathfrak{r}(\mathfrak{v})-\varkappa}{\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})}\right)^{\beta-1} \Upsilon(\varkappa)d\varkappa\right]$$
$$= \frac{\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})}{2} \left[\frac{\Upsilon(l(\mathfrak{v}))}{l(\mathfrak{v})-\mathfrak{r}(\mathfrak{v})} - \frac{\Gamma(\beta+1)}{(\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v}))^{\beta+1}} \mathcal{J}^\beta_{\mathfrak{r}(\mathfrak{v})^-} \Upsilon(l(\mathfrak{v}))\right], \tag{49}$$

From (48) and (49), (47) we have

$$\frac{\Upsilon(l(\mathfrak{v}))+\Upsilon(\mathfrak{r}(\mathfrak{v}))}{2} - \frac{\Gamma(\beta+1)}{2(\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v}))^\beta} \left[\mathcal{J}^\beta_{l(\mathfrak{v})^+} \Upsilon(\mathfrak{r}(\mathfrak{v})) + \mathcal{J}^\beta_{\mathfrak{r}(\mathfrak{v})^-} \Upsilon(l(\mathfrak{v}))\right]$$
$$= \frac{\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})}{2} \int_0^1 [(1-\varphi)^\beta - \varphi^\beta] \Upsilon'(\varphi l(\mathfrak{v}) + (1-\varphi)\mathfrak{r}(\mathfrak{v}))d\varphi.$$

Hence the required result.

Here is new exceptional case of Lemma 3.1, which is also new one.

Remark 4: If in Lemma 1, one takes $\beta = 1$, then on can obtain following inequality:

$$\frac{\Upsilon(l(\mathfrak{v}))+\Upsilon(\mathfrak{r}(\mathfrak{v}))}{2} - \frac{1}{\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})} \int_{l(\mathfrak{v})}^{\mathfrak{r}(\mathfrak{v})} \Upsilon(\varkappa)d\varkappa$$
$$= \frac{\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})}{2} \int_0^1 (1-2\varphi) \Upsilon'(\varphi l(\mathfrak{v}) + (1-\varphi)\mathfrak{r}(\mathfrak{v}))d\varphi. \tag{50}$$

Theorem 3: Let $\Upsilon: \widetilde{\omega} \to \mathbb{R}$ be a real-valued mapping on $\widetilde{\omega}$, whose parametrized form is $[\widetilde{\omega}]^{\mathfrak{v}} = [l(\mathfrak{v}), \mathfrak{r}(\mathfrak{v})]$, for all $\mathfrak{v} \in [0,1]$. Let Υ is differentiable on $(l(\mathfrak{v}), \mathfrak{r}(\mathfrak{v}))$ with $\mathfrak{r}(\mathfrak{v}) > l(\mathfrak{v})$. If $|\Upsilon'|$ is typ-2 convex mapping on $\widetilde{\omega}$, then the following inequality hold for Riemann-Liouville fractional-like integrals:

$$\left|\frac{\Upsilon(l(\mathfrak{v}))+\Upsilon(\mathfrak{r}(\mathfrak{v}))}{2} - \frac{\Gamma(\beta+1)}{2(\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v}))^\beta} \left[\mathcal{J}^\beta_{l(\mathfrak{v})^+} \Upsilon(\mathfrak{r}(\mathfrak{v})) + \mathcal{J}^\beta_{\mathfrak{r}(\mathfrak{v})^-} \Upsilon(l(\mathfrak{v}))\right]\right|$$
$$\leq \frac{\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})}{2(\beta+1)} \left(1 - \frac{1}{2^\beta}\right) [\Upsilon'(l(\mathfrak{v})) + \Upsilon'(\mathfrak{r}(\mathfrak{v}))]. \tag{51}$$

