Numerical investigations of a new singular second order nonlinear coupled functional Lane-Emden model

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Abstract:

The aim of the present study is to design a second order nonlinear Lane-Emden coupled functional differential model and numerically investigate by using the famous spectral collocation method. For validation of the new designed model, three dissimilar variants have been considered and formulated numerically by applying a famous spectral collocation method. Moreover, comparison of the obtained results with the exact/true results endorse the effectiveness and competency of the new designed model, as well as, the present technique.

Keywords: Coupled Lane-Emden; Nonlinear; Functional differential model; singular; spectral collocation scheme; Exact solutions

1 Introduction

The present research work is related to the singular models for the second order nonlinear system of functional differential (FD) equations. These FD equations have a huge variety of applications, some are them are, the growth rate population model [1], electrodynamics [2], infection HIV-1 model [3], growth rate of tumor model [4], chemical kinetics model [5], hepatitis-B virus infection model [6] and gene regulatory system [7]. Few numerical techniques have been applied to solve these FD equations, e.g., Kadalbajoo et al [8] presented a numerical scheme for solving the FD equations, Mirzaee et al [9] implemented a collocation technique, Xu et al [10] used fractional measures and boundary functions for presenting the solution of these equations and Genga et al [11] discussed a numerical approach for solving the singularly perturbed FD equations. Due to the singular point, these models have achieved diverse attention of the research community. One of the significant and historical singular model is Lane-Emden (LE) model introduced by Lane and further investigated by Emden, which has a wider range of applications in science, technology and engineering. The LE model is used in the density field of gaseous star [12], catalytic diffusion reactions [13], mathematical geometry and physics [14], isothermal and polytrophic gas spheres [15], the theory of electromagnetic [16], magnetic field oscillation [17], quantum and classical mechanics [18], isotropic continuous media [19], morphogenesis [20] and dusty fluid models [21]. To the solution of LE model, many numerical and analytic techniques have been applied. Shawagfeh [22] applied method of Adomian decomposition, Bender et al [23] used method of perturbation, Liao [24] proposed an analytic algorithm, Nouh [25] implemented power series technique by using Pade approximation, Mandelzweigand and Tabakin [26] used method of Bellman and Kalabas quasi linearization to solve the LE equation. Recently, some numerical techniques are also broadly implemented to solve the singular LE type of models [27-33].

The present research work is about to model the second kind of singular nonlinear coupled

LE system of FD equations and its modeled form is written as:

$$\begin{aligned}\mathcal{Z}^{''}(ax+\alpha)+\frac{\eta_1}{x}\mathcal{Z}^{'}(bx+\beta)+\mathcal{Q}(x)\mathcal{Z}(cx+\gamma)=\mathcal{F}(x)\\ \mathcal{Q}^{''}(ax+\alpha)+\frac{\eta_2}{x}\mathcal{Q}^{'}(bx+\beta)+\mathcal{Z}(x)\mathcal{Q}(cx+\gamma)=\mathcal{H}(x),\ 0\le x\le\mathcal{L},\end{aligned}\tag{1}$$

with the following conditions

$$\mathcal{Z}(\mathcal{L})=\zeta_1,\quad \mathcal{Z}^{'}(0)=\zeta_2,\quad \mathcal{Q}(\mathcal{L})=\zeta_3,\quad \mathcal{Q}^{'}(0)=\zeta_4.\tag{2}$$

where a, b, c, α, β, γ, η_1, η_2, ζ_1, ζ_2, ζ_3 and ζ_4 are given constants while $\mathcal{F}(x)$, $\mathcal{H}(x)$ are given functions. The FD model is basically the extension of the research study of Sabir et al [34-36], which is applied to solve the singular nonlinear functional differential equations. The designed model is verified by solving the three variants based on nonlinear LE second order coupled FD equations using the numerical spectral collocation scheme. The novel features of the current work are briefly shortened as:

- The mathematical model for the nonlinear LE second order coupled FD equation is successfully presented and verified by solving the three variants of the models using the spectral collocation method.

- The comparison is performed of the obtained numerical results from the spectral collocation method with the true results, which shows the correctness of the presented system, as well as, designed approach.

- Manipulation of the present spectral collocation method by applying the designed model provided brilliance solutions with higher accuracy and greater dependability.

- The consistency of the designed mathematical model is certified from the reliable absolute error of the proposed and exact outcomes.

- The nonlinear LE second order system of FD model is not simple to handle numerically because of the singularity, harder in nature and non-linearity. spectral collocation method is one of the best suggestions, as well as great selection to tackle such kinds of complex systems.

