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On the periodic solutions of a perturbed spatial quantized Hill's problem

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Abstract: In this work, the differences and similarities among some perturbation approaches such as classical perturbation theory, Lindstedt–Poincaré technique, multiple scales method, KB averaging method and, averaging theory are investigated. The necessary conditions to construct the periodic solutions of the spatial quantized Hill's problem are found. In this context, the periodic solutions emerging from the equilibrium points of the spatial Hill's problem are evaluated by using the averaging theory, under the perturbation effect of quantum corrections. This model can be used to develop a Lunar theory and the families of periodic orbits in the frame work of the spatial quantized Hill's problem. Thereby, these applications serve to reinforce the obtained results on these periodic solutions and gain its own significance.

Keywords: Quantized Hill problem; Averaging theory; Periodic solution

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1. Introduction

Three–body problem plays an vital role in space science, in particular the related field of solar system motion, stars, planets and their moons. The model of this problem can be used to characterize the dynamical behavior of the most stellar and planetary systems and to give also accurate pictures on their motion. The three–body problem acquisition is important due to its wildly applications not only in space science but also in many fields such as applied mathematics and mathematical physics. In addition, it is considered the simplest non–integrable dynamical system in space dynamics, and it can be approximated to more simple systems such as, the perturbed two–body problem [1], Robe's restricted problem [2,3] and Hill problem [4,5].

Periodic solutions of a dynamical system are solutions that characterize some repeated phenomena identically at regular intervals. These solutions play a vital role in many branches of science such as Physics and Engineering, but have a special appearance in Celestial Mechanics, to study the dynamical structures of two–body problem [6,7]; analysis the infinitesimal body motion within frame of three–body problem [8–11] or $N-$body problem [12–14].

The periodic solutions of either unperturbed Kepler problem or the perturbed one have been received considerable contributions. For example, in [6], the authors have explored the existence of two periodic orbits at every energy level, in the frame work of the anisotropic Kepler system, which emerges from elliptic orbits of the Kepler motion with high eccentricity, when the parameter of the anisotropy is small. Some interesting works have been constructed to analyze the periodic solutions [15–17].

The periodic solutions have particular significant in three–body problem, that is regard to its extended applications in both Space Dynamics and Celestial Mechanics. Thereby, there are many and various dynamical system that can be studied by considering the problem of the restricted three–body. Some of these systems have applications in space mission for spacecrafts in the Planet–Moons systems (such like: Earth–Moon system). Further, this problem has applications in stellar systems to study the behavior of exoplanets in the proximity of one or both objects of a binary star system [18–20].

In fact, the periodic solutions have a considerable significant, because the most of natural phenomena in physical and engineering sciences as well as celestial mechanics can be characterized by periodic solutions of dynamical system, which is characterized by ordinary or partial differential equations system. These systems have wide variety of applications not only physical, mathematical and engineering sciences but also these systems have greet importance in the fields of biology, chemistry, neural networks [21–24].

There are many methods are developed to analyze periodic solutions, such as averaging method, Lindstedt–Poincaré technique, Krylov–Bogoliubov–Mitropolsky (KBM) and multiple scales methods, [25–28]. Periodic solutions are everywhere in the analysis of dynamical systems. Every field of science in particularly celestial mechanics has its own oscillatory phenomena, which can be described by periodic solution.

In this work, we aim to find the periodic solutions of the dynamical system of spatial quantized Hill problem. So, we will evaluate the equilibria points of linear system, and the necessary conditions will be used to calculate the periodic solutions arising from the equilibria points of the spatial quantized Hill problem by using the averaging theory. This system is constructed in the first time by Abouelmagd et al (2020) [5], this is motivated us to study the dynamical structures of this system through finding its own periodic solutions.

2. Perturbation techniques

The perturbation techniques play substantial role to analyze non–linear dynamics. Which became the driving forces, and pushed the mathematician researchers to extreme efforts, in order to explore and characterize the features of dynamical systems. These methods are considered the excellent tools, which are designed for the objectives have wonderful applicable in many fields.

In this chapter, we will shed light on some techniques, which may be used to construct an appropriate analytical periodic solutions of a perturbed dynamical systems. On the other hand, these techniques could be applied to some problems, where their closed form solutions are not exist or where the exact solutions are either impossible or unrealistic of their physical meaning. In general these systems are often neither linear nor autonomous in nature.

Now we assume that $\mathcal{H}$ is an $n-$dimensional Hamiltonian system defined in terms of conjugate variables $(\mathbf{X}, \mathbf{Y})$, where $\mathbf{X} \in \Gamma$, Γ is an open set of $\mathbb{R}^n$ and $\mathbf{Y} \in T^n$, here T denotes the standard one–dimensional torus. Then a nearly integrable Hamiltonian systems $\mathcal{H}$ can be read as

$$\mathcal{H}(\mathbf{X}, \mathbf{Y}) = h(\mathbf{X}, \mathbf{Y}) + \varepsilon f(\mathbf{X}, \mathbf{Y}) \tag{1}$$

where h and f are analytical functions called an integrable (or unperturbed) Hamiltonian and the perturbing function, respectively. But ε is a small parameter, which measure the size of perturbation force.

In the case of $\varepsilon = 0$ the Hamiltonian function is given by

$$\mathcal{H}(\mathbf{X}, \mathbf{Y}) = h(\mathbf{X}, \mathbf{Y}), \tag{2}$$

using Hamiltonian relation (2), the associated equations of motion are given as

$$\begin{aligned} \dot{\mathbf{X}} &= 0 \\ \dot{\mathbf{Y}} &= \mathbf{w}(\mathbf{X}), \end{aligned} \tag{3}$$

where denoted with a dot is used for the derivatives with respect to time, $\mathbf{w}$ is the frequency vector is defined as $\mathbf{w} = \partial h(\mathbf{X})/\partial \mathbf{X}$.

The integration of System (3) is

$$\begin{aligned} \mathbf{X}(t) &= \mathbf{X}_0 \\ \mathbf{Y} &= \mathbf{w}_0 t + \mathbf{X}_0, \end{aligned} \tag{4}$$

where $\mathbf{X}_0 = \mathbf{X}(0)$ and $\mathbf{w}_0 = \mathbf{w}(\mathbf{X}(0))$, Solution (4) shows that the variable $\mathbf{X}$ is constant, while its conjugate vary linearly with time.