Proof: By using Lemma 1, we have

$$\left|\frac{\Upsilon(l(\mathfrak{v}))+\Upsilon(\mathfrak{r}(\mathfrak{v}))}{2}-\frac{\Gamma(\beta+1)}{2(\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v}))^{\beta}}\left[\mathcal{J}^{\beta}_{l(\mathfrak{v})^{+}}\Upsilon(\mathfrak{r}(\mathfrak{v}))+\mathcal{J}^{\beta}_{\mathfrak{r}(\mathfrak{v})^{-}}\Upsilon(l(\mathfrak{v}))\right]\right|$$
$$=\left|\frac{\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})}{2}\int_0^1[(1-\varphi)^{\beta}-\varphi^{\beta}]\Upsilon'(\varphi l(\mathfrak{v})+(1-\varphi)\mathfrak{r}(\mathfrak{v}))d\varphi\right|$$
$$\leq\frac{\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})}{2}\int_0^1|(1-\varphi)^{\beta}-\varphi^{\beta}||\Upsilon'(\varphi l(\mathfrak{v})+(1-\varphi)\mathfrak{r}(\mathfrak{v}))|d\varphi.$$

Since $|\Upsilon'|$ is typ-2 convex mapping on $\widetilde{\omega}$, then we find

$$\left|\frac{\Upsilon(l(\mathfrak{v}))+\Upsilon(\mathfrak{r}(\mathfrak{v}))}{2}-\frac{\Gamma(\beta+1)}{2(\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v}))^{\beta}}\left[\mathcal{J}^{\beta}_{l(\mathfrak{v})^{+}}\Upsilon(\mathfrak{r}(\mathfrak{v}))+\mathcal{J}^{\beta}_{\mathfrak{r}(\mathfrak{v})^{-}}\Upsilon(l(\mathfrak{v}))\right]\right|$$
$$\leq\frac{\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})}{2}\int_0^1|(1-\varphi)^{\beta}-\varphi^{\beta}|[\varphi|\Upsilon'(l(\mathfrak{v}))|+(1-\varphi)|\Upsilon'(\mathfrak{r}(\mathfrak{v}))|]d\varphi$$
$$=\frac{\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})}{2}\int_0^{\frac{1}{2}}[(1-\varphi)^{\beta}-\varphi^{\beta}][\varphi|\Upsilon'(l(\mathfrak{v}))|+(1-\varphi)|\Upsilon'(\mathfrak{r}(\mathfrak{v}))|]d\varphi$$
$$+\frac{\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})}{2}\int_{\frac{1}{2}}^{1}[\varphi^{\beta}-(1-\varphi)^{\beta}][\varphi|\Upsilon'(l(\mathfrak{v}))|+(1-\varphi)|\Upsilon'(\mathfrak{r}(\mathfrak{v}))|]d\varphi$$
$$=\frac{\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})}{2}\left[|\Upsilon'(l(\mathfrak{v}))|\int_0^{\frac{1}{2}}[\varphi(1-\varphi)^{\beta}-\varphi^{\beta+1}]d\varphi+|\Upsilon'(\mathfrak{r}(\mathfrak{v}))|\int_0^{\frac{1}{2}}[(1-\varphi)^{\beta+1}-\varphi^{\beta}(1-\varphi)]d\varphi\right]$$
$$+\frac{\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})}{2}\left[|\Upsilon'(l(\mathfrak{v}))|\int_{\frac{1}{2}}^{1}[\varphi^{\beta+1}-\varphi(1-\varphi)^{\beta}]d\varphi+|\Upsilon'(\mathfrak{r}(\mathfrak{v}))|\int_{\frac{1}{2}}^{1}[\varphi^{\beta}(1-\varphi)-(1-\varphi)^{\beta+1}]d\varphi\right]$$
$$=\frac{\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})}{2}\left[|\Upsilon'(l(\mathfrak{v}))|\left[\frac{1}{(\beta+1)(\beta+2)}-\frac{\left(\frac{1}{2}\right)^{\beta+1}}{\beta+1}\right]+|\Upsilon'(\mathfrak{r}(\mathfrak{v}))|\left[\frac{1}{\beta+2}-\frac{\left(\frac{1}{2}\right)^{\beta+1}}{\beta+1}\right]\right]$$
$$+\frac{\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})}{2}\left[|\Upsilon'(l(\mathfrak{v}))|\left[\frac{1}{\beta+2}-\frac{\left(\frac{1}{2}\right)^{\beta+1}}{\beta+1}\right]+|\Upsilon'(\mathfrak{r}(\mathfrak{v}))|\left[\frac{1}{(\beta+1)(\beta+2)}-\frac{\left(\frac{1}{2}\right)^{\beta+1}}{\beta+1}\right]\right].$$

by simple simplification, above inequality reduces to the inequality (44). Hence, the result has been proven.

