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A Novel Configuration of the Fuzzy Elzaki Transform for Solving Nonlinear Partial Differential Equations via Fuzzy Fractional Derivative with General Order

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Abstract: The primary goal of this research is to generalize the definition of Caputo fractional derivatives (in short, CFDs) (of order $0 < \alpha < r$) by employing all conceivable configurations of objects with t_1 equal to 1 and t_2 (the others) equal to 2 via fuzzifications. Under $g\mathcal{H}$-differentiability, we also construct fuzzy Elzaki transforms for CFDs for the generic fractional order $\alpha \in (r-1, r)$. Furthermore, a novel decomposition method for obtaining the solutions to nonlinear fuzzy fractional partial differential equations (PDEs) via the fuzzy Elzaki transform is constructed. The aforesaid scheme is a novel correlation of the fuzzy Elzaki transform and the Adomian decomposition method. In terms of CFD, several new results for the general fractional order are obtained via $g\mathcal{H}$-differentiability. By considering the triangular fuzzy numbers of a nonlinear fuzzy fractional PDE, the correctness and capabilities of the proposed algorithm are demonstrated. In the domain of the fractional sense, the schematic representation and tabulated results outcomes indicate that the algorithm technique is precise and straightforward. Subsequently, future directions and concluding remarks are acted upon with the most focused use of references.

Keywords: fuzzy set theory; Elzaki transform; Adomian decomposition method; nonlinear partial differential equation; Caputo fractional derivative

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1. Introduction

The idea of differential and integral calculus is essential for stronger and more comprehensive descriptions of natural reality. It aids in the modeling of early evolution and forecasting the future of respective manifestations. Furthermore, thanks to its capability to express more fascinating ramifications of computer simulations, numerous researchers have subsequently been drawn to the investigation of fractional calculus [1–7].

Fractional calculus is particularly effective at modeling processes or systems relying on hereditary patterns and legacy conceptions, and traditional calculus is a restricted component of fractional calculus. This approach seems to be as ancient as a classical notion, but it has just subsequently been applied to the detection of convoluted frameworks by numerous investigators, and it has been demonstrated by various researchers. Fractional calculus has been advocated by a number of innovators, [8–14]. Many scholars analyze simulations depicting viruses, bifurcation, chaos, control theory, image processing, quantum fluid flow, and several other related disciplines using the underlying concepts and properties of operators shown within the framework of fractional calculus [15–28].

Fuzzy set theory is a valuable tool for modeling unpredictable phenomena. As a result, fuzzy conceptions are often leveraged to describe a variety of natural phenomena. Fuzzy PDEs are an excellent means of modeling vagueness and misinterpretation in certain quantities for specified real-life scenarios, see [29–33]. In recent years, FPDEs have been exploited in a variety of disciplines, notably in control systems, knowledge-based systems, image processing, power engineering, industrial automation, robotics, consumer electronics, artificial intelligence/expert systems, management, and operations research.

Due to its significance in a multitude of scientific domains, fuzzy set theory has a profound correlation with fractional calculus [34]. Kandel and Byatt [35] proposed fuzzy DEs in 1978, while Agarwal et al. [36] were the first to investigate fuzziness and the Riemann–Liouville differentiability concept under the Hukuhara-differentiability concept. Fuzzy set theory and FC both incorporate a number of computational methodologies that allow for a deeper understanding of dynamic structures while also minimizing the computation complexity of solving them. Determining precise analytical solutions in the case of FPDEs is a challenging task.

Owing to the model's intricacy, determining an analytical solution to PDEs is generally problematic. As a result, there is a developing trend in implementing mathematical approaches to obtain an exact solution. The Adomian decomposition method (ADM) is a prominent numerical approach that is widely used. Several researchers have employed different terminologies to address FPDEs. For resolving complex fuzzy PDEs, Nemati and Matinfar [37] constructed an implicit finite difference approach. Moreover, to demonstrate the competence and acceptability of the suggested methodology, experimental investigations incorporating parabolic PDEs were provided. According to Allahviranloo and Kermani [38], an explicit numerical solution to the fuzzy hyperbolic and parabolic equations is provided. The validity and resilience of the proposed system were investigated in order to demonstrate that it is inherently robust. Arqub et al. [39] employed the reproducing kernel algorithm for the solution of two-point fuzzy boundary value problems. The fuzzy Fredholm–Volterra integrodifferential equations were solved by the adaptation of the reproducing kernel algorithm by [40].

When it comes to discovering solutions to significant challenges, researchers prefer integral transformations. The Elzaki transformation [41], proposed by T. Elzaki in 2011, was used on a biological population model, the Fornberg–Whitham Model, and Fisher's models in [42–44].

The focus of this research is to suggest a sophisticated Adomian decomposition method [45] that can handle nonlinear partial fuzzy differential equations employing the fuzzy Elzaki transformation. A novel algorithmic approach is defined to construct the solution of nonlinear fuzzy fractional PDE. The nonlinear components of the problem are then handled using the Adomian polynomial [46] approach to achieve the solution. The fuzzy Elzaki method is the name given to the novel decomposition method.

In this research, CFDs of order $\alpha \in (0, r)$ for a fuzzy-valued mapping by employing all conceivable configurations of objects with t_1 equal to 1 and t_2 (the others) equal to 2 are presented. Moreover, a new result in connection of the Caputo fractional derivative and Elzaki transform via fuzzification is also presented. Taking into consideration $g\mathcal{H}$-differentiabilty for a new algorithm, the fuzzy Elzaki decomposition approach is used to construct the parametric form of the fuzzy mappings, which are considered to be a valuable tool for solving the fuzzy fractional nonlinear PDE under fuzzy initial conditions. The Elzaki transform applied here, in general, is a refinement of the Laplace and Sumudu transforms. A test problem for the proposed algorithm is presented via the different fractional order and uncertainty parameter, $\wp \in [0, 1]$. Furthermore, their 2D and 3D simulations show the applicability of the method over the other methods. As a consequence, each finding generates a pair of solutions that are closely in agreement with the existing one. However, we have the choice of how to attain the appropriate one. Finally, as a part of our concluding remarks, we discussed the accumulated facts of our findings.

The following is a synopsis of the persisting sections with regard to introduction and implementation: Section 2 represents the fundamentals and essential details of fractional calculus and fuzzy set theory. Section 3 concerns problem preparation, initialization, and processing. Section 4 concerns the Caputo fractional derivative formulation via fuzzification in generic order and some further results. Section 5 concerns numerical algorithms and mathematical debates with some tabulation and graphical results. Ultimately, Section 6 concerns conclusions and future highlights.

2. Preliminaries

This section consists of some significant concepts and results from fractional calculus and fuzzy set theory. For more details, see [4,5,13,47].

Here, $\mathbb{C}^F[\hat{a},\hat{b}]$ represents the space of all continuous fuzzy-valued mappings on $[\hat{a},\hat{b}]$. Moreover, the space of all Lebesgue integrable fuzzy-valued mappings on the bounded interval $[\hat{a},\hat{b}] \subset \mathbb{R}$ is represented by $\mathbb{L}^F[\hat{a},\hat{b}]$.

Definition 1 ([48])**.** *A fuzzy number is a mapping $\mathbf{f} : \mathbb{R} \mapsto [0,1]$, that fulfills the subsequent assumptions:*

(i) *$\mathbf{f}$ is upper semi-continuous on $\mathbb{R}$;*
(ii) *$\mathbf{f}(\mathbf{x}) = 0$ for some interval $[\hat{c},\hat{d}]$;*
(iii) *For $\hat{a},\hat{b} \in \mathbb{R}$ having $\hat{c} \leq \hat{a} \leq \hat{b} \leq \hat{d}$ such that $\mathbf{f}$ is increasing on $[\hat{c},\hat{a}]$ and decreasing on $[\hat{b},\hat{d}]$ and $\mathbf{f}(\mathbf{x}) = 1$ for every $\mathbf{x} \in [\hat{a},\hat{b}]$;*
(iv) *$\mathbf{f}(\wp\mathbf{x} + (1-\wp)\mathbf{y}) \geq \min\{\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{y})\}$ for every $\mathbf{x},\mathbf{y} \in \mathbb{R},\ \wp \in [0,1]$.*

The set of all fuzzy numbers is denoted by the letter E^1. If $\hat{a} \in \mathbb{R}$, it can be regarded as a fuzzy number; $\tilde{\hat{a}} = \chi_{\{\hat{a}\}}$ is the characteristic function, and therefore $\mathbb{R} \subset E^1$.

Definition 2 ([49])**.** *The $\wp$-level set of $\mathbf{f}$ is the crisp set $[\mathbf{f}]^{\wp}$, if $\wp \in [0,1]$ and $\mathbf{f} \in E^1$, then*

$$[\mathbf{f}]^{\wp} = \{\mathbf{x} \in \mathbb{R} : \mathbf{f}(\mathbf{x}) \geq \wp\}. \tag{1}$$

In addition, any $\wp$-level set is closed and bounded, signified by $[\underline{\mathbf{f}}(\wp), \bar{\mathbf{f}}(\wp)]$, $\forall \wp \in [0,1]$, where $\underline{\mathbf{f}}, \bar{\mathbf{f}} : [0,1] \mapsto \mathbb{R}$ are the lower and upper bounds of $[\mathbf{f}]^{\wp}$, respectively.

Definition 3 ([49])**.** *For each $\wp \in [0,1]$, a parameterize formulation of fuzzy number $\mathbf{f}$ is an ordered pair $\mathbf{f} = [\underline{\mathbf{f}}(\wp), \bar{\mathbf{f}}(\wp)]$ of mappings $\underline{\mathbf{f}}(\wp)$ and $\bar{\mathbf{f}}(\wp)$ that addresses the basic conditions:*

(i) *The mapping $\underline{\mathbf{f}}(\wp)$ is a bounded left continuous monotonic increasing in $[0,1]$;*
(ii) *The mapping $\bar{\mathbf{f}}(\wp)$ is a bounded left continuous monotonic decreasing in $[0,1]$;*
(iii) *$\underline{\mathbf{f}}(\wp) \leq \bar{\mathbf{f}}(\wp)$.*

Furthermore, the addition and scalar multiplication of fuzzy numbers $\mathbf{f}_1 = [\underline{\mathbf{f}}_1(\wp), \bar{\mathbf{f}}_1(\wp)]$ and $\mathbf{f}_2 = [\underline{\mathbf{f}}_2(\wp), \bar{\mathbf{f}}_2(\wp)]$ are presented as follows:

$$[\mathbf{f}_1 \oplus \mathbf{f}_2]^{\wp} = [\mathbf{f}_1]^{\wp} + [\mathbf{f}_2]^{\wp} = [\underline{\mathbf{f}}_1(\wp) + \underline{\mathbf{f}}_2(\wp), \bar{\mathbf{f}}_1(\wp) + \bar{\mathbf{f}}_2(\wp)] \ \ and \ \ [k \odot \mathbf{f}]^{\wp} = \begin{cases} [k\underline{\mathbf{f}}(\wp), k\bar{\mathbf{f}}(\wp)],\ k > 0, \\ [k\bar{\mathbf{f}}(\wp), k\underline{\mathbf{f}}(\wp)],\ k < 0. \end{cases} \tag{2}$$

As a distance between fuzzy numbers, we employ the Hausdorff metric.

Definition 4 ([48])**.** *Consider the two fuzzy numbers $\mathbf{f}_1 = [\underline{\mathbf{f}}_1(\wp), \bar{\mathbf{f}}_1(\wp)]$ and $\mathbf{f}_2 = [\underline{\mathbf{f}}_2(\wp), \bar{\mathbf{f}}_2(\wp)]$ defined on E^1. Then the distance between two fuzzy numbers is presented as follows:*

$$d(\mathbf{f}_1, \mathbf{f}_2) = \sup_{\wp \in [0,1]} \max\{|\underline{\mathbf{f}_1}(\wp) - \underline{\mathbf{f}}_2(\wp)|, |\bar{\mathbf{f}_1}(\wp) - \bar{\mathbf{f}}_2(\wp)|\}. \tag{3}$$

Definition 5 ([50]). *A fuzzy number* $\mathbf{f}$ *has the following forms:*

(i) *If* $\underline{\mathbf{f}}(1) \geq 0$, *then* $\mathbf{f}$ *is positive;*
(ii) *If* $\underline{\mathbf{f}}(1) > 0$, *then* $\mathbf{f}$ *is strictly positive;*
(iii) *If* $\bar{\mathbf{f}}(1) \leq 0$, *then* $\mathbf{f}$ *is negative;*
(iv) *If* $\underline{\mathbf{f}}(1) < 0$, *then* $\mathbf{f}$ *is strictly negative.*

The sets of positive and negative fuzzy numbers are denoted by E^+ *and* E^-, *respectively.*

Consider $\mathcal{D}$ as the set representing the domain of fuzzy-valued mappings $\mathbf{f}$. Define the mappings $\underline{\mathbf{f}}(.,.;\wp), \bar{\mathbf{f}}(.,.;\wp) : \mathcal{D} \mapsto \mathbb{R}$, $\forall \wp \in [0,1]$. These mappings are known to be the left and right $\wp$-level mappings of the map $\mathbf{f}$.

Definition 6 ([51]). *A fuzzy valued mapping* $\mathbf{f} : \mathcal{D} \mapsto E^1$ *is known to be continuous at* $(s_0, \xi_0) \in \mathcal{D}$ *if for every* $\epsilon > 0$ *exists* $\delta > 0$ *such that* $d(\mathbf{f}(s,\xi), \mathbf{f}(s_0,\xi_0)) < \epsilon$ *whenever* $|s - s_0| + |\xi - \xi_0| < \delta$. *If* $\mathbf{f}$ *is continuous for each* $(s_1, \xi_1) \in \mathcal{D}$, *then* $\mathbf{f}$ *is said to be continuous on* $\mathcal{D}$.

Definition 7 ([52]). *Suppose* $\mathbf{x}_1, \mathbf{x}_2 \in E^1$ *and* $\mathbf{y} \in E^1$ *such that the following holds:*

(i) $\mathbf{x}_1 = \mathbf{x}_2 \oplus \mathbf{y}$; *or*
(ii) $\mathbf{y} = \mathbf{x}_1 \oplus (-1) \odot \mathbf{x}_2$.

Then, $\mathbf{y}$ *is known to be the generalized Hukuhara difference* (${}_{g\mathcal{H}}$*-difference) of fuzzy numbers* $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *and is denoted by* $\mathbf{x}_1 \ominus {}_{g\mathcal{H}} \mathbf{x}_2$.

Again, suppose $\mathbf{x}_1, \mathbf{x}_2 \in E^1$, *then* $\mathbf{x}_1 \ominus {}_{g\mathcal{H}} \mathbf{x}_2 = \mathbf{y} \Leftrightarrow$

(i) $\mathbf{y} = (\underline{\mathbf{x}_1}(\wp) - \underline{\mathbf{x}_2}(\wp), \bar{\mathbf{x}_1}(\wp) - \bar{\mathbf{x}_2}(\wp))$; *or*
(ii) $\mathbf{y} = (\bar{\mathbf{x}_1}(\wp) - \bar{\mathbf{x}_2}(\wp), \underline{\mathbf{x}_1}(\wp) - \underline{\mathbf{x}_2}(\wp))$.