In the case of the perturbing force has its own effect ($\varepsilon \neq 0$), the Hamiltonian function is given as in Relation (1) and the associated equations of motion are

$$\begin{aligned} \dot{\mathbf{X}} &= -\varepsilon \frac{\partial f}{\partial \mathbf{Y}} \\ \dot{\mathbf{Y}} &= \mathbf{w}(\mathbf{X}) + \varepsilon \frac{\partial f}{\partial \mathbf{X}}. \end{aligned} \tag{5}$$

System (5) may not be integrable and chaotic motion could appear.

2.1. On perturbation techniques

2.1.1. Importance of perturbation techniques

In fact, the most of physical phenomena in nature are nonlinear and non–autonomous in their structures. The description of these phenomena within linear sense is not realistic and present inaccurate information about their behaviour. So it is necessary to preform the nonlinear dynamical systems, which describe these phenomena, but there are extra–difficulty to treat these systems by direct methods, and the perturbation techniques are considered the best choice in most cases. The perturbation techniques are employed for the dynamical system which is consists of ordinary or partial differential equations.

We would like to refer that there are numerous and considerable methods, that can be used to get periodic solutions. For example the averaging method, see for details [29–33]. The Liouville–Green method, which is known as LG or WKB method, Lyapunov's theorem, Lindstedt–Poincaré technique and KBM method [34,35]. There are extra methods such that the straightforward expansion technique (Classical perturbation theory), but this methods may fall in removing secular terms. Also the multiple scales method, which is considered one of the most strong techniques of obtaining the periodic solutions [36].

2.1.2. Advantages and disadvantages of perturbation techniques

Exact solutions of the dynamical systems are rare not only in celestial mechanics but also in many branches of applied mathematics: quantum mechanics, fluid mechanics, solid mechanics, and theoretical physics. This concern to nonlinearities behavior of physical phenomenon. Thereby the engineers, the physicists, and the mathematicians are forced to find approximate solutions for the mathematical models, which they are facing. These solutions may be, purely analytical, purely numerical or a combination of analytical and numerical techniques.

Perturbation techniques provide the most multilateral tools obtainable in non–linear a set of differential equations. That can be applied and employed to even more complex models. But perturbation techniques have their own limitations. Which are mainly depend on the presumption, that a very small parameter must appear in the prevailing equations.

2.2. Validity of perturbation techniques

Many applications of perturbation methods are not available without the existence of this small parameter. Actually an overwhelming preponderance of non–linear dynamical systems, in particulary they have "strong non–linearity", have no small parameter. In some cases of estimating this parameter is more the resulting of an technical procedure than scientific methodology. The convenient selection of a small parameter could lead to intelligible results. On the contrary, an incorrect choice for this parameter creates inaccurate or even unrealistic solutions. Even if there exists appropriate small parameter, the perturbation methods provide us by analytical solutions, they are adequate in cases of bounded this parameter.

The structures and analysis of perturbation approaches, such as *classical perturbation theory*, *Lindstedt–poincaré technique*, *multiple scales method*, *Krylov–Bogoliubov (KB) averaging method* and *averaging theory* are familiar in the literatures as we have mentioned in the previous subsections. But we will show the differences and similarities among these methods and rationale the choosing of averaging theory to find the periodic solution of the spatial quantized Hill's problem.

The aforementioned techniques demand that the dynamical system be weakly non–linear or weakly non–autonomous, meaning that those terms in the equation including the non–linearity or non–autonomy are small. Alternatively systems of this structure can be thought of as almost linear, or quasi–linear. A consequence of almost linear systems is that the differential equations will have linear terms and small non–linear or non–autonomous terms separated from each other. These techniques can be applied to nearly integrable Hamiltonian systems as in System (5).

The classical perturbation theory, which is called the straightforward expansion technique. It is used to find analytically solution in the power series of the following form

$$\mathbf{X}(t,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \mathbf{X}_k(t) \tag{6}$$

In general, this methods comes out with secular terms that will provide unbounded solution in the case of long interval. Straightforward application of the classical perturbation theory to periodic nonlinear motion gives a result with secular terms, which proportional with time. In spite of the fact that the behavior of motion is known to be bounded. One of the commonly approaches used to remove those unwanted secular terms is the Lindstedt–Poincaré technique (also called continuation method). For applying this method on the astronomical dynamical systems and illustrating the outcome of secular terms, with the alternative techniques to remove these term by either Lindstedt–Poincaré or KBM method [26].

Multiple scales technique is a generalized formula for the Lindstedt–Poincaré method. The latter method is depending on the angular velocity of the non–linear oscillation may depends upon its amplitude. The angular velocity is expanded in an asymptotic series, and the coefficients of each term in the series are evaluated in such a way the solution has no secular terms, see its application on real system [27]

In a similar manner, multiple scales technique admit the solution vary on fast and slow time separation scales. The first variable "fast time scale" with respect to the first linear order in Lindstedt–Poincaré expansion, and every slow scale matches with the second and higher terms in this expansion. The major difference is that multiple scales supposes that the coefficients of each scale are fixed, equal one, and employee a variation–of–parameters approach, as well as it considers the integration constants which appear in the linear solution be functions in the slow time scales variables. This process results in the dynamical system of partial differential equations at each order of that has to be solved to obtain a uniform solution of the dynamical system. The multiple scales technique is a much more powerful tool than the Lindstedt–Poincaré method, because the former admit for the constants versus with the slow time scales variables, while the latter only introduces one free variable at each order in the perturbed parameter. But this method may result a system, where its solution for the obtaining of periodicity conditions is more difficult than the main system

The Krylov–Bogolyubov averaging method is a technique to find the periodic solution of non–linear system, based on the averaging principle, where the exact system is replaced by an averaged one. In order to obtain the solution of the perturbed system or non–linear motion, using this method we have to admit that the constants of linear motion vary slowly with time t and the perturbation parameter ε. The significance of this method is that a general averaging approach is developed and proved that the solutions of the averaged systems give precise approximation to the original system [7]

Averaging is a mathematical method to replace a given field by its own average over a specified variable such as time or an angular variable to get asymptotic approximation to the original system with aiming of obtaining periodic solution. In dynamical systems, the averaging method or the averaging theory utilizes systems including time–scale separation: fast oscillation versus a slow drift. We propose averaging over a certain interval of time to iron out the fast oscillation and monitor the qualitative behaviour from the resulting dynamics. It turns out to be a familiar problem where there exists the trade off between how perfect is the approximate solution balanced by how much time it holds to be similar to the original solution [31]

In this work, we have used the averaging theory of dynamical systems, because it is special worthy in the case of systems that can have isolated periodic orbits like it is the case. In the next section, we will apply this method to Hill's version of quantized three–body problem to find periodic solutions.