Particular Cases

Here some of the exceptional cases have been discussed which depend upon the triangular fuzzy number and trapezoidal fuzzy number.

Firstly, taking triangular fuzzy number such that

$$[\widetilde{\omega}]^{\mathfrak{v}}=[r-\lambda(1-\mathfrak{v}),r+\gamma(1-\mathfrak{v})],$$

then inequality (50) reduces to the Riemann-Liouville fractional-like integral $\mathcal{H}.\mathcal{H}$-type inequalities over triangular fuzzy number $\widetilde{\omega}$ such that

$$\left|\frac{\Upsilon(r-\lambda(1-\mathfrak{v}))+\Upsilon(r+\gamma(1-\mathfrak{v}))}{2}-\frac{\Gamma(\beta+1)}{2((\gamma+\lambda)(1-\mathfrak{v}))^{\beta}}\left[\mathcal{J}^{\beta}_{(r-\lambda(1-\mathfrak{v}))^{+}}\Upsilon(r+\gamma(1-\mathfrak{v}))+\mathcal{J}^{\beta}_{(r+\gamma(1-\mathfrak{v}))^{-}}\Upsilon(r-\lambda(1-\mathfrak{v}))\right]\right|$$
$$\leq\frac{((\gamma+\lambda)(1-\mathfrak{v}))}{2(\beta+1)}\left(1-\frac{1}{2^{\beta}}\right)[\Upsilon'(r-\lambda(1-\mathfrak{v}))+\Upsilon'(r+\gamma(1-\mathfrak{v}))].\qquad(52)$$

Secondly, taking trapezoidal fuzzy number such that

$$[\widetilde{\omega}]^{\mathfrak{v}}=[r-\lambda(1-\mathfrak{v}),s+\gamma(1-\mathfrak{v})],$$

then inequality (50) reduces to the Riemann-Liouville fractional-like integral $\mathcal{H}.\mathcal{H}$-type inequalities over trapezoidal fuzzy number $\widetilde{\omega}$ such that

$$\left|\frac{\Upsilon(r-\lambda(1-\mathfrak{v}))+\Upsilon(s+\gamma(1-\mathfrak{v}))}{2}-\frac{\Gamma(\beta+1)}{2(s-r+(\gamma+\lambda)(1-\mathfrak{v}))^{\beta}}\left[\mathcal{J}^{\beta}_{(r-\lambda(1-\mathfrak{v}))^{+}}\Upsilon(s+\gamma(1-\mathfrak{v}))+\mathcal{J}^{\beta}_{(s+\gamma(1-\mathfrak{v}))^{-}}\Upsilon(r-\lambda(1-\mathfrak{v}))\right]\right|$$
$$\leq\frac{s-r+(\gamma+\lambda)(1-\mathfrak{v})}{2(\beta+1)}\left(1-\frac{1}{2^{\beta}}\right)[\Upsilon'(r-\lambda(1-\mathfrak{v}))+\Upsilon'(s+\gamma(1-\mathfrak{v}))].\qquad(53)$$

Note that, If $s=r$, then both double inequalities (52) and (53) coincides.

Remark 5: If $\beta = 1$, then from Theorem 3, we obtain the following outcome which is also new one:

$$\left|\frac{\Upsilon(l(\mathfrak{v}))+\Upsilon(\gamma(\mathfrak{v}))}{2} - \frac{1}{\gamma(\mathfrak{v})-l(\mathfrak{v})}\int_{l(\mathfrak{v})}^{\gamma(\mathfrak{v})} \Upsilon(\varkappa)d\varkappa\right|$$
$$\leq \frac{\gamma(\mathfrak{v})-l(\mathfrak{v})}{8}\left[\Upsilon'\big(l(\mathfrak{v})\big) + \Upsilon'\big(\gamma(\mathfrak{v})\big)\right]. \quad (54)$$