The connection between the ${}_{g\mathcal{H}}$-difference and the Housdroff distance is demonstrated by the following Lemma.

Lemma 1 ([52]). *For all* $\mathbf{f}_1, \mathbf{f}_2 \in E^1$, *then*

$$d(\mathbf{f}_1, \mathbf{f}_2) = \sup_{\wp \in [0,1]} \|[\mathbf{f}_1]^{\wp} \ominus {}_{g\mathcal{H}} [\mathbf{f}_2]^{\wp}\|, \tag{4}$$

where, for an interval $[\hat{a}, \hat{b}]$, *the norm is* $\|[\hat{a}, \hat{b}]\| = \max\{|\hat{a}|, |\hat{b}|\}$.

Definition 8 ([53]). *Let* $\mathbf{f} : \mathcal{D} \mapsto E^1$ *and* $(\mathbf{x}_0, \xi) \in \mathcal{D}$. *A mapping* $\mathbf{f}$ *is known as the strongly strongly generalized Hukuhara differentiable on* $(\mathbf{x}_0, \xi)$ (${}_{g\mathcal{H}}$*-differentiable for short) if there exists an element* $\frac{\partial f_2(\mathbf{x}_0,\xi)}{\partial \mathbf{x}} \in E^1$, *then the subsequent holds:*

(i) *The following* ${}_{g\mathcal{H}}$*-differences exist, if* $\forall\, \epsilon > 0$ *sufficiently small, then*

$$\mathbf{f}(\mathbf{x}_0 + \epsilon, \xi) \ominus {}_{g\mathcal{H}} \mathbf{f}(\mathbf{x}_0, \xi), \quad \mathbf{f}(\mathbf{x}_0, \xi) \ominus {}_{g\mathcal{H}} \mathbf{f}(\mathbf{x}_0 + \epsilon, \xi),$$

the following limits hold as:

$$\lim_{\epsilon \mapsto 0} \frac{\mathbf{f}(\mathbf{x}_0 + \epsilon, \xi) \ominus {}_{g\mathcal{H}} \mathbf{f}(\mathbf{x}_0, \xi)}{\epsilon} = \lim_{\epsilon \mapsto 0} \frac{\mathbf{f}(\mathbf{x}_0, \xi) \ominus {}_{g\mathcal{H}} \mathbf{f}(\mathbf{x}_0 + \epsilon, \xi)}{\epsilon} = \frac{\partial \mathbf{f}(\mathbf{x}_0, \xi)}{\partial \mathbf{x}}. \tag{5}$$

(ii) *The following* ${}_{g\mathcal{H}}$*-differences exist, if* $\forall\, \epsilon > 0$ *sufficiently small, then*

$$\mathbf{f}(\mathbf{x}_0, \xi) \ominus {}_{g\mathcal{H}} \mathbf{f}(\mathbf{x}_0 + \epsilon, \xi), \quad \mathbf{f}(\mathbf{x}_0 - \epsilon, \xi) \ominus {}_{g\mathcal{H}} \mathbf{f}(\mathbf{x}_0, \xi),$$

the following limits hold as:

$$\lim_{\epsilon \mapsto 0} \frac{\mathbf{f}(\mathbf{x}_0, \xi) \ominus {}_{g\mathcal{H}} \mathbf{f}(\mathbf{x}_0 + \epsilon, \xi)}{-\epsilon} = \lim_{\epsilon \mapsto 0} \frac{\mathbf{f}(\mathbf{x}_0 - \epsilon, \xi) \ominus {}_{g\mathcal{H}} \mathbf{f}(\mathbf{x}_0, \xi)}{-\epsilon} = \frac{\partial \mathbf{f}(\mathbf{x}_0, \xi)}{\partial \mathbf{x}}. \tag{6}$$

Lemma 2 ([54]). *Suppose a continuous fuzzy-valued mapping* $\mathbf{f} : \mathcal{D} \mapsto E^1$ *and* $\mathbf{f}(\mathbf{x}, \xi) = \left[\underline{\mathbf{f}}(\mathbf{x}, \xi; \wp), \bar{\mathbf{f}}(\mathbf{x}, \xi; \wp)\right]$, $\forall \wp \in [0, 1]$. *Then*

(i) *If* $\mathbf{f}(\mathbf{x}, \xi)$ *is (i)-differentiable for* $\mathbf{x}$ *under Definition 8(i), then we have the following:*

$$\frac{\partial \mathbf{f}(\mathbf{x}_0, \xi)}{\partial \mathbf{x}} = \left(\frac{\partial \underline{\mathbf{f}}(\mathbf{x}_0, \xi)}{\partial \mathbf{x}}, \frac{\partial \bar{\mathbf{f}}(\mathbf{x}_0, \xi)}{\partial \mathbf{x}}\right); \tag{7}$$

(ii) *If* $\mathbf{f}(\mathbf{x}, \xi)$ *is (ii)-differentiable for* $\mathbf{x}$ *under Definition 8(ii), then we have the following:*

$$\frac{\partial \mathbf{f}(\mathbf{x}_0, \xi)}{\partial \mathbf{x}} = \left(\frac{\partial \bar{\mathbf{f}}(\mathbf{x}_0, \xi)}{\partial \mathbf{x}}, \frac{\partial \underline{\mathbf{f}}(\mathbf{x}_0, \xi)}{\partial \mathbf{x}}\right). \tag{8}$$

Theorem 1 ([55]). *Suppose* $\mathbf{f} : \mathbb{R}^+ \mapsto \mathbb{E}^{\Bbbk}$ *and* $\forall\ \wp \in [0, 1]$.

(i) *The mappings* $\underline{\mathbf{f}}(\mathbf{x}; \xi; \wp)$ *and* $\bar{\mathbf{f}}(\mathbf{x}; \xi; \wp)$ *are Riemann-integrable on* $[0, \hat{b}]$ *for every* $\hat{b} \geq 0$.

(ii) $\underline{\mathcal{M}}(\wp) > 0$ *and* $\bar{\mathcal{M}}(\wp) > 0$ *are the constants, then*

$$\int_0^{\hat{b}} |\underline{\mathbf{f}}(\mathbf{x}; \xi; \wp)| d\mathbf{x} \leq \underline{\mathcal{M}}(\wp), \quad \int_0^{\hat{b}} |\bar{\mathbf{f}}(\mathbf{x}; \xi; \wp)| d\mathbf{x} \leq \bar{\mathcal{M}}(\wp), \ \ \forall\ \hat{b} \geq 0.$$

Then, the mapping $\mathbf{f}$ *is improper fuzzy Riemann-integrable on* $[0, \infty)$ *and the following holds:*

$$\mathcal{FR} \int_0^{\infty} \mathbf{f}(\mathbf{x}) d\mathbf{x} = \left(\int_0^{\infty} \underline{\mathbf{f}}(\mathbf{x}; \wp) d\mathbf{x}, \int_0^{\infty} \bar{\mathbf{f}}(\mathbf{x}; \wp) d\mathbf{x}\right). \tag{9}$$

Theorem 2 ([1]). *Suppose there is a positive integer* r *and a continuous mapping* $\mathbf{D}^{r-1}\mathbf{f}$ *defined on* $\mathcal{J} = [0, \infty)$ *and a collection of piece wise continuous mappings* $\mathbb{C}$ *defined on* $\mathcal{J}' = (0, \infty)$ *is integrable on finite sub-interval of* $\mathcal{J} = [0, \infty)$ *and assume that* $\nu > 0$. *Then*

(i) *If* $\mathbf{D}^r\mathbf{f}$ *is in* $\mathbb{C}$, *then*

$$\mathbf{D}^{-\nu}\mathbf{f}(\mathbf{x}) = \mathbf{D}^{-\nu-r}\left[\mathbf{D}^r\mathbf{f}(\mathbf{x})\right] + X_r(\mathbf{x}, \nu)$$

and

(ii) *If there is a continuous mapping* $\mathbf{D}^r\mathbf{f}$ *on* $\mathcal{J}$, *then for* $\mathbf{x} > 0$

$$\mathbf{D}^r\left[\mathbf{D}^{-\nu}\mathbf{f}(\mathbf{x})\right] = \mathbf{D}^{-\nu}\left[\mathbf{D}^r\mathbf{f}(\mathbf{x})\right] + X_r(\mathbf{x}, \nu - r),$$

where

$$X_r(\mathbf{x}, \nu) = \sum_{\kappa=0}^{r-1} \frac{\mathbf{x}^{\nu+\kappa}}{\Gamma(\nu+\kappa+1)} \mathbf{D}^{\kappa}\mathbf{f}(0).$$

3. Fuzzy Elzaki Transform

Definition 9 ([41]). *Suppose a continuous fuzzy-valued mapping* $\mathbf{f} : \mathbb{R}^+ \mapsto E^1$ *and for* $\omega > 0$, *there is an improper fuzzy Riemann-integrable mapping* $\mathbf{f}(\xi) \odot \exp(-\xi/\omega)$ *defined on* $[0, \infty)$. *Then we have*

$$\mathcal{FR} \int_0^{\infty} \omega \mathbf{f}(\xi) \exp(-\xi/\omega) d\xi, \ \ \omega \in (p_1, p_2),$$

which is known as the Fuzzy Elzaki transform and represented as

$$\mathcal{W}(\omega) = \mathbb{E}[\mathbf{f}(\xi)] = \mathcal{FR} \int_0^{\infty} \omega \mathbf{f}(\xi) \exp(-\xi/\omega) d\xi.$$

The parameterized version of fuzzy Elzaki transform:

$$\mathbb{E}\big[\mathbf{f}(\xi)\big] = \Big[\mathcal{E}\big[\underline{\mathbf{f}}(\xi;\wp)\big], \mathcal{E}\big[\bar{\mathbf{f}}(\xi;\wp)\big]\Big],$$

where

$$\mathcal{E}\big[\underline{\mathbf{f}}(\xi;\wp)\big] = \int_0^\infty \omega \underline{\mathbf{f}}(\xi;\wp)\exp(-\xi/\omega)d\xi,$$

$$\mathcal{E}\big[\bar{\mathbf{f}}(\xi;\wp)\big] = \int_0^\infty \omega \bar{\mathbf{f}}(\xi;\wp)\exp(-\xi/\omega)d\xi.$$

4. Fuzzy Elzaki Transform of the Fuzzy CFDs of Orders $r-1<\alpha<r$

This section consists of CFDs of the general fractional order $0<\alpha<r$. Moreover, we obtain the fuzzy Elzaki transform for CFD of the generic order $r-1<\alpha<r$ for fuzzy valued mapping $\mathbf{f}$ under $g\mathcal{H}$-differentiability.

For the sake of simplicity, for $0<\alpha<r$ and $\mathbf{f}(\mathbf{x}) \in \mathbb{C}^F[0,\hat{b}] \bigcap \mathbb{L}^F[0,\hat{b}]$, denoting

$$\mathcal{G}(\mathbf{x}) = \frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)} \int_0^{\mathbf{x}} \frac{\mathbf{f}(\xi)d\xi}{(\mathbf{x}-\xi)^{1-\lceil\alpha\rceil+\alpha}} \ominus \sum_{\kappa=0}^{\lceil\alpha\rceil-\alpha} \frac{\mathbf{D}^\kappa \mathbf{f}(0)\mathbf{x}^{\lceil\alpha\rceil-\alpha+\kappa}}{\Gamma(1+\lceil\alpha\rceil-\alpha+\kappa)}. \tag{10}$$

Definition 10. *Suppose $\mathbf{f}(\mathbf{x}) \in \mathbb{C}^F[0,\hat{b}] \bigcap \mathbb{L}^F[0,\hat{b}]$ and $\lceil\alpha\rceil$ and $\lfloor\alpha\rfloor$ indicates α values that have been rounded forward and descend to the closest integer value, respectively. It is clear that $\mathcal{G}(\mathbf{x})$ and the mappings $\mathcal{G}_{\jmath_1,\jmath_2,\ldots,\jmath_\iota,1}$ and $\mathcal{G}_{\jmath_1,\jmath_2,\ldots,\jmath_\iota,2}$ are stated as*

$$\mathcal{G}_{\jmath_1,\jmath_2,\ldots,\jmath_\iota,1}(\mathbf{x}_0) = \lim_{\epsilon\mapsto 0^+} \frac{\mathcal{G}_{\jmath_1,\jmath_2,\ldots,\jmath_\iota}(\mathbf{x}_0+\epsilon) \ominus \mathcal{G}_{\jmath_1,\jmath_2,\ldots,\jmath_\iota}(\mathbf{x}_0)}{\epsilon} = \lim_{\epsilon\mapsto 0^+} \frac{\mathcal{G}_{\jmath_1,\jmath_2,\ldots,\jmath_\iota}(\mathbf{x}_0) \ominus \mathcal{G}_{\jmath_1,\jmath_2,\ldots,\jmath_\iota}(\mathbf{x}_0-\epsilon)}{\epsilon}, \tag{11}$$

$$\mathcal{G}_{\jmath_1,\jmath_2,\ldots,\jmath_\iota,2}(\mathbf{x}_0) = \lim_{\epsilon\mapsto 0^+} \frac{\mathcal{G}_{\jmath_1,\jmath_2,\ldots,\jmath_\iota}(\mathbf{x}_0) \ominus \mathcal{G}_{\jmath_1,\jmath_2,\ldots,\jmath_\iota}(\mathbf{x}_0+\epsilon)}{-\epsilon} = \lim_{\epsilon\mapsto 0^+} \frac{\mathcal{G}_{\jmath_1,\jmath_2,\ldots,\jmath_\iota}(\mathbf{x}_0-\epsilon) \ominus \mathcal{G}_{\jmath_1,\jmath_2,\ldots,\jmath_\iota}(\mathbf{x}_0)}{-\epsilon}, \tag{12}$$

for $\iota = 0,1,2,\ldots,r-2$ such that $\jmath_1,\jmath_2,\ldots,\jmath_\iota$ are all possible arrangements of ι objects that represents the numbers in the following principal:

$$\iota P_{t_1 t_2} = \frac{\iota!}{t_1! t_2!}, \;\; t_1+t_2=\iota,$$

where t_1 of them equal 1 (means CD in the first version) and t_2 of them equal 2 (means CD in the second version). Furthermore, $\jmath_1,\jmath_2,\ldots,\jmath_0$.