3. Mathematical Model

Hill version of quantized three–body problem is derived and analyzed for first time in [5], and the equations of motion are given by

$$\begin{aligned}
\ddot{\xi} - 2\dot{\eta} &= 3\xi + 2(\alpha_1 - \alpha_{11}) - \frac{1}{r^3}\left[1 + \frac{2\alpha_{21}}{r} + \frac{3\alpha_{22}}{r^2}\right]\xi, \\
\ddot{\eta} + 2\dot{\xi} &= -\frac{1}{r^3}\left[1 + \frac{2\alpha_{21}}{r} + \frac{3\alpha_{22}}{r^2}\right]\eta, \\
\ddot{\zeta} &= -\zeta - \frac{1}{r^3}\left[1 + \frac{2\alpha_{21}}{r} + \frac{3\alpha_{22}}{r^2}\right]\zeta.
\end{aligned} \tag{7}$$

System (7) characterize the perturbed spatial Hill problem, the system is perturbed by quantum corrections, hence this system is called the spatial quantized Hill problem (SQHP). Furthermore, this system is considered a limiting case from the spatial quantized restricted three bodies problem, it was developed in [37]. In addition, this system can be described by the style of writing the restricted three–body problem

$$\begin{aligned} \ddot{\xi} - 2\dot{\eta} &= \Psi_{\xi}(\xi,\, \eta,\, \zeta), \\ \ddot{\eta} + 2\dot{\xi} &= \Psi_{\eta}(\xi,\, \eta,\, \zeta), \\ \ddot{\zeta} &= \Psi_{\zeta}(\xi,\, \eta,\, \zeta), \end{aligned} \tag{8}$$

where

$$\Psi(\xi,\eta,\zeta) = \frac{1}{2}\Big[3\,\xi^2 + 4(\alpha_1 - \alpha_{11})\xi - \zeta^2\Big] + \frac{1}{r}\Big[1 + \frac{\alpha_{21}}{r} + \frac{\alpha_{22}}{r^2}\Big], \tag{9}$$

and Ψ_{ξ}, Ψ_{β}, Ψ_{ζ} refer to the partial derivatives of the potential function with respect to the variables ξ, β and ξ, while the separation distance r is given by $r = \sqrt{\xi^2 + \beta^2 + \zeta^2}$.

Here α_1, α_{11} and α_{21} are very small quantities with order $\mathcal{O}(1/c^2)$, while α_{22} is more smaller with order $\mathcal{O}(1/c^3)$, where c is the speed of light, for a comprehensive details [5,37]. Since α_1, α_{11} and α_{21} have the same order, then $\alpha_1 - \alpha_{11} \cong 0$, and System (8) and the potential function can be simplified and rewritten

$$\begin{aligned} \ddot{\xi} - 2\dot{\eta} &= \bar{\Psi}_{\xi}(\xi,\, \eta,\, \zeta), \\ \ddot{\eta} + 2\dot{\xi} &= \bar{\Psi}_{\eta}(\xi,\, \eta,\, \zeta), \\ \ddot{\zeta} &= \bar{\Psi}_{\zeta}(\xi,\, \eta,\, \zeta), \end{aligned} \tag{10}$$

where

$$\Psi(\xi,\eta,\zeta) = \frac{1}{2}\Big[3\,\xi^2 - \zeta^2\Big] + \frac{1}{r}\Big[1 + \frac{\alpha_{21}}{r} + \frac{\alpha_{22}}{r^2}\Big]. \tag{11}$$

Although many different analysis in celestial mechanics have been accomplished in the frame work of the three–body problem, but there are also many dynamical concepts can be carried out within frame of Hill problem [38] without loosing the required accuracy underlying of using simple model. Furthermore, the perturbed model of this problem can be used to study the effect of some perturbed forces on the dynamical properties, such as the emerging periodic solutions from the equilibria points, which will be analyzed in the next sections.

4. Periodic solutions

Periodic solution or periodic orbits are considered one the major reasons of stability and continuous of our life, for example the periodicity motion of sun an moon. When the Hill model has been constructed, the researchers devoted their work to calculate the families of periodic orbits. More work has been developed to analyze the lunar theory depend on Hill's problem. For the importance of Hill's problem, we intend to explore the presence of periodic solutions emerged from equilibria points by underlying SQHP.

By taking $\alpha_1 = \alpha_{11} = \alpha_{21} = 1.5 \times 10^{-3}, \alpha_{22} = 1.5 \times 10^{-5}$ of the differential System (7), we get the following equilibrium points $E_1 = (\delta, 0, 0)$ and $E_2 = (-\delta, 0, 0)$ where $\delta \approx 0.694035$.

Thus, we study the presence of periodic solutions, which emerge from equilibria points E_1 and the same ones are valid for E_2. In order to study the motion around or in the proximity of the equilibria points E_1 and E_2, We first have to linearize System (8). Thus, we impose that $\xi = x_1 - \delta$, $\eta = y_1$, $\zeta = z_1$, where x_1, y_1 z_1 are very small displacement from the equilibria points, thereby the associated linear system to non–linear System (8) are given by

$$\begin{aligned} \frac{d^2x_1}{dt^2} - 2\frac{dy_1}{dt} - \alpha x_1 = 0, \\ \frac{d^2y_1}{dt^2} + 2\frac{dx_1}{dt} + \beta y_1 = 0, \\ \frac{d^2z_1}{dt^2} + \gamma z_1 = 0, \end{aligned} \tag{12}$$

with $\alpha \approx 9.03$, $\beta \approx 3$ and $\gamma \approx 3$.

Since the means of the averaging theory is a one of the powerful tools for finding the periodic solutions, then we will apply this method to study the existence of periodic solution of the following system

$$\begin{aligned} \frac{d^2x_1}{dt^2} - 2\frac{dy_1}{dt} - \alpha x_1 &= \varepsilon \mathcal{F}_1\left(t, x_1, \frac{dx_1}{dt}, y_1, \frac{dy_1}{dt}, , z_1, \frac{dz_1}{dt}\right), \\ \frac{d^2y_1}{dt^2} + 2\frac{dx_1}{dt} + \beta y_1 &= \varepsilon \mathcal{F}_2\left(t, x_1, \frac{dx_1}{dt}, y_1, \frac{dy_1}{dt}, z_1, \frac{dz_1}{dt}\right), \\ \frac{d^2z_1}{dt^2} + \gamma z_1 &= \varepsilon \mathcal{F}_3\left(t, x_1, \frac{dx_1}{dt}, y_1, \frac{dy_1}{dt}, z_1, \frac{dz_1}{dt}\right), \end{aligned} \tag{13}$$

where ε is the perturbation parameter, it is very small quantity and the functions $\mathcal{F}_1, \mathcal{F}_2$, $\mathcal{F}_3$ represent the non–linear terms which will be ignored when $\varepsilon = 0$, but this functions satisfies the following properties:

- $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ are smooth functions,
- $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ are periodic functions in variable t ,
- $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ are resonance in $\iota_1 : \iota_2$ with periodic solutions for System (12),

where ι_1 and ι_2 are primes numbers.