A Large amount of work to model the physical systems has been restricted to ordinary differential equations. Therefore, the urgent requirement to achieve the exact solutions or simply the approximate ones to these problems has emerged. Since, the finding of the exact solutions is not possible for these fractional differential models mostly. Hence, the numerical techniques have been implemented to find the approximate solutions to solve them. Some local numerical techniques are introduced for solving such systems and this method may become computationally heavy. Moreover, the local schemes listed the approximate solution at particular points, whereas the global approaches provide the approximate results in whole interval. Hence, the global behavior of the solution can be naturally taken into account. The spectral collocation technique is a global numerical metod that is a particular kind of famous spectral methods, which is widely applicable for almost each type of differential equations. Recently, there are more interest of appointing the spectral collocation method to treat with various types of integral and differential models [37, 38], due to its importance to finite/infinite ranges [39, 40]. The convergence speed is the major advantages of the spectral collocation method. This method has exponential convergence rates as well as a high accuracy level. The spectral method has been classified into four classes, collocation [42], tau [43], Galerkin [44] and Petrov Galerkin [45] method. The collocation approach is a particular kind of spectral technique, that is widely suitable for almost all kinds of differential systems.

The other parts of the paper are organized as: A few relevant properties of Jacobi shift polynomials, designed scheme, detailed result discussions, conclusions and future research guidance are described in the remaining sections.

2 Shifted Jacobi polynomials

The Jacobi polynomials (JP) known as the eigen functions based on the singular form of the Sturm-Liouville equation. In view of this, many particular cases exist, like as Legendre, the four type of Gegenbauer and Chebyshev polynomials. Furthermore, the

JP have been applied in extensive applications because of its wider ability to approximate the general categories of the functions. Few of them are the Gibbs' phenomenon resolution, data compression electrocardiogram and to solve differential models. For $[0, L]$ interval, a shifted Jacobi polynomials (SJP) is indeed applied with the freedom to select the Jacobi indexes θ and ϑ, the method can be calibrated for a wide variety of problems. To consider the SJP $\mathcal{J}_k^{(\rho,\sigma)}(x)$, which satisfy the following properties:

$$\begin{aligned}
&\mathcal{J}_{k+1}^{(\rho,\sigma)}(x) = (a_k^{(\rho,\sigma)}x - b_k^{(\rho,\sigma)})\mathcal{J}_k^{(\rho,\sigma)}(x) - c_k^{(\rho,\sigma)}\mathcal{J}_{k-1}^{(\rho,\sigma)}(x), \quad k \geq 1,\\
&\mathcal{J}_0^{(\rho,\sigma)}(x) = 1, \quad \mathcal{J}_1^{(\rho,\sigma)}(x) = \frac{1}{2}(\rho+\sigma+2)x + \frac{1}{2}(\rho-\sigma),\\
&\mathcal{J}_k^{(\rho,\sigma)}(-x) = (-1)^k\mathcal{J}_k^{(\rho,\sigma)}(x), \quad \mathcal{J}_k^{(\rho,\sigma)}(-1) = \frac{(-1)^k\Gamma(k+\sigma+1)}{k!\Gamma(\sigma+1)},
\end{aligned} \tag{3}$$

where $\rho,\ \sigma > -1$, $x \in [-1, 1]$ and

$$\begin{aligned}
a_k^{(\rho,\sigma)} &= \frac{(2k+\rho+\sigma+1)(2k+\rho+\sigma+2)}{2(k+1)(k+\rho+\sigma+1)},\\
b_k^{(\rho,\sigma)} &= \frac{(\sigma^2-\rho^2)(2k+\rho+\sigma+1)}{2(k+1)(k+\rho+\sigma+1)(2k+\rho+\sigma)},\\
c_k^{(\rho,\sigma)} &= \frac{(k+\rho)(k+\sigma)(2k+\rho+\sigma+2)}{(k+1)(k+\rho+\sigma+1)(2k+\rho+\sigma)}.
\end{aligned}$$

Moreover, the rth derivative of $\mathcal{J}_j^{(\rho,\sigma)}(x)$, is formulated as

$$D^r\mathcal{J}_j^{(\rho,\sigma)}(x) = \frac{\Gamma(j+\rho+\sigma+q+1)}{2^r\Gamma(j+\rho+\sigma+1)}\mathcal{J}_{j-r}^{(\rho+r,\sigma+r)}(x), \tag{4}$$

where $'r'$ represents an integer value. For the SJP $\mathcal{J}_{\mathcal{L},k}^{(\rho,\sigma)}(x) = \mathcal{J}_k^{(\rho,\sigma)}(\frac{2x}{\mathcal{L}} - 1),\ \mathcal{L} > 0$, the analytic explicit form is given as:

$$\begin{aligned}
P_{\mathcal{L},k}^{(\rho,\sigma)}(x) &= \sum_{j=0}^{k}(-1)^{k-j}\frac{\Gamma(k+\sigma+1)\Gamma(j+k+\rho+\sigma+1)}{\Gamma(j+\sigma+1)\Gamma(k+\rho+\sigma+1)(k-j)!j!\mathcal{L}^j}x^j\\
&= \sum_{j=0}^{k}\frac{\Gamma(k+\rho+1)\Gamma(k+j+\rho+\sigma+1)}{j!(k-j)!\Gamma(j+\rho+1)\Gamma(k+\rho+\sigma+1)\mathcal{L}^j}(x-\mathcal{L})^j.
\end{aligned} \tag{5}$$