Now, $\mathbf{f}(\mathbf{x})$ is the Caputo fractional type fuzzy differentiable mapping of order $0<\alpha<r$, $\alpha \neq 1,2,\ldots r-1$ at $\mathbf{x}_0 \in (0,\hat{b})$ if $\exists$ an element $\big({}^c\mathbf{D}^\alpha\mathbf{f}\big)(\mathbf{x}_0) \in \mathbb{C}^F$ such that $\forall\wp \in [0,1]$ and for $\epsilon>0$ close to zero. Then

(i) *If $\jmath_{\lceil\alpha\rceil}=1$, then*

$$\big({}^c\mathbf{D}^\alpha\mathbf{f}\big)(\mathbf{x}_0) = \lim_{\epsilon\mapsto 0^+} \frac{\mathcal{G}_{\jmath_1,\jmath_2,\ldots,\jmath_{\lceil\alpha\rceil}}(\mathbf{x}_0+\epsilon) \ominus \mathcal{G}_{\jmath_1,\jmath_2,\ldots,\jmath_\iota}(\mathbf{x}_0)}{\epsilon} = \lim_{\epsilon\mapsto 0^+} \frac{\mathcal{G}_{\jmath_1,\jmath_2,\ldots,\jmath_{\lceil\alpha\rceil}}(\mathbf{x}_0) \ominus \mathcal{G}_{\jmath_1,\jmath_2,\ldots,\jmath_{\lceil\alpha\rceil}}(\mathbf{x}_0-\epsilon)}{\epsilon}; \tag{13}$$

(ii) *If $\jmath_{\lceil\alpha\rceil}=2$, then*

$$\big({}^c\mathbf{D}^\alpha\mathbf{f}\big)(\mathbf{x}_0) = \lim_{\epsilon\mapsto 0^+} \frac{\mathcal{G}_{\jmath_1,\jmath_2,\ldots,\jmath_{\lceil\alpha\rceil}}(\mathbf{x}_0) \ominus \mathcal{G}_{\jmath_1,\jmath_2,\ldots,\jmath_\iota}(\mathbf{x}_0+\epsilon)}{-\epsilon} = \lim_{\epsilon\mapsto 0^+} \frac{\mathcal{G}_{\jmath_1,\jmath_2,\ldots,\jmath_{\lceil\alpha\rceil}}(\mathbf{x}_0-\epsilon) \ominus \mathcal{G}_{\jmath_1,\jmath_2,\ldots,\jmath_{\lceil\alpha\rceil}}(\mathbf{x}_0)}{-\epsilon}, \tag{14}$$

for $\alpha \in (\kappa - 1, \kappa)$, $\kappa = 1, 2, \ldots, r$ *such that* $\jmath_1, \jmath_2, \ldots, \jmath\lceil\alpha\rceil$ *are all the suitable arrangements of* $\lceil\alpha\rceil$ *objects that have the following representation:*

$$\lceil\alpha\rceil P_{t_1 t_2} = \frac{\lceil\alpha\rceil!}{t_1! t_2!}, \; t_1 + t_2 = \lceil\alpha\rceil.$$

Theorem 3. *Suppose* $\mathbf{f}(\mathbf{x}) \in \mathbb{C}^F[0, \hat{b}] \bigcap \mathbb{L}^F[0, \hat{b}]$ *be a fuzzy-valued mapping such that* $\mathbf{f}(\mathbf{x}) = \left[\underline{\mathbf{f}}(\mathbf{x}; \wp), \bar{\mathbf{f}}(\mathbf{x}; \wp)\right]$ *for* $\wp \in [0, 1]$, $\mathbf{x}_0 \in (0, \hat{b})$ *and* $\mathcal{G}(\mathbf{x})$ *is stated in* (10).

Assume that $0 < \alpha < r$ *and* ℓ *is the number of repetitions of 2 among* $\jmath_1, \jmath_2, \ldots, \jmath_{\lceil\alpha\rceil}$ *for* $\alpha \in (\kappa - 1, \kappa)$, $\kappa = 1, 2, \ldots, r$, *say,* $\jmath_{\kappa_1}, \jmath_{\kappa_2}, \ldots, \jmath_{\kappa_\ell}$ *such that* $\kappa_1 < \kappa_2 < \ldots < \kappa_\ell$, *i.e.,* $\jmath_{\kappa_1} = \jmath_{\kappa_2} = \ldots = \jmath_{\kappa_\ell} = 2$ *and* $0 \leq \ell \leq \lceil\alpha\rceil$. *Then we have the following*

If ℓ *is even number, then*

$$\left({}^c\mathbf{D}^{\beta}_{\jmath_1, \jmath_2, \ldots \jmath_{\lceil\alpha\rceil}} \mathbf{f}\right)(\mathbf{x}_0) = \left[\left({}^c\mathbf{D}^{\alpha}\underline{\mathbf{f}}\right)(\mathbf{x}_0; \wp), \left({}^c\mathbf{D}^{\alpha}\bar{\mathbf{f}}\right)(\mathbf{x}_0; \wp)\right]. \tag{15}$$

If ℓ *is odd number, then*

$$\left({}^c\mathbf{D}^{\beta}_{\jmath_1, \jmath_2, \ldots \jmath_{\lceil\alpha\rceil}} \mathbf{f}\right)(\mathbf{x}_0) = \left[\left({}^c\mathbf{D}^{\alpha}\bar{\mathbf{f}}\right)(\mathbf{x}_0; \wp), \left({}^c\mathbf{D}^{\alpha}\underline{\mathbf{f}}\right)(\mathbf{x}_0; \wp)\right], \tag{16}$$

where

$$\begin{aligned}
\left({}^c\mathbf{D}^{\alpha}\underline{\mathbf{f}}\right)(\mathbf{x}_0; \wp) &= \left[\frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \int\limits_0^{\mathbf{x}} \frac{\mathbf{D}^{\lceil\alpha\rceil}\underline{\mathbf{f}}(\xi; \wp) d\xi}{(\mathbf{x} - \xi)^{1 - \lceil\alpha\rceil + \alpha}}\right]_{\mathbf{x} = \mathbf{x}_0}, \\
\left({}^c\mathbf{D}^{\alpha}\bar{\mathbf{f}}\right)(\mathbf{x}_0; \wp) &= \left[\frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \int\limits_0^{\mathbf{x}} \frac{\mathbf{D}^{\lceil\alpha\rceil}\bar{\mathbf{f}}(\xi; \wp) d\xi}{(\mathbf{x} - \xi)^{1 - \lceil\alpha\rceil + \alpha}}\right]_{\mathbf{x} = \mathbf{x}_0}, \quad \mathbf{D}^{\kappa}\mathbf{f}(\xi) = \frac{d^{\kappa}\mathbf{f}(\xi)}{d\xi^{\kappa}}.
\end{aligned} \tag{17}$$

Proof. Let ℓ be an even number and then $\ell = 2s_1$, $s_1 \in \mathbb{N}$. Here, we have two assumptions as follows:

The first assumption is $\left({}^c\mathbf{D}^{\beta}_{\jmath_1, \ldots, \jmath_{\kappa_1}, \ldots, \jmath_{\kappa_2}, \ldots, \jmath_{\lceil\alpha\rceil}} \mathbf{f}\right)(\mathbf{x}_0)$ is the Caputo type fuzzy fractional differentiable mapping in the first form ($\jmath_{\lceil\alpha\rceil} = 1$) and in view of (13) from Definition 10, we have

$$\begin{aligned}
&\mathcal{G}_{\jmath_1, \ldots, \jmath_{\lceil\alpha\rceil}}(\mathbf{x}_0 + \epsilon) \ominus \mathcal{G}_{\jmath_1, \ldots, \jmath_{\lceil\alpha\rceil}}(\mathbf{x}_0) \\
&= \left[\underline{\mathcal{G}}_{\jmath_1, \ldots, \jmath_{\lceil\alpha\rceil}}(\mathbf{x}_0 + \epsilon; \wp) - \underline{\mathcal{G}}_{\jmath_1, \ldots, \jmath_{\lceil\alpha\rceil}}(\mathbf{x}_0; \wp), \bar{\mathcal{G}}_{\jmath_1, \ldots, \jmath_{\lceil\alpha\rceil}}(\mathbf{x}_0 + \epsilon; \wp) - \bar{\mathcal{G}}_{\jmath_1, \ldots, \jmath_{\lceil\alpha\rceil}}(\mathbf{x}_0; \wp)\right], \\
&\mathcal{G}_{\jmath_1, \ldots, \jmath_{\lceil\alpha\rceil}}(\mathbf{x}_0) \ominus \mathcal{G}_{\jmath_1, \ldots, \jmath_{\lceil\alpha\rceil}}(\mathbf{x}_0 - \epsilon) \\
&= \left[\underline{\mathcal{G}}_{\jmath_1, \ldots, \jmath_{\lceil\alpha\rceil}}(\mathbf{x}_0; \wp) - \underline{\mathcal{G}}_{\jmath_1, \ldots, \jmath_{\lceil\alpha\rceil}}(\mathbf{x}_0 - \epsilon; \wp), \bar{\mathcal{G}}_{\jmath_1, \ldots, \jmath_{\lceil\alpha\rceil}}(\mathbf{x}_0; \wp) - \bar{\mathcal{G}}_{\jmath_1, \ldots, \jmath_{\lceil\alpha\rceil}}(\mathbf{x}_0 - \epsilon; \wp)\right].
\end{aligned} \tag{18}$$

Conducting product on both sides by $1/\epsilon$, $\epsilon > 0$, and then applying $\epsilon \mapsto^+$, yields

$$\left({}^{\mathcal{RL}}\mathbf{D}^{\alpha}\mathbf{f}\right)(\mathbf{x}_0) = \left[\frac{d}{d\mathbf{x}}\underline{\mathcal{G}}_{\jmath_1, \ldots, \jmath_{\lceil\alpha\rceil}}(\mathbf{x}_0; \wp), \frac{d}{d\mathbf{x}}\bar{\mathcal{G}}_{\jmath_1, \ldots, \jmath_{\lceil\alpha\rceil}}(\mathbf{x}_0; \wp)\right]. \tag{19}$$

Thus, $\mathcal{G}_{\jmath_1, \ldots, \jmath_{\kappa_1 - 1}}$ is identical to the specified restrictions mentioned in (11) of Definition 10, then by employing (11) for $(\kappa_1 - 1)$ times, we have that

$$\mathcal{G}_{\jmath_1, \ldots, \jmath_{\kappa_1} - 1}(\mathbf{x}_0) = \left[\underline{\mathcal{G}}^{(\kappa_1 - 1)}(\mathbf{x}_0; \wp), \bar{\mathcal{G}}^{(\kappa_1 - 1)}(\mathbf{x}_0; \wp)\right]. \tag{20}$$

Since $\mathcal{G}_{\jmath_1}, .., \jmath_{\kappa_1}(\mathbf{x}_0)$ is identical to the specified restrictions stated in (12) of Definition 10, then by employing (12), we have

$$\mathcal{G}_{\jmath_1, \ldots, \jmath_{\kappa_1 - 1}}(\mathbf{x}_0) = \left[\bar{\mathcal{G}}^{(\kappa_1)}(\mathbf{x}_0; \wp), \underline{\mathcal{G}}^{(\kappa_1)}(\mathbf{x}_0; \wp)\right]. \tag{21}$$

Since $\mathcal{G}_{\jmath_1}, \ldots, \jmath_{\kappa_2 - 1}(\mathbf{x}_0)$ is identical to the specified restrictions stated in (11) of Definition 10 then by employing (11), we have

$$\mathcal{G}_{\jmath_1,\ldots,\jmath_{\kappa_2-1}}(\mathbf{x}_0) = \left[\bar{\mathcal{G}}^{(\kappa_2-1)}(\mathbf{x}_0;\wp), \underline{\mathcal{G}}^{(\kappa_2-1)}(\mathbf{x}_0;\wp)\right]. \tag{22}$$

Since $\mathcal{G}_{\jmath_1}, \ldots, \jmath_{\kappa_2}(\mathbf{x}_0)$ is identical to the specified restrictions stated in (12) of Definition 10 then by employing (12), we have

$$\mathcal{G}_{\jmath_1,\ldots,\jmath_{\kappa_2}}(\mathbf{x}_0) = \left[\underline{\mathcal{G}}^{(\kappa_2-1)}(\mathbf{x}_0;\wp), \bar{\mathcal{G}}^{(\kappa_2-1)}(\mathbf{x}_0;\wp)\right]. \tag{23}$$

On the other hand, from (23) we notice that we will have a similar equation, following the application of (11) and (12) for any even number of $\jmath_{\kappa_1}, \jmath_{\kappa_2}, \ldots, \jmath_{\kappa_m}$ of (23). Thus, for $\mathcal{G}_{\jmath_1,\ldots,\jmath_{2s_1}}(\mathbf{x}_0)$, we have

$$\mathcal{G}_{\jmath_1,\ldots,\jmath_{\kappa_{2s_1}}}(\mathbf{x}_0) = \left[\underline{\mathcal{G}}^{(\kappa_{2s_1})}(\mathbf{x}_0;\wp), \bar{\mathcal{G}}^{(\kappa_{2s_1})}(\mathbf{x}_0;\wp)\right], \tag{24}$$

where $2s_1$ is an even number.

Consequently, since $\mathcal{G}_{\jmath_1}, \ldots, \jmath_{\lceil\alpha\rceil}(\mathbf{x}_0)$ is identical to the specified restrictions stated in (11) of Definition 10 then by employing (11) for $(\lceil\alpha\rceil - \kappa_{2s_1})$, we have

$$\mathcal{G}_{\jmath_1,\ldots,\jmath_{\lceil\alpha\rceil}}(\mathbf{x}_0) = \left[\underline{\mathcal{G}}^{(\lceil\alpha\rceil)}(\mathbf{x}_0;\wp), \bar{\mathcal{G}}^{(\lceil\alpha\rceil)}(\mathbf{x}_0;\wp)\right], \tag{25}$$

then, we have

$$\begin{aligned} \underline{\mathcal{G}}_{\jmath_1,\ldots,\jmath_{\lceil\alpha\rceil}}(\mathbf{x}_0;\wp) &= \underline{\mathcal{G}}^{(\lceil\alpha\rceil)}(\mathbf{x}_0;\wp), \\ \bar{\mathcal{G}}_{\jmath_1,\ldots,\jmath_{\lceil\alpha\rceil}}(\mathbf{x}_0;\wp) &= \bar{\mathcal{G}}^{(\lceil\alpha\rceil)}(\mathbf{x}_0;\wp). \end{aligned} \tag{26}$$

Plugging (26) and (19) gives the subsequent

$$({}^{c}\mathbf{D}^{\alpha}\mathbf{f})(\mathbf{x}_0) = \left[\mathbf{D}^{\lceil\beta\rceil}\underline{\mathcal{G}}(\mathbf{x}_0;\wp), \mathbf{D}^{\lceil\beta\rceil}\bar{\mathcal{G}}(\mathbf{x}_0;\wp)\right], \quad \mathbf{D} = d/d\mathbf{x}. \tag{27}$$

Thus,

$$\begin{aligned} ({}^{c}\mathbf{D}^{\alpha}\mathbf{f})(\mathbf{x}_0) = & \left[\mathbf{D}^{\lceil\beta\rceil}\left(\frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}\int_0^{\mathbf{x}}\frac{\underline{\mathbf{f}}(\xi;\wp)d\xi}{(\mathbf{x}-\xi)^{1-\lceil\alpha\rceil+\alpha}} - \sum_{\kappa=0}^{\lceil\alpha\rceil-\alpha}\frac{\mathbf{D}^{\kappa}\underline{\mathbf{f}}(0)\mathbf{x}^{\lceil\alpha\rceil-\alpha+\kappa}}{\Gamma(1+\lceil\alpha\rceil-\alpha+\kappa)}\right)\Bigg|_{\mathbf{x}=\mathbf{x}_0},\right. \\ & \left.\mathbf{D}^{\lceil\beta\rceil}\left(\frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}\int_0^{\mathbf{x}}\frac{\bar{\mathbf{f}}(\xi;\wp)d\xi}{(\mathbf{x}-\xi)^{1-\lceil\alpha\rceil+\alpha}} - \sum_{\kappa=0}^{\lceil\alpha\rceil-\alpha}\frac{\mathbf{D}^{\kappa}\bar{\mathbf{f}}(0)\mathbf{x}^{\lceil\alpha\rceil-\alpha+\kappa}}{\Gamma(1+\lceil\alpha\rceil-\alpha+\kappa)}\right)\Bigg|_{\mathbf{x}=\mathbf{x}_0}\right]. \end{aligned} \tag{28}$$