Let one takes $\boldsymbol{\beta = 1}$ and $\widetilde{\omega} = \widetilde{[\tau,\varsigma]}$. Then from Remark 1 and Theorem 3, we obtain the classical inequality, see [65]:

$$\left|\frac{\Upsilon(\tau)+\Upsilon(\varsigma)}{2} - \frac{1}{\varsigma-\tau}\int_{\tau}^{\varsigma} \Upsilon(\varkappa)d\varkappa\right| \leq \frac{\varsigma-\tau}{8}[\Upsilon'(\tau) + \Upsilon'(\varsigma)]. \quad (55)$$

Example 1: Consider the trapezoidal fuzzy numbers $\widetilde{\omega} = \left(1,2;\frac{1}{2},2\right)$, that is

$$\widetilde{\omega}(\varkappa) = \begin{cases} 1, & \omega \in [1,2] \\ \frac{\omega-\frac{1}{2}}{\lambda}, & \omega \in \left[1-\frac{1}{2},1\right] \\ \frac{4-\omega}{2}, & \omega \in [2,2+2] \\ 0, & \text{otherwise,} \end{cases} \quad (56)$$

whose parametrized form is $[\widetilde{\omega}]^{\mathfrak{v}} = \left[1-\frac{1}{2}(1-\mathfrak{v}), 2+2(1-\mathfrak{v})\right]$, for all $\mathfrak{v} \in [0,1]$. Let $\beta = \frac{1}{2}$, and $\Upsilon(\varkappa) = \varkappa^2$ be a real-valued mapping on fuzzy domain $[\widetilde{\omega}]^{\mathfrak{v}} = \left[1-\frac{1}{2}(1-\mathfrak{v}), 2+2(1-\mathfrak{v})\right]$. Then, it can be easily seen that $\Upsilon(\varkappa) = \varkappa^2$ is a convex-like real-valued mapping, for all $\mathfrak{v} \in [0,1]$. Now we compute the following

$$\Upsilon\left(\frac{l(\mathfrak{v})+\gamma(\mathfrak{v})}{2}\right) = \Upsilon\left(\frac{6+3(1-\mathfrak{v})}{4}\right) = \frac{1}{16}\big(6+3(1-\mathfrak{v})\big)^2.$$

$$\frac{\Gamma(\beta+1)}{2\big(\gamma(\mathfrak{v})-l(\mathfrak{v})\big)^{\beta}}\left[\mathcal{J}^{\beta}_{l(\mathfrak{v})^+}\Upsilon\big(\gamma(\mathfrak{v})\big) + \mathcal{J}^{\beta}_{\gamma(\mathfrak{v})^-}\Upsilon\big(l(\mathfrak{v})\big)\right]$$
$$= \frac{\sqrt{\pi}}{4\left(3+\frac{5}{2}(1-\mathfrak{v})\right)^{\frac{1}{2}}}\frac{1}{\sqrt{\pi}}\int_{1-\frac{1}{2}(1-\mathfrak{v})}^{2+2(1-\mathfrak{v})}\big(2+2(1-\mathfrak{v})-\varkappa\big)^{\frac{-1}{2}}.\varkappa^2 d\varkappa$$
$$+ \frac{\sqrt{\pi}}{4\left(3+\frac{5}{2}(1-\mathfrak{v})\right)^{\frac{1}{2}}}\frac{1}{\sqrt{\pi}}\int_{1-\frac{1}{2}(1-\mathfrak{v})}^{2+2(1-\mathfrak{v})}\left(\varkappa-1+\frac{1}{2}(1-\mathfrak{v})\right)^{\frac{-1}{2}}.\varkappa^2 d\varkappa$$
$$= \frac{(\sqrt{7-5\mathfrak{v}})(155\mathfrak{v}^2-650\mathfrak{v}+779)}{120\sqrt{2}\left(1+\frac{5}{2}(1-\mathfrak{v})\right)^{\frac{1}{2}}}$$

$$\frac{\Upsilon\big(l(\mathfrak{v})\big)+\Upsilon\big(\gamma(\mathfrak{v})\big)}{2} = \frac{\left(\frac{2-(1-\mathfrak{v})}{2}\right)^2+\big(2+2(1-\mathfrak{v})\big)^2}{2}$$
$$= \frac{1}{8}[4(2(1-\mathfrak{v})+2)^2 + (1+\mathfrak{v})^2],$$