There is a unique singular point for the unperturbed System (12) at the origin with eigenvalues $\pm\Omega$, $\pm\omega_1 i$, $\pm\omega_2 i$, where $\Omega \approx 2.51$, $\omega_1 \approx 2.07$ and $\omega_2 \approx 1.73$. In the phase space

$$\left(x_1, \frac{dx_1}{dt}, y_1, \frac{dy_1}{dt}, z_1, \frac{dz_1}{dt}\right)$$

the aforementioned system (the unperturbed system) has two planes filed of periodic solutions with the exception of the origin, where the periods of solutions are

$$T_1 = 2\pi/\omega_1 \text{ or } 2\pi/\omega_2,$$

here the periods T_1 and T_2 are related to the eigenvalues $\pm\omega_1 i$ or $\pm\omega_2 i$, respectively. We will explore which of the periodic solutions continue for the perturbed System (13) where the parameter of perturbation ε is enough small and there are two periods either $\iota_1 T_1/\iota_2$ or $\iota_1 T_2/\iota_2$, for the perturbed function $\mathcal{F}_i$ for $i \in \{1, 2, 3\}$.

Consider $Z^0 = (Z_1^0, Z_2^0)$, and $\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2)$ for the System (13), with

$$\begin{aligned}\mathcal{H}_1\left(Z^0\right) &= \frac{1}{\iota_1 T_2}\int_0^{\iota_1 T_2} \langle(\cos(\omega_2 t), -\sin(\omega_2 t)), (\mathcal{F}_5^*(t), \mathcal{F}_6^*(t))\rangle dt \\ &= \frac{1}{\iota_1 T_2}\int_0^{\iota_1 T_2} -\sin(\omega_2 t)\mathcal{F}_6^*(t)dt, \\ \mathcal{H}_2\left(Z^0\right) &= \frac{1}{pT_2}\int_0^{pT_2} \langle(\sin(\omega_2 t), \cos(\omega_2 t)), (\mathcal{F}_5^*(t), \mathcal{F}_6^*(t))\rangle dt \\ &= \frac{1}{pT_2}\int_0^{pT_2} \cos(\omega_2 t)\mathcal{F}_6^*(t)dt,\end{aligned} \tag{14}$$

where $\langle , \rangle$ is the scalar product and $\mathcal{F}_5^*(t) = 0$, $\mathcal{F}_6^* = 1.12\,\mathcal{F}_3$, $\mathcal{F}_3 = \mathcal{F}_3\left(\eta_1^2(t), \ldots, \eta_6^2(t)\right)$ and $\eta_j^2(t) = 0, j = 1, \ldots, 4$, while

$$\begin{aligned}\eta_5^2(t) &= 0.50\Big(Z_1^0\cos(\omega_2 t) + Z_2^0\sin(\omega_2 t)\Big), \\ \eta_6^2(t) &= 0.86\Big(Z_2^0\cos(\omega_2 t) - Z_1^0\sin(\omega_2 t)\Big).\end{aligned}$$

Now, we impose that $Z^{0*} = (Z_1^{0*}, Z_2^{0*})$ is the zero of non–linear system $\mathcal{H}(Z^0) = 0$ where

$$\left|\frac{\partial \mathcal{H}}{\partial Z^0}\right| \neq 0 \text{ when } Z^0 = Z^{0*},$$

Then we can state that the system has a simple zero [39]. We would to remark that the expression of simple zero or pole is used to describe the zero or pole of order one, and sometimes the term of "degree" is used instead of "order". The property of this zero or pole leads to this zero can be isolated and its neighbourhood has no other zero.

We emphasize that if the Malkin bifurcation function $\mathcal{H}$ has a simple zero Z^{0*} and the solution of the unperturbed system has a period T_2 by using initial value Z^{0*}, then the perturbed system will has also T_2-periodic solution.

The periodic solution of the dynamical System (13) is considered the main first result in this work, where this solution will bifurcate from the T_2-periodic solution of the unperturbed system, hence we will present the following theorem:

Theorem 1. *We impose that ι_1 and ι_2 primes numbers and*

- *$\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$ are smooth functions of System* (13)
- *$\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$ are periodic with period $\iota_1 T_2/\iota_2$ in variable t*

For each simple zero $Z^{0*} \neq 0$ of the non–linear system $\mathcal{H}(Z^0) = 0$ when $\varepsilon \neq 0$ and enough small, we can find a periodic solution for the perturbed System (13) *takes the form $(x_1(t, \varepsilon)$, $y_1(t, \varepsilon), z_1(t, \varepsilon))$ and tend to the periodic solution $(x_1(t), y_1(t), z_1(t)) = (\eta_1^2(t),\, \eta_3^2(t),\, \eta_5^2(t))|_{Z^0 Z^{0*}}$ of the unperturbed System* (12) *traveled ι_1 times.*

We will presented the proof of Theorem 1 in Sec. 5. Further, the following corollary is considered the application of this theorem and its proof will be presented in Sec. 6.

Corollary 1. *Considering that $\mathcal{F}_1(t, x_1, \dot{x}_1, y_1, \dot{y}_1, z_1, \dot{z}_1) = 0$, $\mathcal{F}_2(t, x_1, \dot{x}_1, y_1, \dot{y}_1, z_1, \dot{z}_1) = 0$, $\mathcal{F}_3(t, x_1, \dot{x}_1, y_1, \dot{y}_1, z_1, \dot{z}_1) = z^5 + \sin(\omega_2 t)\dot{z}^2$. Thus, the System* (13) *with $\varepsilon \neq 0$ and it is enough small, has one periodic solution $(x_1(t, \varepsilon), y_1(t, \varepsilon), z_1(t, \varepsilon))$ approximating to the periodic solutions $(x_1(t), y_1(t), z_1(t)) = \left(\eta_1^2(t), \eta_3^2(t), \eta_5^2(t)\right)|_{Z^0 = Z^{0*}}$ of* (2) *when $\varepsilon \to 0$, given by $Z^{0*} = (0, -2.11)$.*