Utilizing the fact of (10) we have

$$\begin{aligned} ({}^{c}\mathbf{D}^{\alpha}\mathbf{f})(\mathbf{x}_0) = & \left[\mathbf{D}^{\lceil\alpha\rceil}(\mathbf{D}^{-(\lceil\alpha\rceil-\alpha)}\underline{\mathbf{f}})(\mathbf{x}_0;\wp) - \left(\sum_{\kappa=0}^{\lceil\alpha\rceil}\frac{\mathbf{D}^{\kappa}\underline{\mathbf{f}}(0;\wp)\mathbf{D}^{\lceil\alpha\rceil}\mathbf{x}^{\lceil\alpha\rceil-\alpha+\kappa}}{\Gamma(1+\lceil\alpha\rceil-\alpha+\kappa)}\right)\Bigg|_{\mathbf{x}=\mathbf{x}_0},\right. \\ & \left.\mathbf{D}^{\lceil\alpha\rceil}(\mathbf{D}^{-(\lceil\alpha\rceil-\alpha)}\bar{\mathbf{f}})(\mathbf{x}_0;\wp) - \left(\sum_{\kappa=0}^{\lceil\alpha\rceil}\frac{\mathbf{D}^{\kappa}\bar{\mathbf{f}}(0;\wp)\mathbf{D}^{\lceil\alpha\rceil}\mathbf{x}^{\lceil\alpha\rceil-\alpha+\kappa}}{\Gamma(1+\lceil\alpha\rceil-\alpha+\kappa)}\right)\Bigg|_{\mathbf{x}=\mathbf{x}_0}\right], \end{aligned} \tag{29}$$

where $(\mathbf{D}^{-(\lceil\alpha\rceil-\alpha)}\underline{\mathbf{f}})(\mathbf{x}_0;\wp)$ and $(\mathbf{D}^{-(\lceil\alpha\rceil-\alpha)}\bar{\mathbf{f}})(\mathbf{x}_0;\wp)$ are the $\mathcal{RL}$ fractional integrals of the mappings $\underline{\mathbf{f}}(\mathbf{x}_0;\wp)$ and $\bar{\mathbf{f}}(\mathbf{x}_0;\wp)$ at $\mathbf{x} = \mathbf{x}_0$, respectively. By the use of continuity of $\mathbf{D}^r\mathbf{f}$ having $r = \lceil\alpha\rceil$, $\nu = \lceil\alpha\rceil - \alpha$ and by the virtue of Theorem 2, $\mathbf{D}^r\mathbf{x}^{\ell} = \frac{\Gamma(\ell+1)}{\Gamma(\ell+1-r)}\mathbf{x}^{\ell-r}$, we have

$$\left({}^{c}\mathbf{D}^{\alpha}\mathbf{f}\right)(\mathbf{x}_0)= \left[\mathbf{D}^{-(\lceil\alpha\rceil-\alpha)}\left(\mathbf{D}^{\lceil\alpha\rceil}\underline{\mathbf{f}}(\mathbf{x}_0;\wp)\right)+\underline{\mathcal{Q}}(\mathbf{x}_0,-\alpha)-\sum_{\kappa=0}^{\lceil\alpha\rceil}\frac{\mathbf{D}^{\kappa}\underline{\mathbf{f}}(0;\wp)\mathbf{x}^{\kappa-\alpha}}{\Gamma(1-\alpha+\kappa)}\bigg|_{\mathbf{x}=\mathbf{x}_0},\right.$$
$$\left.\mathbf{D}^{-(\lceil\alpha\rceil-\alpha)}\left(\mathbf{D}^{\lceil\alpha\rceil}\bar{\mathbf{f}}(\mathbf{x}_0;\wp)\right)+\bar{\mathcal{Q}}(\mathbf{x}_0,-\alpha)-\sum_{\kappa=0}^{\lceil\alpha\rceil}\frac{\mathbf{D}^{\kappa}\bar{\mathbf{f}}(0;\wp)\mathbf{x}^{\kappa-\alpha}}{\Gamma(1-\alpha+\kappa)}\bigg|_{\mathbf{x}=\mathbf{x}_0}\right]. \quad (30)$$

Thus

$$\begin{aligned}\left({}^{c}\mathbf{D}^{\alpha}\mathbf{f}\right)(\mathbf{x}_0) &= \left[\mathbf{D}^{-(\lceil\alpha\rceil-\alpha)}\left(\mathbf{D}^{\lceil\alpha\rceil}\underline{\mathbf{f}}(\mathbf{x}_0;\wp)\right),\mathbf{D}^{-(\lceil\alpha\rceil-\alpha)}\left(\mathbf{D}^{\lceil\alpha\rceil}\bar{\mathbf{f}}\right)(\mathbf{x}_0;\wp)\right]\\ &= \left[\left({}^{c}\mathbf{D}^{\alpha}\underline{\mathbf{f}}\right)(\mathbf{x}_0;\wp),\left({}^{c}\mathbf{D}^{\alpha}\bar{\mathbf{f}}\right)(\mathbf{x}_0;\wp)\right]. \end{aligned}\quad (31)$$

If ℓ is odd, the solution is similar as the one found before. □

Theorem 4. *Assume that there is a fuzzy-valued mapping $\mathbf{f}(\mathbf{x})\in\mathbb{C}^F[0,\infty)\bigcap\mathbb{L}^F[0,\infty)$ such that $\mathbf{f}(\mathbf{x})=\left[\underline{\mathbf{f}}(\mathbf{x};\wp),\bar{\mathbf{f}}(\mathbf{x},\wp)\right]$ for $\wp\in[0,1]$. In addition, let $r-1<\alpha<r$ and ℓ be the quantity replicated of two amongst $\jmath_1,\jmath_2,\jmath_3,\ldots,\jmath_r$ say $\jmath_{\kappa_1},\jmath_{\kappa_2},\jmath_{\kappa_3},\ldots,\jmath_{\kappa_\ell}$ such that $\kappa_1<\kappa_2<\ldots<\kappa_m$; i.e., $\jmath_{\kappa_1},\jmath_{\kappa_2},\jmath_{\kappa_3},\ldots,\jmath_{\kappa_\ell}=2$ and $0\le\ell\le r$.*

If ℓ is an even number, then

$$\mathbb{E}\left[\left({}^{c}\mathbf{D}^{\alpha}_{\jmath_1,\jmath_2,\ldots,\jmath_r}\mathbf{f}\right)(\mathbf{x})\right]=\omega^{-\alpha}\mathbb{E}[\mathbf{f}(\mathbf{x})]\ominus\omega^{2-\alpha}\mathbf{f}(0)\otimes\sum_{\kappa=1}^{r-1}\omega^{2-\alpha+\kappa}\mathbf{f}^{(\kappa)}(0), \quad (32)$$

then

$$\otimes=\begin{cases}\ominus, \textit{ if such quantity is replication of two amongest } \jmath_1,\jmath_2,\ldots\jmath_{r-(\kappa+1)} \text{ is an even number},\\ -, \textit{ if such quantity is replication of two amongest} \jmath_1,\jmath_2,\ldots\jmath_{r-(\kappa+1)} \text{ is an odd number}.\end{cases} \quad (33)$$

If ℓ is an odd number, we have

$$\mathbb{E}\left[\left({}^{c}\mathbf{D}^{\alpha}_{\jmath_1,\jmath_2,\ldots,\jmath_r}\mathbf{f}\right)(\mathbf{x})\right]=-\omega^{2-\alpha}\mathbf{f}(0)\ominus(-\omega^{-\alpha})\mathbb{E}[\mathbf{f}(\mathbf{x})]\otimes\sum_{\kappa=0}^{r-1}\omega^{2-\alpha+\kappa}\mathbf{f}(0), \quad (34)$$

$$\otimes=\begin{cases}\ominus, \textit{ if such quantity is replication of two amongest } \jmath_1,\jmath_2,\ldots\jmath_{r-(\kappa+1)} \text{ is an odd number},\\ -, \textit{ if such quantity is replication of two amongest } \jmath_1,\jmath_2,\ldots\jmath_{r-(\kappa+1)} \text{ is an even number}.\end{cases} \quad (35)$$

Proof. Considering $\left({}^{c}\mathbf{D}^{\alpha}_{\jmath_1,\jmath_2,\ldots,\jmath_r}\mathbf{f}\right)(\mathbf{x})$, that can be expressed as $\left({}^{c}\mathbf{D}^{\alpha}_{\jmath_1,\ldots,\jmath_{\kappa_1},\ldots,\jmath_{\kappa_2},\ldots,\jmath_{\kappa_\ell},\ldots,\jmath_r}\mathbf{f}\right)(\mathbf{x})$. Moreover, assume that ℓ is an odd number, then by means of Theorem 3 and $r-1<\alpha<r$, we have

$$\left({}^{c}\mathbf{D}^{\alpha}_{\jmath_1,\jmath_2,\ldots,\jmath_r}\mathbf{f}\right)(\mathbf{x})=\left[\left({}^{c}\mathbf{D}^{\alpha}\bar{\mathbf{f}}\right)(\mathbf{x};\wp),\left({}^{c}\mathbf{D}^{\alpha}\underline{\mathbf{f}}\right)(\mathbf{x};\wp)\right]. \quad (36)$$

Thus, we have

$$\begin{aligned}\underline{\left({}^{c}\mathbf{D}^{\alpha}\mathbf{f}\right)}(\mathbf{x};\wp)&=\left({}^{c}\mathbf{D}^{\alpha}\bar{\mathbf{f}}\right)(\mathbf{x};\wp),\\ \overline{\left({}^{c}\mathbf{D}^{\alpha}\mathbf{f}\right)}(\mathbf{x};\wp)&=\left({}^{c}\mathbf{D}^{\alpha}\underline{\mathbf{f}}\right)(\mathbf{x};\wp).\end{aligned} \quad (37)$$

Using the fact of (37), we have

$$\begin{aligned}\mathbb{E}\left[\left({}^{c}\mathbf{D}^{\alpha}_{\jmath_1,\jmath_2,\ldots,\jmath_r}\mathbf{f}\right)(\mathbf{x})\right]&=\mathbb{E}\left[\underline{\left({}^{c}\mathbf{D}^{\alpha}\mathbf{f}\right)}(\mathbf{x};\wp),\overline{\left({}^{c}\mathbf{D}^{\alpha}\mathbf{f}\right)}(\mathbf{x};\wp)\right]\\ &=\left[\mathcal{E}\left[\left({}^{c}\mathbf{D}^{\alpha}\bar{\mathbf{f}}\right)(\mathbf{x};\wp)\right],\mathcal{E}\left[\left({}^{c}\mathbf{D}^{\alpha}\underline{\mathbf{f}}\right)(\mathbf{x};\wp)\right]\right].\end{aligned} \quad (38)$$

In view of the Elzaki transform of the Caputo fractional derivative of order α ([56]), we have

$$\begin{aligned}
\mathcal{E}\left[\left({}^{c}\mathbf{D}^{\alpha}\underline{\mathbf{f}}\right)(\mathbf{x};\wp)\right] &= \omega^{-\alpha}\mathcal{E}\left[\underline{\mathbf{f}}(\mathbf{x};\wp)\right] - \sum_{\kappa=0}^{r-1}\omega^{2-\alpha+\kappa}\underline{\mathbf{f}}^{(\kappa)}(0;\wp) \\
&= \omega^{-\alpha}\mathcal{E}\left[\underline{\mathbf{f}}(\mathbf{x};\wp)\right] - \omega^{2-\alpha}\underline{\mathbf{f}}(0;\wp) - \sum_{\kappa=1}^{r-1}\omega^{2-\alpha+\kappa}\underline{\mathbf{f}}^{(\kappa)}(0;\wp). \qquad (39)
\end{aligned}$$

The aforementioned expression can be represented as

$$\begin{aligned}
\mathcal{E}\left[\left({}^{c}\mathbf{D}^{\alpha}\underline{\mathbf{f}}\right)(\mathbf{x};\wp)\right] &= \omega^{-\alpha}\mathcal{E}\left[\underline{\mathbf{f}}(\mathbf{x};\wp)\right] - \omega^{2-\alpha}\underline{\mathbf{f}}(0;\wp) \\
&\quad - \sum_{\kappa=1}^{\kappa_1-1}\omega^{2-\alpha+\kappa}\underline{\mathbf{f}}^{(\kappa)}(0;\wp) - \sum_{\kappa=\kappa_1}^{\kappa_2-1}\omega^{2-\alpha+\kappa}\underline{\mathbf{f}}^{(\kappa)}(0;\wp) - \ldots \\
&\quad - \sum_{\kappa=\kappa_{\ell-1}}^{\kappa_\ell-1}\omega^{2-\alpha+\kappa}\underline{\mathbf{f}}^{(\kappa)}(0;\wp) - \sum_{\kappa=\kappa_\ell}^{r-1}\omega^{2-\alpha+\kappa}\underline{\mathbf{f}}(0;\wp). \qquad (40)
\end{aligned}$$

Repeating the same process, we can write

$$\begin{aligned}
\mathcal{E}\left[\left({}^{c}\mathbf{D}^{\alpha}\bar{\mathbf{f}}\right)(\mathbf{x};\wp)\right] &= \omega^{-\alpha}\mathcal{E}\left[\bar{\mathbf{f}}(\mathbf{x};\wp)\right] - \omega^{2-\alpha}\bar{\mathbf{f}}(0;\wp) \\
&\quad - \sum_{\kappa=1}^{\kappa_1-1}\omega^{2-\alpha+\kappa}\bar{\mathbf{f}}(0;\wp) - \sum_{\kappa=\kappa_1}^{\kappa_2-1}\omega^{2-\alpha+\kappa}\bar{\mathbf{f}}(0;\wp) - \ldots \\
&\quad - \sum_{\kappa=\kappa_{\ell-1}}^{\kappa_\ell-1}\omega^{2-\alpha+\kappa}\underline{\mathbf{f}}^{(\kappa)}(0;\wp) - \sum_{\kappa=\kappa_\ell}^{r-1}\omega^{2-\alpha+\kappa}\underline{\mathbf{f}}^{(\kappa)})(0;\wp). \qquad (41)
\end{aligned}$$

Even though $J_{\kappa_1} = J_{\kappa_2} = \ldots = J_{\kappa_\ell} = 2$ and ℓ is an odd number, we then have the subsequent forms

$$\begin{aligned}
&\underline{\mathbf{f}}^{(\kappa)}(0;\wp) = \underline{\mathbf{f}^{(\kappa)}}(0;\wp), \\
&\bar{\mathbf{f}}^{(\kappa)}(0;\wp) = \underline{\mathbf{f}^{(\kappa)}}(0;\wp), \ \ \forall\, \kappa \in [1, \kappa_1 - 1], \\
&\underline{\mathbf{f}}^{(\kappa)}(0;\wp) = \overline{\mathbf{f}^{(\kappa)}}(0;\wp) \\
&\bar{\mathbf{f}}^{(\kappa)}(0;\wp) = \underline{\mathbf{f}^{(\kappa)}}(0;\wp), \ \ \forall\, \kappa \in [\kappa_1, \kappa_2 - 1], \\
&\vdots \\
&\underline{\mathbf{f}}^{(\kappa)}(0;\wp) = \underline{\mathbf{f}^{(\kappa)}}(0;\wp), \\
&\mathbf{f}^{(\bar{\kappa})}(0;\wp) = \overline{\mathbf{f}^{(\kappa)}}(0;\wp), \ \ \forall\, \kappa \in [\kappa_{\ell-1}, \kappa_\ell - 1], \\
&\underline{\mathbf{f}}^{(\kappa)}(0;\wp) = \overline{\mathbf{f}^{(\kappa)}}(0;\wp), \\
&\bar{\mathbf{f}}^{(\kappa)}(0;\wp) = \underline{\mathbf{f}^{(\kappa)}}(0;\wp), \ \ \forall\, \kappa \in [\kappa_\ell, r - 1]. \qquad (42)
\end{aligned}$$

When ℓ is an odd number and utilizing Theorem 3, we obtain the aforementioned equations.