that is

$$\frac{1}{16}\big(6+3(1-\mathfrak{v})\big)^2 \leq \frac{(\sqrt{7-5\mathfrak{v}})(155\mathfrak{v}^2-650\mathfrak{v}+779)}{120\sqrt{2}\left(1+\frac{5}{2}(1-\mathfrak{v})\right)^{\frac{1}{2}}} \leq \frac{1}{8}[8(2(1-\mathfrak{v})+2)^2 + (1+\mathfrak{v})^2],$$

for all $\mathfrak{v} \in [0,1]$.
Hence Theorem 2 has been verified.
For Theorem 3, we have

$$\left|\frac{\Upsilon\big(l(\mathfrak{v})\big)+\Upsilon\big(\gamma(\mathfrak{v})\big)}{2} - \frac{\Gamma(\beta+1)}{2\big(\gamma(\mathfrak{v})-l(\mathfrak{v})\big)^{\beta}}\left[\mathcal{J}^{\beta}_{l(\mathfrak{v})^+}\Upsilon\big(\gamma(\mathfrak{v})\big) + \mathcal{J}^{\beta}_{\gamma(\mathfrak{v})^-}\Upsilon\big(l(\mathfrak{v})\big)\right]\right|$$

$$= \left|\frac{1}{8}[4(2(1-\mathfrak{v})+2)^2 + (1+\mathfrak{v})^2] - \frac{(\sqrt{7-5\mathfrak{v}})(155\mathfrak{v}^2-650\mathfrak{v}+779)}{120\sqrt{2}\left(1+\frac{5}{2}(1-\mathfrak{v})\right)^{\frac{1}{2}}}\right|,$$

$$\frac{\mathfrak{r}(\mathfrak{v})-l(\mathfrak{v})}{2(\beta+1)}\left(1-\frac{1}{2^{\beta}}\right)\left[\Upsilon'\left(l(\mathfrak{v})\right)+\Upsilon'\left(\mathfrak{r}(\mathfrak{v})\right)\right]$$
$$=\frac{1}{12}\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)[4(2(1-\mathfrak{v})+2)^2+(1+\mathfrak{v})^2][6-3(1-\mathfrak{v})],$$

$$\left|\frac{1}{8}[4(2(1-\mathfrak{v})+2)^2+(1+\mathfrak{v})^2]-\frac{(\sqrt{7-5\mathfrak{v}})(155\mathfrak{v}^2-650\mathfrak{v}+779)}{120\sqrt{2}\left(1+\frac{5}{2}(1-\mathfrak{v})\right)^{\frac{1}{2}}}\right|$$
$$\leq\frac{1}{12}\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)[4(2(1-\mathfrak{v})+2)^2+(1+\mathfrak{v})^2][6-3(1-\mathfrak{v})],$$

for all $\mathfrak{v}\in[0,1]$.
Hence, Theorem 3 has been verified.

V. Conclusion

In order to calculate the newly defined Riemann-Liouville fractional-like integrals for scalar-valued functions over a fuzzy spatial domain, we have discussed some concepts and examples. These concepts can be used to estimate the size of a region satisfying a particular property, and we have verified that a comparable idea can be defined for a variety of fuzzy numbers. This fact has implications for estimating magnitudes for fuzzily known field functions. Moreover, some applications of Riemann-Liouville fractional-like integral operators are also discussed in the field of inequalities and validate it with nontrivial examples. In future, we will try to explore these concepts for interval-valued and fuzzy-number-valued mappings. Moreover, we will discuss some different types of fractional integral over fuzzy region. Contrarily, the idea of a convex body's width-integral enables a significant application of Beckenbach's inequality to convex geometry on $\mathbb{R}^n$ (see [81]). We would like to apply these concepts in this context to the investigation of a few issues pertaining to convex geometry in the space $\mathbb{E}_C$, a subject that will be the focus of future research.

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JLGG stated model and solved, M.B.K. validated model and presents the results.