Now we impose that $Y^0 = (Y_1^0, Y_2^0)$, and considering the Malkin bifurcation function $\bar{\mathcal{H}} = (\mathcal{H}_3, \mathcal{H}_4)$ for the System (13) controlled by

$$\begin{aligned}
\mathcal{H}_3\left(Y^0\right) &= \frac{1}{pT_1}\int_0^{pT_1} \langle(\cos(\omega_1 t), -\sin(\omega_1 t)), (\mathcal{F}_3^*(t), \mathcal{F}_4^*(t))\rangle dt \\
&= \frac{1}{pT_1}\int_0^{pT_1} (\cos(\omega_1 t)\mathcal{F}_3^*(t) - \sin(\omega_1 t)\mathcal{F}_4^*(t))dt, \\
\mathcal{H}_4\left(Y^0\right) &= \frac{1}{pT_1}\int_0^{pT_1} \langle(\sin(\omega_1 t), \cos(\omega_1 t)), (\mathcal{F}_3^*(t), \mathcal{F}_4^*(t))\rangle dt \\
&= \frac{1}{pT_1}\int_0^{pT_1} (\cos(\omega_1 t)\mathcal{F}_4^*(t) + \sin(\omega_1 t)\mathcal{F}_3^*(t))dt,
\end{aligned}$$

where

$$\mathcal{F}_3^* = -0.45\,\mathcal{F}_1, \quad \mathcal{F}_4^* = -1.46\,\mathcal{F}_2,$$

with $\mathcal{F}_i = \mathcal{F}_i(\eta_1^1(t), \ldots, \eta_6^1(t)), i \in \{1,2\}$ and $\eta_j^2(t) = 0, j = 5, 6,$

$$\begin{aligned}
\eta_1^1(t) &= 0.12\left(Y_2^0\cos(\omega_1 t) - Y_1^0\sin(\omega_1 t)\right), \\
\eta_2^1(t) &= -0.26\left(Y_1^0\cos(\omega_1 t) + Y_2^0\sin(\omega_1 t)\right), \\
\eta_3^1(t) &= -0.41\left(Y_1^0\cos(\omega_1 t) + Y_2^0\sin(\omega_1 t)\right), \\
\eta_4^1(t) &= -0.85\left(Y_2^0\cos(\omega_1 t) - Y_1^0\sin(\omega_1 t)\right),
\end{aligned}$$

As we aforementioned with the existing of a simple zero Y^{0*} of the Malkin bifurcation function $\bar{\mathcal{H}}$, one will obtain form T_1-periodic solution of the unperturbed system with initial value Y^{0*} emerges the solution of perturbed system with T_1-periodic solution, because he simple zero zero can be isolated and its neighbourhood has no other zero.

The main second result related to the periodic solutions, and which is associated to unperturbed System (13) will be stated in the following theorem:

Theorem 2. *We impose that ι_1 and ι_2 are primes numbers and*

- *$\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$ are smooth functions of System (13)*
- *$\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$ are periodic with period $\iota_1 T_1/\iota_2$ in variable t*

For each simple zero $Y^{0*} \neq 0$ of the non–linear system $\bar{\mathcal{H}}(Y^0) = 0$, when $\varepsilon \neq 0$ is enough small, then the perturbed System (13) has a periodic solution $(x_1(t,\varepsilon), y_1(t,\varepsilon), z_1(t,\varepsilon))$ going to the periodic solution $(x_1(t), y_1(t), z_1(t)) = (\eta_1^1(t), \eta_3^1(t), \eta_5^1(t))|_{Y^0=Y^{0*}}$ of the unperturbed System (12) traveled ι_1 times.

We also remark that the application of Theorem 2 can be stated in the following corollary:

Corollary 2. *Considering that $\mathcal{F}_1(t, x_1, \dot{x}_1, y_1, \dot{y}_1, z_1, \dot{z}_1) = \sin(\omega_1 t) + x_1 + 3x_1^2 y_1 + y_1$, $\mathcal{F}_2(t, x_1, \dot{x}_1, y_1, \dot{y}_1, z_1, \dot{z}_1) = \cos(\omega_1 t) - (\dot{y}_1)\hat{}2 + x_1$, $\mathcal{F}_3(t, x_1, \dot{x}_1, y_1, \dot{y}_1, z_1, \dot{z}_1) = 0$. Then the System (13) for $\varepsilon \neq 0$ sufficiently small has one periodic solution $(x_1(t,\varepsilon), y_1(t,\varepsilon), z_1(t,\varepsilon))$ tending to the periodic solutions $(x_1(t), y_1(t), z_1(t)) = \left(\eta_1^1(t), \eta_3^1(t), \eta_5^1(t)\right)|_{Y^0=Y^{0*}}$ of System (12) when $\varepsilon \to 0$, given by $Y^{0*} = (2.59\,,\,9.21)$.*

The prof of Corollary 2 will be given in Sec. 6.

267 **5. Proof of the theorems 1 and 2**

To accomplish the proof of Theorems (1, 2), we will use the following variables:

$$(x_1, x_2, y_1, y_2, z_1, z_2) = \left(x_1, \frac{dx_1}{dt}, y_1, \frac{dy_1}{dt}, z_1, \frac{dz_1}{dt} \right)$$

Thus, the dynamical System (13) can be rewritten in the form of a first order in $\mathbb{R}^6$

$$\begin{aligned} \frac{dx_1}{dt} &= x_2\,, & \frac{dx_2}{dt} &= \alpha x_1 + 2y_2 + \varepsilon \mathcal{F}_1(x_1, x_2, y_1, y_2, z_1, z_2), \\ \frac{dy_1}{dt} &= y_2\,, & \frac{dy_2}{dt} &= -2x_2 - \beta y_1 + \varepsilon \mathcal{F}_2(x_1, x_2, y_1, y_2, z_1, z_2), \\ \frac{dz_1}{dt} &= z_2\,, & \frac{dz_2}{dt} &= -\gamma z_1 + \varepsilon \mathcal{F}_3(x_1, x_2, y_1, y_2, z_1, z_2). \end{aligned} \tag{15}$$