In view of (41), (38) and (40) reduces to

$$\mathbb{E}\left[\left({}^{c}\mathbf{D}^{\alpha}_{J_1,J_2,\ldots,J_r}\mathbf{f}\right)(\mathbf{x})\right] = -\omega^{2-\alpha}\mathbf{f}(0) \ominus (-\omega^{-\alpha})\mathbb{E}\left[\mathbf{f}(\mathbf{x})\right] \otimes \sum_{\kappa=1}^{r-1}\omega^{2-\alpha+\kappa}f^{(\kappa)}(0;\wp). \qquad (43)$$

where $\otimes$ defined in (35).

Adopting the same method, we can prove ℓ to be an even number on parallel lines. $\square$

Corollary 1. *Assume that* $\mathbf{f}(\mathbf{x}) \in \mathbb{C}^F[0,\infty) \bigcap \mathbb{L}^\infty[0,\infty)$. *Moreover, let* $\alpha \in (2,3)$. *Then we obtain the following*

If $\left({}^{c}\mathbf{D}^{\alpha}_{1,1}\mathbf{f}\right)(\mathbf{x})$ *is* ${}^{c}[(i)-\alpha]$*-differentiable fuzzy-valued mapping, then*

$$\mathbb{E}\Big[\Big({}^{c}\mathbf{D}^{\alpha}_{1,1,1}\mathbf{f}\Big)(\mathbf{x})\Big] = \omega^{-\alpha}\mathbb{E}\big[\mathbf{f}(\mathbf{x})\big] \ominus \omega^{-\alpha+2}\mathbf{f}(0) \ominus \omega^{-\alpha+3}\mathbf{f}'(0) \ominus \omega^{-\alpha+4}\mathbf{f}''(0).$$

If $\left({}^{c}\mathbf{D}^{\alpha}_{1,1}\mathbf{f}\right)(\mathbf{x})$ *is* ${}^{c}[(ii)-\alpha]$*-differentiable fuzzy-valued mapping, then*

$$\mathbb{E}\Big[\Big({}^{c}\mathbf{D}^{\alpha}_{1,1,1}\mathbf{f}\Big)(\mathbf{x})\Big] = -\omega^{-\alpha+2}\mathbf{f}(0) \ominus (-\omega^{-\alpha})\mathbb{E}\big[\mathbf{f}(\mathbf{x})\big] - \omega^{-\alpha+3}\mathbf{f}'(0) - \omega^{-\alpha+4}\mathbf{f}''(0).$$

If $\left({}^{c}\mathbf{D}^{\alpha}_{1,2}\mathbf{f}\right)(\mathbf{x})$ *is* ${}^{c}[(i)-\alpha]$*-differentiable fuzzy-valued mapping, then*

$$\mathbb{E}\Big[\Big({}^{c}\mathbf{D}^{\alpha}_{1,2,1}\mathbf{f}\Big)(\mathbf{x})\Big] = -\omega^{-\alpha+2}\mathbf{f}(0) \ominus (-\omega^{-\alpha})\mathbb{E}\big[\mathbf{f}(\mathbf{x})\big] - \omega^{-\alpha+3}\mathbf{f}'(0) \ominus \omega^{-\alpha+4}\mathbf{f}''(0).$$

If $\left({}^{c}\mathbf{D}^{\alpha}_{1,2}\mathbf{f}\right)(\mathbf{x})$ *is* ${}^{c}[(ii)-\alpha]$*-differentiable fuzzy-valued mapping, then*

$$\mathbb{E}\Big[\Big({}^{c}\mathbf{D}^{\alpha}_{1,2,2}\mathbf{f}\Big)(\mathbf{x})\Big] = \omega^{-\alpha}\mathbb{E}\big[\mathbf{f}(\mathbf{x})\big] \ominus \omega^{-\alpha+2}\mathbf{f}(0) \ominus \omega^{-\alpha+3}\mathbf{f}'(0) - \omega^{-\alpha+4}\mathbf{f}''(0).$$

If $\left({}^{c}\mathbf{D}^{\alpha}_{2,1}\mathbf{f}\right)(\mathbf{x})$ *is* ${}^{c}[(i)-\alpha]$*-differentiable fuzzy-valued mapping, then*

$$\mathbb{E}\Big[\Big({}^{c}\mathbf{D}^{\alpha}_{2,1,1}\mathbf{f}\Big)(\mathbf{x})\Big] = -\omega^{-\alpha+2}\mathbf{f}(0) \ominus (-\omega)^{-\alpha}\mathbb{E}\big[\mathbf{f}(\mathbf{x})\big] \ominus \omega^{-\alpha+3}\mathbf{f}'(0) \ominus \omega^{-\alpha+4}\mathbf{f}''(0).$$

If $\left({}^{c}\mathbf{D}^{\alpha}_{2,1}\mathbf{f}\right)(\mathbf{x})$ *is* ${}^{c}[(ii)-\alpha]$*-differentiable fuzzy-valued mapping, then*

$$\mathbb{E}\Big[\Big({}^{c}\mathbf{D}^{\alpha}_{2,1,2}\mathbf{f}\Big)(\mathbf{x})\Big] = \omega^{-\alpha}\mathbb{E}\big[\mathbf{f}(\mathbf{x})\big] \ominus \omega^{-\alpha+2}\mathbf{f}(0) - \omega^{-\alpha+3}\mathbf{f}'(0) - \omega^{-\alpha+4}\mathbf{f}''(0).$$

If $\left({}^{c}\mathbf{D}^{\alpha}_{2,2}\mathbf{f}\right)(\mathbf{x})$ *is* ${}^{c}[(i)-\alpha]$*-differentiable fuzzy-valued mapping, then*

$$\mathbb{E}\Big[\Big({}^{c}\mathbf{D}^{\alpha}_{2,2,1}\mathbf{f}\Big)(\mathbf{x})\Big] = \omega^{-\alpha}\mathbb{E}\big[\mathbf{f}(\mathbf{x})\big] \ominus \omega^{-\alpha+2}\mathbf{f}(0) - \omega^{-\alpha+3}\mathbf{f}'(0) \ominus \omega^{-\alpha+4}\mathbf{f}''(0).$$

If $\left({}^{c}\mathbf{D}^{\alpha}_{2,2}\mathbf{f}\right)(\mathbf{x})$ *is* ${}^{c}[(ii)-\alpha]$*-differentiable fuzzy-valued mapping, then*

$$\mathbb{E}\Big[\Big({}^{c}\mathbf{D}^{\alpha}_{2,2,2}\mathbf{f}\Big)(\mathbf{x})\Big] = -\omega^{-\alpha+2}\mathbf{f}(0) \ominus (-\omega)^{-\alpha}\mathbb{E}\big[\mathbf{f}(\mathbf{x})\big] \ominus \omega^{-\alpha+3}\mathbf{f}'(0) - \omega^{-\alpha+4}\mathbf{f}''(0).$$

5. The Fuzzy Elzaki Decomposition Method for Finding the Solution of the Nonlinear Fuzzy Partial Differential Equation

In this note, we coupled the fuzzy Elzaki transform and the ADM to obtain the solution of the NFPDE. The generic form of the NFPDE is presented as follows:

$$\sum_{\iota=0}^{p} c_\iota \odot \mathbf{D}^{\alpha}_{\xi}\mathbf{f}(\mathbf{x},\xi) \oplus \sum_{j=1}^{q} c_j \odot \frac{\partial^j \mathbf{f}(\mathbf{x},\xi)}{\partial \mathbf{x}^j} \oplus \sum_{\eta=0}^{2}\sum_{\sigma=\eta}^{2} c_{\eta\sigma} \odot \frac{\partial^\eta \mathbf{f}(\mathbf{x},\xi)}{\partial \mathbf{x}^\eta} \odot \frac{\partial^\sigma \mathbf{f}(\mathbf{x},\xi)}{\partial \mathbf{x}^\sigma} = \mathbf{g}(\mathbf{x},\xi), \quad (44)$$

subject to initial conditions

$$\frac{\partial^\iota \mathbf{f}(\mathbf{x},0)}{\partial \xi^\iota} = \psi_\iota(\mathbf{x}),\ \iota = 0,1,\ldots,p-1, \quad (45)$$

where $\mathbf{f},\mathbf{g} : [0,\hat{b}] \times [0,\hat{d}] \mapsto E^1, \psi_\iota : [0,\hat{b}] \mapsto E^1$ are continuous fuzzy mappings and $c_\iota,\ \iota = 1,2,\ldots,p,\ c_j,\ j = 1,2,\ldots,q,\ c_{\eta\sigma}, \eta = 0,1,2, \sigma = 0,1,2,$ are positive constants.

Implementing the fuzzy Elzaki transform on both sides of (44), yields

$$\sum_{\iota=0}^{p} c_\iota \odot \mathbb{E}\big[\mathbf{D}^{\alpha}_{\xi}\mathbf{f}(\mathbf{x},\xi)\big] \oplus \sum_{j=1}^{q} c_j \odot \mathbb{E}\left[\frac{\partial^j \mathbf{f}(\mathbf{x},\xi)}{\partial \mathbf{x}^j}\right] \oplus \sum_{\eta=0}^{2}\sum_{\sigma=\eta}^{2} c_{\eta\sigma} \odot \mathbb{E}\left[\frac{\partial^\eta \mathbf{f}(\mathbf{x},\xi)}{\partial \mathbf{x}^\eta}\right] \odot \mathbb{E}\left[\frac{\partial^\sigma \mathbf{f}(\mathbf{x},\xi)}{\partial \mathbf{x}^\sigma}\right] = \mathbb{E}\big[\mathbf{g}(\mathbf{x},\xi)\big]. \quad (46)$$

Consider $\frac{\partial^\eta \mathbf{f}(\mathbf{x},\xi)}{\partial \xi^\eta}$, $\eta = 0,1,2$ as positive fuzzy-valued mappings.
Then, the parametric version of (46) is as follows:

$$\sum_{\iota=0}^{p} c_\iota \mathcal{E}\left[\mathbf{D}_\xi^\alpha \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)\right] + \sum_{j=1}^{q} c_j \mathcal{E}\left[\frac{\partial^j \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^j}\right] + \sum_{\eta=0}^{2}\sum_{\sigma=\eta}^{2} c_{\eta\sigma}\mathcal{E}\left[\frac{\partial^\eta \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\sigma}\right] = \mathcal{E}\left[\underline{\mathbf{g}}(\mathbf{x},\xi;\wp)\right], \tag{47}$$

and

$$\sum_{\iota=0}^{p} c_\iota \mathcal{E}\left[\mathbf{D}_\xi^\alpha \bar{\mathbf{f}}(\mathbf{x},\xi;\wp)\right] + \sum_{j=1}^{q} c_j \mathcal{E}\left[\frac{\partial^j \bar{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^j}\right] + \sum_{\eta=0}^{2}\sum_{\sigma=\eta}^{2} c_{\eta\sigma}\mathcal{E}\left[\frac{\partial^\eta \bar{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \bar{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\sigma}\right] = \mathcal{E}\left[\bar{\mathbf{g}}(\mathbf{x},\xi;\wp)\right]. \tag{48}$$

Case I. Consider the mapping $\mathbf{f}(\mathbf{x},\xi;\wp)$ as $[(i)-\alpha]$-differentiable of the qth-order with respect to $\mathbf{x}$.

In view of (47), then from (40) and (41) and IC, we have

$$\begin{aligned}\frac{1}{\omega^\alpha}\sum_{\iota=0}^{p} c_\iota \mathcal{E}\left[\underline{\mathbf{f}}(\mathbf{x},\xi;\wp)\right] &= \mathcal{E}\left[\mathbf{g}(\mathbf{x},\xi;\wp)\right] + \sum_{\iota=1}^{p}\omega^2 \underline{\psi}_0(\mathbf{x};\wp) - \sum_{j=1}^{q} c_j \mathcal{E}\left[\frac{\partial^j \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^j}\right] \\ &\quad - \sum_{\eta=0}^{2}\sum_{\sigma=\eta}^{2} c_{\eta\sigma}\mathcal{E}\left[\frac{\partial^\eta \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\sigma}\right].\end{aligned}$$

It follows that

$$\begin{aligned}\mathcal{E}\left[\underline{\mathbf{f}}(\mathbf{x},\xi;\wp)\right] &= \left(\sum_{\iota=0}^{p}\frac{c_\iota}{\omega^\alpha}\right)^{-1}\Bigg[\mathcal{E}\left[\underline{\mathbf{g}}(\mathbf{x},\xi;\wp)\right] + \sum_{\iota=1}^{p}\omega^2 \underline{\psi}_0(\mathbf{x};\wp) - \sum_{j=1}^{q} c_j \mathcal{E}\left[\frac{\partial^j \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^j}\right] \\ &\quad - \sum_{\eta=0}^{2}\sum_{\sigma=\eta}^{2} c_{\eta\sigma}\mathcal{E}\left[\frac{\partial^\eta \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\sigma}\right]\Bigg].\end{aligned}$$

Now, employing the inverse fuzzy Elzaki transform to both sides of the above equation, we obtain

$$\begin{aligned}\underline{\mathbf{f}}(\mathbf{x},\xi;\wp) &= \mathcal{E}^{-1}\left[\left(\sum_{\iota=0}^{p}\frac{c_\iota}{\omega^\alpha}\right)^{-1}\left(\mathcal{E}\left[\underline{\mathbf{g}}(\mathbf{x},\xi;\wp)\right] + \sum_{\iota=1}^{p}\omega^2 \underline{\psi}_0(\mathbf{x};\wp)\right)\right] - \mathcal{E}^{-1}\Bigg[\left(\sum_{\iota=0}^{p}\frac{c_\iota}{\omega^\alpha}\right)^{-1}\Bigg(\sum_{j=1}^{q} c_j \mathcal{E}\left[\frac{\partial^j \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^j}\right] \\ &\quad + \sum_{\eta=0}^{2}\sum_{\sigma=\eta}^{2} c_{\eta\sigma}\mathcal{E}\left[\frac{\partial^\eta \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\sigma}\Bigg)\right]\Bigg].\end{aligned} \tag{49}$$

In view of the Adomian decomposition method, this method has infinite series solutions for the subsequent unknown mappings:

$$\underline{\mathbf{f}}(\mathbf{x},\xi;\wp) = \sum_{r=0}^{\infty}\underline{\mathbf{f}}_r(\mathbf{x},\xi;\wp). \tag{50}$$

The non-linearity is addressed by an infinite series of the Adomian polynomials $\underline{A}_r^{\eta\sigma}$, $\eta = 0,1,2$, $\sigma = 0,1,2$ and has the subsequent representation:

$$\frac{\partial^\eta \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\sigma} = \sum_{r=0}^{\infty}\underline{\mathcal{A}}_r^{\eta\sigma}, \tag{51}$$

where

$$\underline{\mathcal{A}}_r^{\eta\sigma} = \begin{cases} \frac{\partial^\eta \underline{\mathbf{f}}_0}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \underline{\mathbf{f}}_0}{\partial \mathbf{x}^\sigma}, & r=0, \\ \frac{\partial^\eta \underline{\mathbf{f}}_0}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \underline{\mathbf{f}}_1}{\partial \mathbf{x}^\sigma} + \frac{\partial^\eta \underline{\mathbf{f}}_1}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \underline{\mathbf{f}}_0}{\partial \mathbf{x}^\sigma}, & r=1, \\ \frac{\partial^\eta \underline{\mathbf{f}}_0}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \underline{\mathbf{f}}_2}{\partial \mathbf{x}^\sigma} + \frac{\partial^\eta \underline{\mathbf{f}}_1}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \underline{\mathbf{f}}_1}{\partial \mathbf{x}^\sigma} + \frac{\partial^\eta \underline{\mathbf{f}}_2}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \underline{\mathbf{f}}_0}{\partial \mathbf{x}^\sigma}, & r=2, \\ \vdots \end{cases} \tag{52}$$

Inserting (51), (52) in (50) refers to the following equation:

$$\begin{aligned} \sum_{r=0}^{\infty} \underline{\mathbf{f}}_r(\mathbf{x},\xi;\wp) &= \mathcal{E}^{-1}\left[\left(\sum_{\iota=0}^{p}\frac{c_\iota}{\omega^\alpha}\right)^{-1}\left(\mathcal{E}\left[\underline{\mathbf{g}}(\mathbf{x},\xi;\wp)\right] + \sum_{\iota=1}^{p}\omega^2 \underline{\psi}_0(\mathbf{x};\wp)\right)\right] \\ &-\mathcal{E}^{-1}\left[\left(\sum_{\iota=0}^{p}\frac{c_\iota}{\omega^\alpha}\right)^{-1}\left(\sum_{j=1}^{q} c_j \mathcal{E}\left[\sum_{r=0}^{\infty}\frac{\partial^j \underline{\mathbf{f}}_r(\mathbf{x};\wp)}{\partial \mathbf{x}^j}\right] + \sum_{\eta=0}^{2}\sum_{\sigma=\eta}^{2} c_{\eta\sigma}\mathcal{E}\left[\sum_{r=0}^{\infty}\underline{\mathcal{A}}_r^{\eta\sigma}\right]\right)\right]. \end{aligned} \tag{53}$$

The recursive terms of the Elzaki decomposition method can be computed for $r \geq 0$ as follows:

$$\begin{aligned} \underline{\mathbf{f}}_0(\mathbf{x},\xi;\wp) &= \mathcal{E}^{-1}\left[\left(\sum_{\iota=0}^{p}\frac{c_\iota}{\omega^\alpha}\right)^{-1}\left(\mathcal{E}\left[\underline{\mathbf{g}}(\mathbf{x},\xi;\wp)\right] + \sum_{\iota=1}^{p}\omega^2 \underline{\psi}_0(\mathbf{x},\xi;\wp)\right)\right], \\ \underline{\mathbf{f}}_{r+1}(\mathbf{x},\xi;\wp) &= -\mathcal{E}^{-1}\left[\left(\sum_{\iota=0}^{p}\frac{c_\iota}{\omega^\alpha}\right)^{-1}\left(\sum_{j=1}^{q} c_j \mathcal{E}\left[\sum_{r=0}^{\infty}\frac{\partial^j \underline{\mathbf{f}}_r(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^j}\right] + \sum_{\eta=0}^{2}\sum_{\sigma=\eta}^{2} c_{\eta\sigma}\mathcal{E}\left[\sum_{r=0}^{\infty}\underline{\mathcal{A}}_r^{\eta\sigma}\right]\right)\right]. \end{aligned} \tag{54}$$

Case II. Suppose the mapping $\mathbf{f}(\mathbf{x},\xi;\wp)$ is $[(i)-\alpha]$-differentiable of the qth order with respect to $\mathbf{x}$ and $[(ii)-\alpha]$-differentiable of the $2pth$ order with respect to ξ. Then, the parametric version of (46) has the following representation:

$$\begin{aligned} &\sum_{\iota=0}^{p} c_{2\iota}\mathcal{E}\left[\mathbf{D}_\xi^{2\alpha}\underline{\mathbf{f}}(\mathbf{x},\xi;\wp)\right] + \sum_{\iota=1}^{p} c_{2\iota-1}\mathcal{E}\left[\mathbf{D}_\xi^{\alpha}\underline{\mathbf{f}}(\mathbf{x},\xi;\wp)\right] \\ &\quad + \sum_{j=1}^{q} c_j \mathcal{E}\left[\frac{\partial^j \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^j}\right] + \sum_{\eta=0}^{2}\sum_{\sigma=\eta}^{2} c_{\eta\sigma}\mathcal{E}\left[\frac{\partial^\eta \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\sigma}\right] = \mathcal{E}\left[\underline{\mathbf{g}}(\mathbf{x},\xi;\wp)\right], \end{aligned}$$

and

$$\begin{aligned} &\sum_{\iota=0}^{p} c_{2\iota}\mathcal{E}\left[\mathbf{D}_\xi^{2\alpha}\bar{\mathbf{f}}(\mathbf{x},\xi;\wp)\right] + \sum_{\iota=1}^{p} c_{2\iota-1}\mathcal{E}\left[\mathbf{D}_\xi^{\alpha}\bar{\mathbf{f}}(\mathbf{x},\xi;\wp)\right] \\ &\quad + \sum_{j=1}^{q} c_j \mathcal{E}\left[\frac{\partial^j \bar{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^j}\right] + \sum_{\eta=0}^{2}\sum_{\sigma=\eta}^{2} c_{\eta\sigma}\mathcal{E}\left[\frac{\partial^\eta \bar{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \bar{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\sigma}\right] = \mathcal{E}\left[\bar{\mathbf{g}}(\mathbf{x},\xi;\wp)\right]. \end{aligned}$$

Utilizing the fact of Theorem 4 and ICs, we have

$$\begin{aligned} &\mathcal{B}\mathcal{E}\left[\underline{\mathbf{f}}(\mathbf{x},\xi;\wp)\right] + \mathcal{C}\mathcal{E}\left[\mathbf{D}_\xi^{\alpha}\underline{\mathbf{f}}(\mathbf{x},\xi;\wp)\right] \\ &= \mathcal{E}\left[\bar{\mathbf{g}}(\mathbf{x},\xi;\wp) + \mathcal{F}_1(\mathbf{x};\wp)\right] - \sum_{j=1}^{q} c_j \mathcal{E}\left[\frac{\partial^j \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^j}\right] - \sum_{\eta=0}^{2}\sum_{\sigma=\eta}^{2} c_{\eta\sigma}\mathcal{E}\left[\frac{\partial^\eta \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\sigma}\right], \end{aligned} \tag{55}$$

and

$$\begin{aligned} &\mathcal{B}\mathcal{E}\left[\bar{\mathbf{f}}(\mathbf{x},\xi;\wp)\right] + \mathcal{C}\mathcal{E}\left[\bar{\mathbf{f}}(\mathbf{x},\xi;\wp)\right] \\ &= \mathcal{E}\left[\bar{\mathbf{g}}(\mathbf{x},\xi;\wp) + \mathcal{F}_2(\mathbf{x};\wp)\right] - \sum_{j=1}^{q} c_j \mathcal{E}\left[\frac{\partial^j \bar{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^j}\right] - \sum_{\eta=0}^{2}\sum_{\sigma=\eta}^{2} c_{\eta\sigma}\mathcal{E}\left[\frac{\partial^\eta \bar{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \bar{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\sigma}\right]. \end{aligned} \tag{56}$$

where $\mathcal{B} = \sum\limits_{\iota=0}^{p} c_{2\iota}\omega^{\alpha}$, $\mathcal{C} = \sum\limits_{\iota=1}^{p} c_{2\iota-1}\omega^{2-\alpha}$,

$$\mathcal{F}_1(\mathbf{x};\wp) = \sum_{\iota=0}^{p} c_{2\iota}\left(\omega^{2-2\alpha}\underline{\psi}_0(\mathbf{x};\wp) + \omega^{3-2\alpha}\bar{\psi}_1(\mathbf{x};\wp)\right) + \sum_{\iota=0}^{p} c_{2\iota-1}\left(\omega^{3-2\alpha}\bar{\psi}_0(\mathbf{x};\wp) + \omega^{2-2\alpha}\underline{\psi}_0(\mathbf{x};\wp)\right),$$

and

$$\mathcal{F}_2(\mathbf{x};\wp) = \sum_{\iota=0}^{p} c_{2\iota}\left(\omega^{2-2\alpha}\bar{\psi}_0(\mathbf{x};\wp) + \omega^{3-2\alpha}\underline{\psi}_1(\mathbf{x};\wp)\right) + \sum_{\iota=0}^{p} c_{2\iota-1}\left(\omega^{3-2\alpha}\underline{\psi}_0(\mathbf{x};\wp) + \omega^{2-2\alpha}\bar{\psi}_0(\mathbf{x};\wp)\right).$$

For the aforementioned Equations (55) and (56), we obtain $\mathcal{E}\left[\underline{\mathbf{f}}(\mathbf{x},\xi;\wp)\right]$ and $\mathcal{E}\left[\bar{\mathbf{f}}(\mathbf{x},\xi;\wp)\right]$ similar to **Case I,** and we find the the general solution $\mathbf{f}(\mathbf{x};\wp) = \left[\underline{\mathbf{f}}(\mathbf{x},\xi;\wp), \bar{\mathbf{f}}(\mathbf{x},\xi;\wp)\right]$.

Example 1. *Consider the fuzzy fractional partial differential equation as follows:*

$$\mathbf{D}_{\xi}^{2\alpha}\mathbf{f}(\mathbf{x},\xi) \oplus \frac{\partial \mathbf{f}(\mathbf{x},\xi)}{\partial \mathbf{x}} \odot \frac{\partial \mathbf{f}^2(\mathbf{x},\xi)}{\partial \mathbf{x}^2} = g_3(\mathbf{x},\xi),\ \mathbf{x} \geq 0,\ \xi > 0, \tag{57}$$

subject to ICs

$$\mathbf{f}(\mathbf{x},0) = \left(\frac{\mathbf{x}^2}{2}\wp, \frac{\mathbf{x}^2}{2}(2-\wp)\right),\ \ \mathbf{f}'_{\xi}(\mathbf{x},0) = (0,0),\ \mathbf{x} > 0, \tag{58}$$

and $g_3(\mathbf{x},\xi) = \left(\wp + \mathbf{x}\wp^2, 2 - \wp + \mathbf{x}(2-\wp)^2\right)$.

In order to find solution of (57), we have the following three cases.
Case I. If $\mathbf{f}(\mathbf{x},\xi)$ is $[(i)-\alpha]$-differentiable.
Employing the Elzaki transform on (57), then we have

$$\frac{1}{\omega^{2\alpha}}\mathcal{E}\left[\underline{\mathbf{f}}(\mathbf{x},\xi;\wp)\right] - \omega^{2-2\alpha}\underline{\mathbf{f}}(\mathbf{x},0;\wp) = \mathcal{E}\left[g_3(\mathbf{x},\xi) - \frac{\partial \mathbf{f}(\mathbf{x},\xi)}{\partial \mathbf{x}}\frac{\partial \mathbf{f}^2(\mathbf{x},\xi)}{\partial \mathbf{x}^2}\right].$$

or equivalently, we have

$$\mathcal{E}\left[\underline{\mathbf{f}}(\mathbf{x},\xi;\wp)\right] - \omega^{2}\underline{\mathbf{f}}(\mathbf{x},0;\wp) = \omega^{2\alpha}\mathcal{E}\left[g_3(\mathbf{x},\xi) - \frac{\partial \mathbf{f}(\mathbf{x},\xi)}{\partial \mathbf{x}}\frac{\partial \mathbf{f}^2(\mathbf{x},\xi)}{\partial \mathbf{x}^2}\right].$$

Further, implementing the inverse fuzzy Elzaki transform, we have

$$\underline{\mathbf{f}}(\mathbf{x},\xi;\wp) = \mathcal{E}^{-1}\left[\omega^{2}\underline{\mathbf{f}}(\mathbf{x},0;\wp) + \omega^{2\alpha}\mathcal{E}\left[g_3(\mathbf{x},\xi) - \frac{\partial \mathbf{f}(\mathbf{x},\xi)}{\partial \mathbf{x}}\frac{\partial \mathbf{f}^2(\mathbf{x},\xi)}{\partial \mathbf{x}^2}\right]\right].$$

Furthermore, applying the scheme described in Section 4, we have

$$\sum_{r=0}^{\infty}\underline{\mathbf{f}}_r(\mathbf{x},\xi;\wp) = \mathcal{E}^{-1}\left[\omega^{2}\underline{\mathbf{f}}(\mathbf{x},0;\wp) + \omega^{2\alpha}\mathcal{E}\left[g_3(\mathbf{x},\xi)\right] - \omega^{2\alpha}\mathcal{E}\left[\sum_{r=0}^{\infty}\underline{\mathcal{A}}_r\right]\right]. \tag{59}$$

Utilizing the iterative procedure defined in (54), we have

$$\begin{aligned}\underline{\mathbf{f}}_0(\mathbf{x},\xi;\wp) \quad &= \mathcal{E}^{-1}\left[\omega^{2}\underline{\mathbf{f}}(\mathbf{x},0;\wp) + \omega^{2\alpha}\mathcal{E}\left[g_3(\mathbf{x},\xi)\right]\right] \\ &= \wp\frac{\mathbf{x}^2}{2} + (\wp + \mathbf{x}\wp^2)\frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)},\end{aligned} \tag{60}$$

In addition,

$$\underline{\mathbf{f}}_{r+1}(\mathbf{x},\xi;\wp)=\mathcal{E}^{-1}\Big[\omega^{2\alpha}\mathcal{E}\big[\sum_{r=0}^{\infty}\underline{\mathcal{A}}_r\big]\Big]. \tag{61}$$