It is clear that the perturbed System (15) ($\varepsilon \neq 0$) can be reduced to the unperturbed System (12) when $\varepsilon = 0$. Now we write the perturbed System (15) with the style that the linear part at the origin point will take the real Jordan expression, after that we can change the variables to the following form:

$$(x_1, x_2, y_1, y_2, z_1, z_2) \to (X_1, X_2, Y_1, Y_2, Z_1, Z_2)$$

given by

$$\begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -0.32 & 0.32 & 0.0 & 0.12 & 0.0 & 0.0 \\ -0.81 & -0.81 & -0.26 & 0 & 0.0 & 0.0 \\ 0.17 & 0.17 & -0.41 & 0 & 0.0 & 0.0 \\ 0.44 & -0.44 & 0.0 & -0.85 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.86 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ Y_1 \\ Y_2 \\ Z_1 \\ Z_2 \end{pmatrix}, \tag{16}$$

and

$$\begin{pmatrix} X_1 \\ X_2 \\ Y_1 \\ Y_2 \\ Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} -1.92 & -0.53 & 0.34 & -0.28 & 0.0 & 0.0 \\ 1.92 & -0.53 & 0.34 & 0.28 & 0.0 & 0.0 \\ 0.0 & -0.45 & -2.11 & 0.0 & 0.0 & 0.0 \\ -1.98 & 0.0 & 0.0 & -1.46 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 2.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.15 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ z_1 \\ z_2 \end{pmatrix},$$

the differential System (15) becomes

$$\begin{aligned} \dot{X}_1 &= \Omega X_1 + \varepsilon \mathcal{F}_1^*, \\ \dot{X}_2 &= -\,\Omega X_2 + \varepsilon \mathcal{F}_2^*, \\ \dot{Y}_1 &= \omega_1 Y_2 + \varepsilon \mathcal{F}_3^*, \\ \dot{Y}_2 &= -\,\omega_1 Y_1 + \varepsilon \mathcal{F}_4^*, \\ \dot{Z}_1 &= \omega_2 Z_2 + \varepsilon \mathcal{F}_5^*, \\ \dot{Z}_2 &= -\,\omega_2 Z_1 + \varepsilon \mathcal{F}_6^*, \end{aligned} \tag{17}$$

where

$$\begin{aligned}\mathcal{F}_1^* &= -0.53\mathcal{F}_1 - 0.28\mathcal{F}_2,\\ \mathcal{F}_2^* &= -0.53\mathcal{F}_1 + 0.28\mathcal{F}_2,\\ \mathcal{F}_3^* &= -0.45\mathcal{F}_1,\\ \mathcal{F}_4^* &= -1.46\mathcal{F}_2,\\ \mathcal{F}_5^* &= 0,\\ \mathcal{F}_6^* &= 1.15\mathcal{F}_3,\end{aligned}$$

268 with $\mathcal{F}_i = \mathcal{F}_i(\eta_1, \ldots, \eta_6), i \in \{1, 2, 3\}$ and

$$\begin{aligned}\eta_1 &= -0.32X_1 + 0.32X_2 + 0.12Y_2,\\ \eta_2 &= -0.81X_1 - 0.81X_2 - 0.26Y_1,\\ \eta_3 &= 0.17X_1 + 0.17X_2 - 0.41Y_1,\\ \eta_4 &= 0.44X_1 - 0.44X_2 - 0.85Y_2,\\ \eta_5 &= 0.5Z_1,\\ \eta_6 &= 0.86Z_2,\end{aligned}$$

To prove Theorems 1 and 2, we first depict the periodic solution of the unperturbed system in through the following *Lemma*

Lemma 1. *The periodic solutions* $(X_1(t), X_2(t), Y_1(t), Y_2(t), Z_1(t), Z_2(t))$ *of System* (17) *when* $\varepsilon = 0$ *are*

$$\left(0, 0, Y_1^0 \cos(\omega_1 t) + Y_2^0 \sin(\omega_1 t), Y_2^0 \cos(\omega_1 t) - Y_1^0 \sin(\omega_1 t), 0, 0\right), \tag{18}$$

where T_1 *is the period of motion, and*

$$\left(0, 0, 0, 0, Z_1^0 \cos(\omega_2 t) + Z_2^0 \sin(\omega_2 t), Z_2^0 \cos(\omega_2 t) - Z_1^0 \sin(\omega_2 t)\right), \tag{19}$$

here T_2 *is the period of motion.*

Proof of Lemma 1

Since System (17) is linear when $\varepsilon = 0$ thereby, the proof can be easily established.

Proof of Theorem 1

We impose that $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$ of (13) are periodic functions in t with period $\iota_1 T_1/\iota_2$ where ι_1 and ι_2 are primes numbers. Hence, the same periodicity features are same for the System (17) and the periodic Solutions (19) with period $\iota_1 T_2$. By applying Theorem A1 in [39] and using the same notation and terminology to the System (17), then we can write the System (17) in the following form

$$\dot{X}(t) = H_0(t, X) + \varepsilon\, H_1(t, X) + \varepsilon^2\, H_2(t, X, \varepsilon)$$

Then, we can consider

$$X = \begin{pmatrix} X_1 \\ X_2 \\ Y_1 \\ Y_2 \\ Z_1 \\ Z_2 \end{pmatrix}, \quad H_0(t,X) = \begin{pmatrix} \Omega X_1 \\ -\Omega X_2 \\ \omega_1 Y_2 \\ -\omega_1 Y_1 \\ \omega_2 Z_2 \\ -\omega_2 Z_1 \end{pmatrix}, \quad H_1(t,X) = \begin{pmatrix} \mathcal{F}_1^* \\ \mathcal{F}_2^* \\ \mathcal{F}_3^* \\ \mathcal{F}_4^* \\ \mathcal{F}_5^* \\ \mathcal{F}_6^* \end{pmatrix}, \quad H_2(t,X,\varepsilon) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

In this context, the periodic solution of the unperturbed System (17) ($\varepsilon = 0$) will be studied within the Type (19) to continue as a periodic solution for the perturbed system (when $\varepsilon \neq 0$ is enough small). First, we characterize the different parameters, which are stated in Theorem A1 (see the Appendix in [39] for details) due to the certain case of the System (17). Now we assume that $\rho_1 > 0$ and $\rho_2 > 0$, where $\rho_1 > 0$ is chosen to be small, while $\rho_2 > 0$ is chosen to be large. We also assume that V is bounded and open subset of the plane $X_1 = X_2 = Y_1 = Y_2 = 0$ of the form

$$V = \left\{ \left(0,0,0,0,Z_1^0,Z_2^0\right) \in \mathbb{R}^6 : \rho_1 < \sqrt{\left(Z_1^0\right)^2 + \left(Z_2^0\right)^2} < \rho_2 \right\}.$$

Since V is bounded and open subset of $\mathbb{R}^2$, we can choose two numbers $\rho_1 > 0, \rho_2 > 0$ such that

$$V = \left\{ (\beta(\alpha),\alpha) \in \mathbb{R}^2 : \rho_1 < \sqrt{\left(Z_1^0\right)^2 + \left(Z_2^0\right)^2} < \rho_2 \right\}.$$

279 where $\alpha \in R^2$ and $\beta(\alpha) \in R^4$.