Utilizing the first few Adomian polynomials mentioned in (52), we have

$$\begin{aligned}
\underline{\mathbf{f}}(\mathbf{x},\xi;\wp) &= \mathcal{E}^{-1}\Big[\omega^{2\alpha}\big[\underline{\mathcal{A}}_0\big]\Big] \\
&= -\wp^2\mathbf{x}\frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)}-\wp^3\frac{\xi^{4\alpha}}{\Gamma(4\alpha+1)}, \\
\underline{\mathbf{f}}_2(\mathbf{x},\xi;\wp) &= \mathcal{E}^{-1}\Big[\omega^{2\alpha}\big[\underline{\mathcal{A}}_1\big]\Big]=-\wp^3\frac{\xi^{4\alpha}}{\Gamma(4\alpha+1)}, \\
\underline{\mathbf{f}}_3(\mathbf{x},\xi;\wp) &= 0, \\
&\vdots
\end{aligned} \tag{62}$$

In a similar way we obtained the upper solutions as follows:

$$\begin{aligned}
\bar{\mathbf{f}}_0(\mathbf{x},\xi;\wp) &= \frac{\mathbf{x}^2}{2}(2-\wp)+\big(2-\wp+\mathbf{x}(2-\wp)^2\big)\frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)}, \\
\bar{\mathbf{f}}(\mathbf{x},\xi;\wp) &= -\mathbf{x}(2-\wp)^2\frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)}-(2-\wp)^3\frac{\xi^{4\alpha}}{\Gamma(4\alpha+1)}, \\
\bar{\mathbf{f}}_2(\mathbf{x},\xi;\wp) &= -(2-\wp)^3\frac{\xi^{4\alpha}}{\Gamma(4\alpha+1)}, \\
\bar{\mathbf{f}}_3(\mathbf{x},\xi;\wp) &= 0, \\
&\vdots
\end{aligned} \tag{63}$$

The series form solution of Example 1 is presented as follows:

$$\mathbf{f}(\mathbf{x},\xi)=\left(\left(\frac{\mathbf{x}^2}{2}+\frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)}\right)\wp,\left(\frac{\mathbf{x}^2}{2}+\frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)}\right)(2-\wp)\right). \tag{64}$$

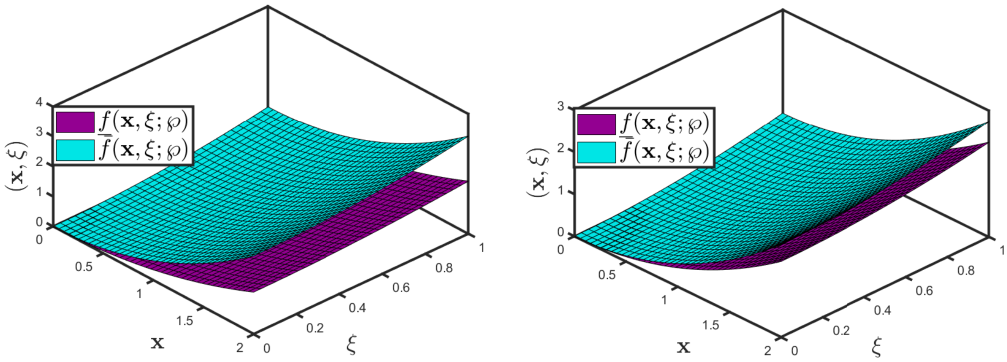
The numerical solution to the fuzzy fractional nonlinear PDE is presented in this section. Incorporating all of the data with respect to the numerous parameters involved in the related equation is a monumental task. Uncertain responses subject to Caputo fractional order derivatives have been considered, as previously stated.

- Table 1 represents the obtained findings with $\mathbf{x}=0.4$ and $\xi=0.7$. Table 1 also comprises the outcomes of Georgieva and Pavlova [57]. As a consequence, the findings acquired by fuzzy Elzaki decomposition method are the same if $\alpha=1$, as those reported by Georgieva and Pavlova [57].
- Figure 1a,b demonstrates the three-dimensional illustration of the lower and upper estimates for different uncertainties $\wp\in[0,1]$.
- Figure 2a,b shows the fuzzy responses for different fractional orders.
- Figure 3a,b illustrates the fuzzy responses for different uncertainty parameters.
- The aforementioned representations illustrate that all graphs are substantially identical in their perspectives and have good agreement with one another, especially when integer-order derivatives are taken into account.

Finally, this generic approach for dealing with nonlinear PDEs is more accurate and powerful than the method applied by [57]. Our findings for the fuzzy Elzaki decomposition method, helpful for fuzzy initial value problems, demonstrate the consistency and strength of the offered solutions.

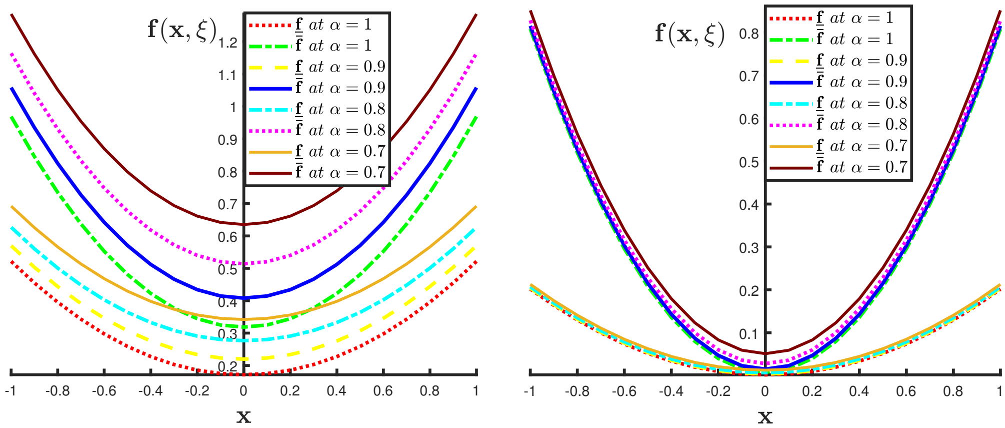
Table 1. Lower and upper solutions of **Case I** of Example 1 for various fractional orders in comparison with the solution derived by [57].

$\wp$	$\underline{\mathbf{f}}(\alpha = 0.7)$	$\bar{\mathbf{f}}(\alpha = 0.7)$	$\underline{\mathbf{f}}(\alpha = 1)$	$\bar{\mathbf{f}}(\alpha = 1)$	$\underline{\mathbf{f}}$ [57]	$\bar{\mathbf{f}}$ [57]
0.1	1.9420×10^{-2}	3.6899×10^{-1}	9.0000×10^{-3}	1.7100×10^{-1}	9.0000×10^{-3}	1.7100×10^{-1}
0.2	3.8841×10^{-2}	3.4957×10^{-1}	1.8000×10^{-2}	1.6200×10^{-1}	1.8000×10^{-2}	1.6200×10^{-1}
0.3	5.8262×10^{-2}	3.30152×10^{-1}	2.7000×10^{-2}	1.5300×10^{-1}	2.7000×10^{-2}	1.5300×10^{-1}
0.4	7.7682×10^{-2}	3.1073×10^{-1}	3.6000×10^{-2}	1.4400×10^{-1}	3.6000×10^{-2}	1.4400×10^{-1}
0.5	9.7103×10^{-2}	2.9131×10^{-1}	4.5000×10^{-2}	1.3500×10^{-1}	4.5000×10^{-2}	1.3500×10^{-1}
0.6	1.1652×10^{-2}	2.71890×10^{-1}	5.4000×10^{-2}	1.2600×10^{-1}	5.4000×10^{-2}	1.2600×10^{-1}
0.7	1.3594×10^{-2}	2.5246×10^{-1}	6.3000×10^{-2}	1.1700×10^{-1}	6.3000×10^{-2}	1.1700×10^{-1}
0.8	1.5536×10^{-2}	2.3304×10^{-1}	7.2000×10^{-2}	1.0800×10^{-1}	7.2000×10^{-2}	1.0800×10^{-1}
0.9	1.7478×10^{-2}	2.1362×10^{-1}	8.1000×10^{-2}	9.9000×10^{-2}	8.1000×10^{-2}	9.9000×10^{-2}
1.0	1.9420×10^{-1}	1.9420×10^{-1}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}



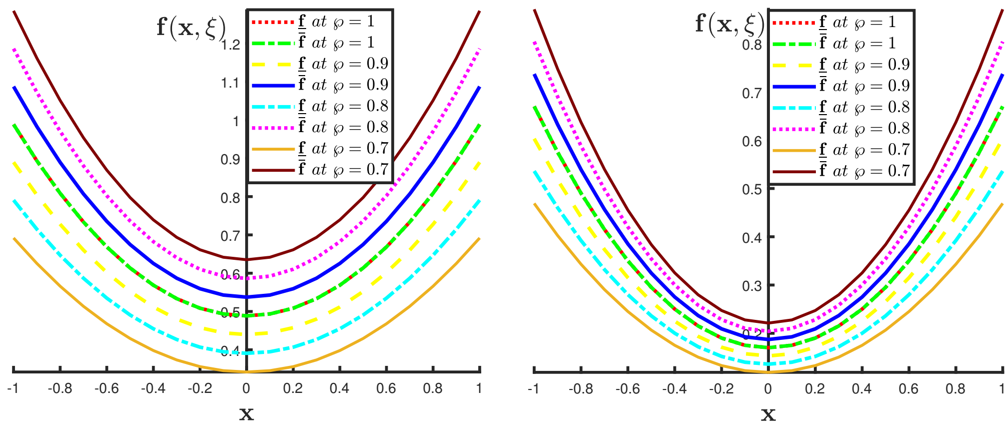
(**a**) (**b**)

Figure 1. Three-dimensional fuzzy responses of Example 1 for **Case I** at (**a**) $\wp = 0.7$, (**b**) $\wp = 0.9$ with fractional order $\alpha = 1$.



(**a**) (**b**)

Figure 2. Two-dimensional fuzzy responses of Example 1 for **Case I** at (**a**) $\wp = 0.7$ and $\xi = 0.7$, (**b**) $\wp = 0.4$ and $\xi = 0.1$ with varying fractional orders.



(a) (b)

Figure 3. Two-dimensional fuzzy responses of Example 1 for **Case I** at (**a**) $\alpha = 0.7$ and $\xi = 0.7$, (**b**) $\alpha = 0.4$ and $\xi = 0.1$ with varying uncertainty parameters $\wp \in [0, 1]$.

Case II. If $\mathbf{f}(\mathbf{x}, \xi)$ is $[(ii) - \alpha]$-differentiable, taking into account (55) and (56), we find

$$\begin{aligned} \frac{1}{\omega^{2\alpha}} \mathcal{E}\left[\underline{\mathbf{f}}(\mathbf{x}, \xi; \wp)\right] &= \omega^{2-2\alpha} \underline{\mathbf{f}}(\mathbf{x}, 0; \wp) + \mathcal{E}\left[\underline{g_3}(\mathbf{x}, \xi)\right] - \mathcal{E}\left[\frac{\partial \underline{\mathbf{f}}(\mathbf{x}, \xi)}{\partial \mathbf{x}} \frac{\partial \underline{\mathbf{f}}^2(\mathbf{x}, \xi)}{\partial \mathbf{x}^2}\right], \\ \frac{1}{\omega^{2\alpha}} \mathcal{E}\left[\bar{\mathbf{f}}(\mathbf{x}, \xi; \wp)\right] &= \omega^{2-2\alpha} \bar{\mathbf{f}}(\mathbf{x}, 0; \wp) + \mathcal{E}\left[\bar{g}_3(\mathbf{x}, \xi)\right] - \mathcal{E}\left[\frac{\partial \bar{\mathbf{f}}(\mathbf{x}, \xi)}{\partial \mathbf{x}} \frac{\partial \bar{\mathbf{f}}^2(\mathbf{x}, \xi)}{\partial \mathbf{x}^2}\right]. \end{aligned} \tag{65}$$

Employing the inverse fuzzy Elzaki transform to both sides of the aforementioned equations and incorporation of the Elzaki decomposition method, we find the solution on same lines as we did in **Case I**.

Case III. If $\mathbf{f}(\mathbf{x}, \xi)$ is $[(i) - \alpha]$-differentiable and $\mathbf{f}'(\mathbf{x}, \xi)$ is $[(ii) - \alpha]$-differentiable, then

$$\mathbb{E}(\mathbf{f}'(\mathbf{x}, \xi)) = \left[\mathcal{E}\left(\underline{\mathbf{f}}'(\mathbf{x}, \xi; \wp)\right), \mathcal{E}\left(\bar{\mathbf{f}}'(\mathbf{x}, \xi; \wp)\right)\right] \tag{66}$$

and

$$\mathbb{E}(\mathbf{f}''(\mathbf{x}, \xi)) = \left[\mathcal{E}\left(\bar{\mathbf{f}}''(\mathbf{x}, \xi; \wp)\right), \mathcal{E}\left(\underline{\mathbf{f}}''(\mathbf{x}, \xi; \wp)\right)\right]. \tag{67}$$

In view of (54) and Theorem 4 with IC, we follow the iterative process:
Employing the Elzaki transform on (57), then we have

$$\frac{1}{\omega^{2\alpha}} \mathcal{E}\left[\underline{\mathbf{f}}(\mathbf{x}, \xi; \wp)\right] - \omega^{2-2\alpha} \underline{\mathbf{f}}(\mathbf{x}, 0; \wp) = \mathcal{E}\left[\bar{g}_3(\mathbf{x}, \xi) - \frac{\partial \bar{\mathbf{f}}(\mathbf{x}, \xi)}{\partial \mathbf{x}} \frac{\partial \bar{\mathbf{f}}^2(\mathbf{x}, \xi)}{\partial \mathbf{x}^2}\right].$$

or equivalently, we have

$$\mathcal{E}\left[\underline{\mathbf{f}}(\mathbf{x}, \xi; \wp)\right] - \omega^2 \underline{\mathbf{f}}(\mathbf{x}, 0; \wp) = \omega^{2\alpha} \mathcal{E}\left[\bar{g}_3(\mathbf{x}, \xi) - \frac{\partial \bar{\mathbf{f}}(\mathbf{x}, \xi)}{\partial \mathbf{x}} \frac{\partial \bar{\mathbf{f}}^2(\mathbf{x}, \xi)}{\partial \mathbf{x}^2}\right].$$

Furthermore, implementing the inverse fuzzy Elzaki transform, we have

$$\underline{\mathbf{f}}(\mathbf{x}, \xi; \wp) = \mathcal{E}^{-1}\left[\omega^2 \underline{\mathbf{f}}(\mathbf{x}, 0; \wp) + \omega^{2\alpha} \mathcal{E}\left[\bar{g}_3(\mathbf{x}, \xi) - \frac{\partial \bar{\mathbf{f}}(\mathbf{x}, \xi)}{\partial \mathbf{x}} \frac{\partial \bar{\mathbf{f}}^2(\mathbf{x}, \xi)}{\partial \mathbf{x}^2}\right]\right].$$

In addition, applying the scheme described in Section 4, we have

$$\sum_{r=0}^{\infty} \underline{\mathbf{f}}_r(\mathbf{x}, \xi; \wp) = \mathcal{E}^{-1}\left[\omega^2 \underline{\mathbf{f}}(\mathbf{x}, 0; \wp) + \omega^{2\alpha} \mathcal{E}\left[\bar{g}_3(\mathbf{x}, \xi)\right] - \omega^{2\alpha} \mathcal{E}\left[\sum_{r=0}^{\infty} \bar{\mathcal{A}}_r\right]\right].$$

Utilizing the iterative procedure defined in (54), we have

$$
\begin{aligned}
\underline{\mathbf{f}}_0(\mathbf{x},\xi;\wp) &= \mathcal{E}^{-1}\Big[\omega^2\underline{\mathbf{f}}(\mathbf{x},0;\wp)+\omega^{2\alpha}\mathcal{E}\big[\bar{g}_3(\mathbf{x},\xi)\big]\Big] \\
&= \wp\frac{\mathbf{x}^2}{2}+\big[(2-\wp)+\mathbf{x}(2-\wp)^2\big]\frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)},
\end{aligned}
$$

Moreover,

$$
\underline{\mathbf{f}}_{r+1}(\mathbf{x},\xi;\wp)=\mathcal{E}^{-1}\Big[\omega^{2\alpha}\mathcal{E}\big[\sum_{r=0}^{\infty}\bar{\mathcal{A}}_r\big]\Big].
$$

Utilizing the first few Adomian polynomials as follows

$$
\underline{\mathcal{A}}_r=\begin{cases}\frac{\partial^\eta\underline{\mathbf{f}}_0}{\partial\mathbf{x}^\eta}\frac{\partial^\sigma\underline{\mathbf{f}}_0}{\partial\mathbf{x}^\sigma}, & r=0,\\ \frac{\partial^\eta\underline{\mathbf{f}}_0}{\partial\mathbf{x}^\eta}\frac{\partial^\sigma\underline{\mathbf{f}}_1}{\partial\mathbf{x}^\sigma}+\frac{\partial^\eta\underline{\mathbf{f}}_1}{\partial\mathbf{x}^\eta}\frac{\partial^\sigma\underline{\mathbf{f}}_0}{\partial\mathbf{x}^\sigma}, & r=1,\\ \frac{\partial^\eta\underline{\mathbf{f}}_0}{\partial\mathbf{x}^\eta}\frac{\partial^\sigma\underline{\mathbf{f}}_2}{\partial\mathbf{x}^\sigma}+\frac{\partial^\eta\underline{\mathbf{f}}_1}{\partial\mathbf{x}^\eta}\frac{\partial^\sigma\underline{\mathbf{f}}_1}{\partial\mathbf{x}^\sigma}+\frac{\partial^\eta\underline{\mathbf{f}}_2}{\partial\mathbf{x}^\eta}\frac{\partial^\sigma\underline{\mathbf{f}}_0}{\partial\mathbf{x}^\sigma}, & r=2,\\ \vdots\end{cases}
\quad
\bar{\mathcal{A}}_r=\begin{cases}\frac{\partial^\eta\bar{\mathbf{f}}_0}{\partial\mathbf{x}^\eta}\frac{\partial^\sigma\bar{\mathbf{f}}_0}{\partial\mathbf{x}^\sigma}, & r=0,\\ \frac{\partial^\eta\bar{\mathbf{f}}_0}{\partial\mathbf{x}^\eta}\frac{\partial^\sigma\bar{\mathbf{f}}_1}{\partial\mathbf{x}^\sigma}+\frac{\partial^\eta\bar{\mathbf{f}}_1}{\partial\mathbf{x}^\eta}\frac{\partial^\sigma\bar{\mathbf{f}}_0}{\partial\mathbf{x}^\sigma}, & r=1,\\ \frac{\partial^\eta\bar{\mathbf{f}}_0}{\partial\mathbf{x}^\eta}\frac{\partial^\sigma\bar{\mathbf{f}}_2}{\partial\mathbf{x}^\sigma}+\frac{\partial^\eta\bar{\mathbf{f}}_1}{\partial\mathbf{x}^\eta}\frac{\partial^\sigma\bar{\mathbf{f}}_1}{\partial\mathbf{x}^\sigma}+\frac{\partial^\eta\bar{\mathbf{f}}_2}{\partial\mathbf{x}^\eta}\frac{\partial^\sigma\bar{\mathbf{f}}_0}{\partial\mathbf{x}^\sigma}, & r=2,\\ \vdots\end{cases}
$$

$$
\begin{aligned}
\underline{\mathbf{f}}(\mathbf{x},\xi;\wp) &= \mathcal{E}^{-1}\Big[\omega^{2\alpha}\big[\bar{\mathcal{A}}_0\big]\Big] \\
&= -\wp^2(2-\wp)\frac{\xi^{4\alpha}}{\Gamma(4\alpha+1)}-\mathbf{x}(2-\wp)^2\frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)}, \\
\underline{\mathbf{f}}_2(\mathbf{x},\xi;\wp) &= \mathcal{E}^{-1}\Big[\omega^{2\alpha}\big[\bar{\mathcal{A}}_1\big]\Big]=\wp^2(2-\wp)\frac{\xi^{4\alpha}}{\Gamma(4\alpha+1)}, \\
\underline{\mathbf{f}}_3(\mathbf{x},\xi;\wp) &= 0, \\
&\vdots
\end{aligned}
$$

In a similar way we obtained the upper solutions as follows:

$$
\begin{aligned}
\bar{\mathbf{f}}_0(\mathbf{x},\xi;\wp) &= \frac{\mathbf{x}^2}{2}(2-\wp)+\big(2-\wp+\mathbf{x}(2-\wp)^2\big)\frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)}, \\
\bar{\mathbf{f}}(\mathbf{x},\xi;\wp) &= -\mathbf{x}(2-\wp)^2\frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)}-(2-\wp)^3\frac{\xi^{4\alpha}}{\Gamma(4\alpha+1)}, \\
\bar{\mathbf{f}}_2(\mathbf{x},\xi;\wp) &= -(2-\wp)^3\frac{\xi^{4\alpha}}{\Gamma(4\alpha+1)}, \\
\bar{\mathbf{f}}_3(\mathbf{x},\xi;\wp) &= 0, \\
&\vdots
\end{aligned}
$$

The series form solution of Example 1 is presented as follows:

$$
\mathbf{f}(\mathbf{x},\xi)=\left(\left(\frac{\mathbf{x}^2}{2}\wp+(2-\wp)\frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)}\right),\left(\frac{\mathbf{x}^2}{2}(2-\wp)+\wp\frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)}\right)\right).
$$

The results show that perfect fractional order precision and uncertainty for fuzzy numerical solutions of the function $\mathbf{f}(\mathbf{x},\xi)$ are highly correlated to stuffing time and the fractional order used, whereas additional precision solutions can be obtained by using more redundancy and iterative development.

- Table 2 represents the obtained findings with $\mathbf{x}=0.4$ and $\xi=0.7$. Table 2 also comprises the outcomes of Georgieva and Pavlova [57]. As a consequence, the findings acquired by the fuzzy Elzaki decomposition method are the same if $\alpha = 1$, as those reported by Georgieva and Pavlova [57].

- Figure 4a,b demonstrates the three-dimensional illustration of the lower and upper estimates for different uncertainties $\wp \in [0, 1]$.
- Figure 5a,b shows the fuzzy responses for different fractional orders. Figure 6a,b illustrates the fuzzy responses for different uncertainty parameters.
- The aforementioned representations illustrate that all graphs are substantially identical in their perspectives and have good agreement with one another, especially when integer-order derivatives are taken into account.

Finally, this generic approach for dealing with nonlinear PDEs is more accurate and powerful than the method applied by [57]. Our findings for the fuzzy Elzaki decomposition method, helpful for fuzzy initial value problems, demonstrate the consistency and strength of the offered solutions.

Table 2. Lower and upper solutions of **Case II** of Example 1 for various fractional orders in comparison with the solution derived by [57].

$\wp$	$\underline{\mathbf{f}}(\alpha = \mathbf{0.7})$	$\bar{\mathbf{f}}(\alpha = \mathbf{0.7})$	$\underline{\mathbf{f}}(\alpha = \mathbf{1})$	$\bar{\mathbf{f}}(\alpha = \mathbf{1})$	$\underline{\mathbf{f}}$ [57]	$\bar{\mathbf{f}}$ [57]
0.1	2.8799×10^{-1}	1.0042×10^{-1}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}
0.2	2.7757×10^{-1}	1.1084×10^{-1}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}
0.3	2.6715×10^{-1}	1.2126×10^{-1}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}
0.4	2.5673×10^{-1}	1.3168×10^{-1}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}
0.5	2.4631×10^{-1}	1.4210×10^{-1}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}
0.6	2.3589×10^{-1}	1.5252×10^{-1}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}
0.7	2.2546×10^{-1}	1.6294×10^{-1}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-1}
0.8	2.1504×10^{-1}	1.7336×10^{-1}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-1}
0.9	2.0462×10^{-1}	1.8378×10^{-1}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}
1.0	1.9420×10^{-1}	1.9420×10^{-1}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}

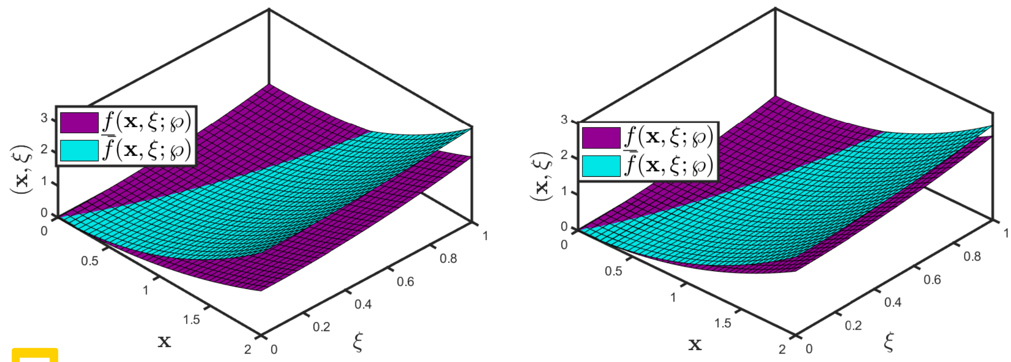
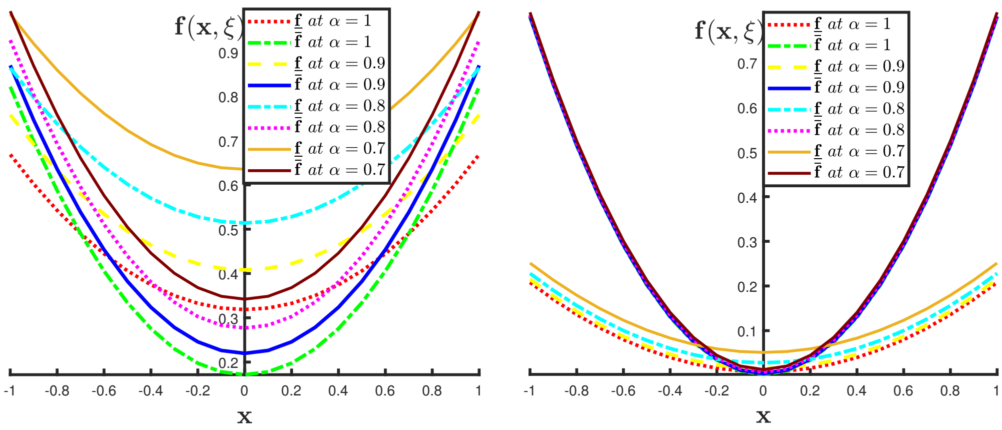
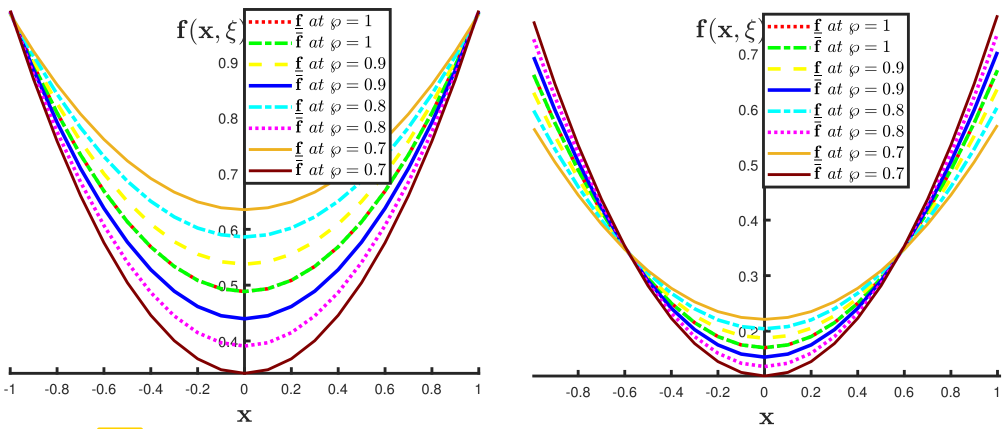


Figure 4. Three-dimensional fuzzy responses of Example 1 for **Case II** at (**a**) $\wp = 0.7$, (**b**) $\wp = 0.9$ with fractional order $\alpha = 1$.



(**a**) (**b**)

Figure 5. Two-dimensional fuzzy responses of Example 1 for **Case II** at (**a**) $\wp = 0.7$ and $\xi = 0.7$, (**b**) $\wp = 0.4$ and $\xi = 0.1$ with varying fractional orders.



(**a**) (**b**)

Figure 6. Two-dimensional fuzzy responses of Example 1 for **Case II** at (**a**) $\alpha = 0.7$ and $\xi = 0.7$, (**b**) $\alpha = 0.4$ and $\xi = 0.1$ with varying uncertainty parameters $\wp \in [0, 1]$.

6. Conclusions

In this investigation, the fuzzy Caputo fractional problem formalism, homogenized fuzzy initial condition, partial differential equation, exemplification of fuzzy Caputo fractional derivative and numerical solutions under $g\mathcal{H}$ are the main significations of the following subordinate part.

- The generic formulation of fuzzy CFDs pertaining to the generic order of $0 < \alpha < r$ is derived by combining all conceivable groupings of items such that t_1 equals one and t_2 (the others) equals two and is utilized for the first time.
- The generic formulas for CFDs regarding the order $\alpha \in (r-1, r)$ are generated under the $g\mathcal{H}$-difference.
- Under $\mathcal{H}$-differentiability, a semi-analytical approach for finding the solution of nonlinear fuzzy fractional PDE was applied. Furthermore, this methodology offers a series of solutions as an analytical expression is its significant aspect.
- A test problem is solved to demonstrate the proposed approach. The simulation results can solve nonlinear partial fuzzy differential equations in a flexible and efficient manner, whilst the frame of numerical programming is natural and the computations are very swift in terms of fractional orders and uncertainty parameters $\wp \in [0, 1]$.
- The results of the projected methodology are more general and fractional in nature than the results provided by [57].

- For futuristic research, a similar method can be applied to Fitzhugh–Nagumo–Huxley by formulating the Henstock integrals (fuzzy integrals in the Lebesgue notion) at infinite intervals [58,59]. Furthermore, one can explore the implementation of this strategy for relatively intricate challenges, such as the eigenproblem [60] and maximum likelihood estimation [61].

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