Now we assume that $\alpha = Z^0 = (Z_1^0, Z_2^0)$, then we characterize V with the set $\{\alpha \in \mathbb{R}^2 : \rho_1 < ||\alpha|| < \rho_2\}$, being $||\cdot||$ the Euclidean norm in $\mathbb{R}^2$, while the function $\beta(\alpha)$ is defined as $\beta : C1(V) \to \mathbb{R}^4$ such that $\beta(\alpha) = (0,0,0,0)$, here $C1(V)$ refers to the closure of V. Hence, for the proposed system one obtains

$$\begin{aligned} \mathcal{Z} &= \{\mathbf{z}_\alpha = (\beta(\alpha),\alpha), \alpha \in C1(V)\} \\ &= \left\{ \left(0,0,0,0,Z_1^0,Z_2^0\right) \in \mathbb{R}^6 : \rho_1 \leq \sqrt{\left(Z_1^0\right)^2 + \left(Z_2^0\right)^2} \leq \rho_2 \right\}. \end{aligned}$$

We take for each $\mathbf{z}_\alpha \in \mathcal{Z}$ the periodic solution

$$\mathbf{x}(t,\mathbf{z}_\alpha) = (0,0,0,0,Z_1(t),Z_2(t)),$$

controlled by System (19) of period $\iota_1 T_2$. Calculating the matrix $M_{\mathbf{z}_\alpha}(t)$ of the linear System (17), which is called the fundamental matrix where $\varepsilon = 0$ related to the $\iota_1 T_2$ periodic solution $\mathbf{z}_\alpha = (0,0,0,0,Z_1^0,Z_2^0)$ and also $M_{\mathbf{z}_\alpha}(0)$ is the identity element in space $\mathbb{R}^6$, thus one obtains

$$M_{\mathbf{z}_\alpha}(t) = M(t) = \begin{pmatrix} e^{\Omega t} & 0 & 0 & 0 & 0 & 0 \\ 0 & -e^{-\Omega t} & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(\omega_1 t) & \sin(\omega_1 t) & 0 & 0 \\ 0 & 0 & -\sin(\omega_1 t) & \cos(\omega_1 t) & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(\omega_2 t) & \sin(\omega_2 t) \\ 0 & 0 & 0 & 0 & -\sin(\omega_2 t) & \cos(\omega_2 t) \end{pmatrix}.$$

We remark that there is no correlation between the matrix $M_{\mathbf{z}_\alpha}(t)$ and particular periodic solution $\mathbf{x}(t, \mathbf{z}_\alpha, 0)$.

Now, we impose that the matrix $\hbar$ is defined by

$$\hbar = M^{-1}(0) - M^{-1}(\iota_1\, T_2)$$

then the matrix $\hbar$ is satisfied the stated assumptions (ii) in Theorem A1 (see the Appendix in [39] for details), where the matrix $\hbar$ and its determinant are given by

$$\hbar = \begin{pmatrix} 1 - e^{-2\pi\,\Omega\,\iota_1\,/\omega_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 - e^{2\pi\,\Omega\,\iota_1\,/\omega_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\sin^2(\pi\,\iota_1\,\omega_1/\omega_2) & \sin(2\pi\,\iota_1\,\omega_1/\omega_2) & 0 & 0 \\ 0 & 0 & -\sin(2\pi\,\iota_1\,\omega_1/\omega_2) & 2\sin^2(\pi\,\iota_1\,\omega_1/\omega_2) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\sin^2(\pi\,\iota_1) & \sin(2\pi\,\iota_1) \\ 0 & 0 & 0 & 0 & -\sin(2\pi\,\iota_1) & 2\sin^2(\pi\,\iota_1) \end{pmatrix},$$

$$|\hbar| = \begin{vmatrix} 1 - e^{-2\pi\,\Omega\,\iota_1/\omega_2} & 0 & 0 & 0 \\ 0 & 1 - e^{2\pi\,\Omega\iota_1/\omega_2} & 0 & 0 \\ 0 & 0 & 2\sin^2(\pi\,\iota_1\,\omega_1/\omega_2) & \sin(2\pi\,\iota_1\,\omega_1/\omega_2) \\ 0 & 0 & -\sin(2\pi\,\iota_1\,\omega_1/\omega_2) & 2\sin^2(\pi\,\iota_1\,\omega_1/\omega_2) \end{vmatrix}$$

hence

$$|\hbar| = -16\sinh^2(\pi\,\Omega\,\iota_1/\omega_2)\sin^2(\pi\,\iota_1\omega_1/\omega_2) \neq 0,$$

because the ratio of the frequencies is non–resonant with π. In small word, all the stated assumptions in Theorem A1 are satisfied by the System (17).

In the proposed system, the map $\xi : \mathbb{R}^6 \longrightarrow \mathbb{R}^2$ can be written as

$$\xi(X_1, X_2, Y_1, Y_2, Z_1, Z_2) = (Z_1, Z_2),$$

by evaluating the function

$$\mathcal{H}\Big(\mathcal{Z}_1^0, \mathcal{Z}_2^0\Big) = \mathcal{H}(\alpha) = \xi\left(\frac{1}{pT_2}\int_0^{pT_2} M_{\mathbf{z}_\alpha}^{-1}(t) H_1(t, \mathbf{x}(t, \mathbf{z}_\alpha, 0))dt\right),$$

we get $\mathcal{H}(\mathcal{Z}^0) = (\mathcal{H}_1(\mathcal{Z}^0), \mathcal{H}_2(\mathcal{Z}^0))$, where the functions $\mathcal{H}_k$ for $k = 1, 2$ are the ones given in (14). Then, by Theorem A1 we have that for every simple zero $\mathcal{Z}^{0*} \in V$ of the system of non–linear functions $\mathcal{H}(Z^0) = 0$, we have a periodic solution

$$(X_1, X_2, Y_1, Y_2, Z_1, Z_2)(t, \varepsilon)$$

of System (17) such that

$$(X_1, X_2, Y_1, Y_2, Z_1, Z_2)(0, \varepsilon) \longrightarrow \Big(0, 0, 0, 0, Z_1^{0*}, Z_2^{0*}\Big) \text{ when } \varepsilon \longrightarrow 0$$

Let us changes the variables in System (16), then, one obtains a periodic solution

$$(x_1, x_2, y_1, y_2, z_1, z_2)(t, \varepsilon)$$

of System (17) where

$$\begin{pmatrix} x_1(t,\varepsilon) \\ x_2(t,\varepsilon) \\ y_1(t,\varepsilon) \\ y_2(t,\varepsilon) \\ z_1(t,\varepsilon) \\ z_2(t,\varepsilon) \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.5\left(Z_1^{0*}\cos(\omega_2 t) + Z_2^{0*}\sin(\omega_2 t)\right) \\ 0.86\left(Z_2^{0}\cos(\omega_2 t) - Z_1^{0}\sin(\omega_2 t)\right) \end{pmatrix} \text{ when } \varepsilon \longrightarrow 0$$

Thus, periodic solution of System (17) $(x_1(t,\varepsilon), y_1(t,\varepsilon)$, and $z_1(t,\varepsilon)$ can be written as

$$(x_1, y_1, z_1)(t,\varepsilon) \to \begin{pmatrix} 0 \\ 0 \\ 0.5\left(Z_1^{0*}\cos(\omega_2 t) + Z_2^{0*}\sin(\omega_2 t)\right) \end{pmatrix} \text{ when } \varepsilon \longrightarrow 0$$

The pervious steps gives the complete proof of Theorem 1

Proof of Theorem 2

To prove this theorem we will follow the same steps of proving Theorem 1. Thus, the periodic solution can be written in the following form

$$\begin{pmatrix} 0.12\Big(Y_2^0\cos(\omega_1 t) - Y_1^0\sin(\omega_1 t)\Big) \\ -0.26\Big(Y_1^0\cos(\omega_1 t) + Y_2^0\sin(\omega_1 t)\Big) \\ -0.41\Big(Y_1^0\cos(\omega_1 t) + Y_2^0\sin(\omega_1 t)\Big) \\ -0.85\Big(Y_2^0\cos(\omega_1 t) - Y_1^0\sin(\omega_1 t)\Big) \\ 0 \\ 0 \end{pmatrix} \text{ when } \varepsilon \longrightarrow 0.$$

Hence, we get a periodic solution $(x_1, y_1, z_1)(t,\varepsilon)$ of System (13) such that

$$(x_1, y_1, z_1)(t,\varepsilon) \to \begin{pmatrix} 0.12\left(Y_2^0\cos(\omega_1 t) - Y_1^0\sin(\omega_1 t)\right) \\ -0.41\Big(Y_1^0\cos(\omega_1 t) + Y_2^0\sin(\omega_1 t)\Big) \\ 0 \end{pmatrix} \text{ when } \varepsilon \longrightarrow 0.$$

6. Proof of the corollaries 1 and 2

Proof of corollary 1

Under the aforementioned assumptions in Corollary 1, the non–linear System (14) can be written as

$$\begin{aligned} \mathcal{H}_1\left(Z_1^0, Z_2^0\right) = & -0.005468749997\left(Z_1^0\right)^4 Z_2^0 - 0.05177199997\left(Z_2^0\right)^2 - 0.01093750000\left(Z_1^0\right)^2\left(Z_2^0\right)^3 \\ & -0.1553160000\left(Z_1^0\right)^2 - 0.005468749997\left(Z_2^0\right)^5 \end{aligned}$$

$$\mathcal{H}_2\left(Z_1^0, Z_2^0\right) = 3.183098861 \times 10^{-12}\, Z_1^0 (6872233931.0 \left(Z_1^0\right)^2 \left(Z_2^0\right)^2 + 3436116965.0 \left(Z_1^0\right)^4 + 3436116965.0 \left(Z_2^0\right)^4 - 65058613960.0\, Z_2^0$$

Then the solution of the above system is

$$Z^{0*} = (0, -2.11).$$

Since

$$\left|\frac{\partial \mathcal{H}}{\partial Z^0}\right| = 0.21 \neq 0, \text{ when } Z^{0*} = (0\,,\, -2.11)$$

this solution is simple. Finally, by Theorem 1., we only have one periodic solution for this system and the proof is over.

Proof of corollary 2

Again under the aforementioned assumptions in Corollary 2, the non–linear system $\bar{\mathcal{H}}\left(Y^0\right) = 0$ can be written as

$$\mathcal{H}_3\left(Y_1^0, Y_2^0\right) = -0.02699999999\, Y_2^0 + 0.004649999999\, Y_1^0 + 0.0009962999998 \left(Y_1^0\right)^3 + 0.0009962999998 \left(Y_2^0\right)^2 Y_1^0,$$

$$\mathcal{H}_4\left(Y_1^0, Y_2^0\right) = -0.9549999997 + 0.004649999999\, Y_2^0 + 0.0009962999998 \left(Y_1^0\right)^2 Y_2^0 + 0.0009962999998 \left(Y_2^0\right)^3 + 0.02699999999\, Y_1^0,$$

The above system satisfy the following solution

$$Y^{0*} = (2.590\,,\, 9.210),$$

Moreover, since

$$\left|\frac{\partial \bar{\mathcal{H}}}{\partial Y^0}\right| = 0.020 \neq 0, \text{ when } Y^{0*} = (2.590\,,\, 9.210)$$

the obtained solution is simple, using Theorem 2, one can obtain only one periodic solution for this system, which gives a complete proof.

7. Conclusion

The averaging theory is one of the most important perturbation methods, which can be used to study the existence and stability of periodic solutions for the ordinary differential equations systems. It is a powerful tool as it has been proven its effectiveness many times in the literature by examining the existence and stability of periodicity of dynamical systems in both Physical and Engineering Sciences.

In this work, the dynamical system of the perturbed spatial Hill's problem by quantum corrections, which is called the spatial quantized Hill's problem is analyzed to find the possible periodic solutions. First, the importance of this problem is stated in the introduction section. While the differences and similarities among some perturbation approaches such as classical perturbation theory, Lindstedt—poincaré technique, multiple scales method, KB averaging method and, averaging theory are investigated in the second section. Then the equilibrium points of linear system are evaluated. The necessary conditions are analyzed to calculate the periodic solutions emerging from the equilibrium points of the SQHP by using the averaging theory. The application of this theory on the quantized Hill's problem has given interesting and important results about the periodic solution through the proof of Theorems (1, 2) and their associated corollaries.

We applied a known theory on a new model to state new results on such model. The difficulty of our proofs is to show that all hypotheses of the averaging theory of dynamical systems hold for the perturbed spatial quantized Hill's problem, in order to be able to apply the theorems of this theory. Changes of variables, obtaining the normal form of this theory, and many technical tricks are needed in this aim, which allows us to state dynamical information on the perturbed spatial quantized Hill's model. Furthermore, this model can be used to develop a Lunar theory and the families of periodic orbits in the frame work of the spatial quantized Hill's problem. Thereby, these applications serve to reinforce the obtained results about these periodic solutions and gain its own significance.

Conflicts of interest

The authors declare no conflict of interest.

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