To deduce the following

$$\begin{aligned}
P_{\mathcal{L},k}^{(\rho,\sigma)}(0) &= (-1)^k\frac{\Gamma(k+\sigma+1)}{\Gamma(\sigma+1)\ k!},\\
\mathcal{J}_{\mathcal{L},k}^{(\rho,\sigma)}(\mathcal{L}) &= \frac{\Gamma(k+\rho+1)}{\Gamma(\rho+1)\ k!},
\end{aligned} \tag{6}$$

$$D^r \mathcal{J}_{\mathcal{L},k}^{(\rho,\sigma)}(0) = \frac{(-1)^{k-r}\Gamma(k+\sigma+1)(k+\rho+\sigma+1)_r}{L^r\Gamma(k-r+1)\Gamma(r+\sigma+1)}, \tag{7}$$

$$D^r \mathcal{J}_{\mathcal{L},k}^{(\rho,\sigma)}(\mathcal{L}) = \frac{\Gamma(k+\rho+1)(k+\rho+\sigma+1)_r}{L^r\Gamma(k-r+1)\Gamma(r+\rho+1)}, \tag{8}$$

$$D^r \mathcal{J}_{\mathcal{L},k}^{(\rho,\sigma)}(x) = \frac{\Gamma(r+k+\rho+\sigma+1)}{\mathcal{L}^r\Gamma(k+\rho+\sigma+1)} \mathcal{J}_{\mathcal{L},k-r}^{(\rho+r,\sigma+r)}(x). \tag{9}$$

3 Methodology of Shifted Jacobi collocation method

The collocation technique is an easy weighted residuals approach. Lanczos [46] first time introduced the proper trial function form together with the collocation point distributions that is considered fundamental to the precision of the obtained outcomes. Further, this research work is revived by the Clenshaw et al [47, 48] and Wright [49]. These studies involve the applications of the expansions of the Chebyshev polynomial to the initial value problems. Here, in this section, a numerical method based on shifted Jacobi collocation approach is presented to solve a new nonlinear singular second kind of coupled functional LE differential model, given as:

$$\begin{aligned} &\mathcal{Z}''(ax+\alpha) + \frac{\eta_1}{x}\mathcal{Z}'(bx+\beta) + \mathcal{Q}(x)\mathcal{Z}(cx+\gamma) = \mathcal{F}(x) \\ &\mathcal{Q}''(ax+\alpha) + \frac{\eta_2}{x}\mathcal{Q}'(bx+\beta) + \mathcal{Z}(x)\mathcal{Q}(cx+\gamma) = \mathcal{H}(x),\ 0 \le x \le \mathcal{L}, \end{aligned} \tag{10}$$

with the following conditions

$$\mathcal{Z}(\mathcal{L}) = \zeta_1, \quad \mathcal{Z}'(0) = \zeta_2, \quad \mathcal{Q}(\mathcal{L}) = \zeta_3, \quad \mathcal{Q}'(0) = \zeta_4, \tag{11}$$

where a, b, c, α, β, γ, η_1, η_2, ζ_1, ζ_2, ζ_3 and ζ_4 are given constants while $\mathcal{F}(x)$, $\mathcal{H}(x)$ are given functions. The solution of Eq. (10) is approximated as.

$$\mathcal{Z}_{\mathcal{K}}(x) = \sum_{j=0}^{\mathcal{K}} \varsigma_j \mathcal{J}_{\mathcal{L},j}^{(\rho,\sigma)}(x) = \Delta_{\mathcal{L},\mathcal{K}}^{(\rho,\sigma)}(x), \quad \mathcal{Q}_{\mathcal{K}}(x) = \sum_{j=0}^{\mathcal{K}} \varepsilon_j \mathcal{J}_{\mathcal{L},j}^{(\rho,\sigma)}(x) = \Omega_{\mathcal{L},\mathcal{K}}^{(\rho,\sigma)}(x). \tag{12}$$

The approximate independent variables by applying the shifted Jacobi collocation scheme at $x_{\mathcal{L},\mathcal{K},j}^{(\rho,\sigma)}$ grids. These grids are the point set in a indicated range, where the values of dependent variable are estimated. Generally, the performance of the nodes location become optional using $x_{\mathcal{L},\mathcal{K},j}^{(\rho,\sigma)}$ as a Jacobi-Gauss-Lobatto nodes. Thus, we can approximate the functions $\mathcal{Z}(cx+\gamma)$, $\mathcal{Q}(cx+\gamma)$ as:

$$\begin{aligned}\mathcal{Z}_{\mathcal{K}}(cx+\gamma) &= \sum_{j=0}^{\mathcal{K}} \varsigma_j \mathcal{J}_{\mathcal{L},j}^{(\rho,\sigma)}(cx+\gamma) = \Delta_{\mathcal{L},\mathcal{K}}^{(\rho,\sigma)}(cx+\gamma)\\ \mathcal{Q}_{\mathcal{K}}(cx+\gamma) &= \sum_{j=0}^{\mathcal{K}} \varepsilon_j \mathcal{J}_{\mathcal{L},j}^{(\rho,\sigma)}(cx+\gamma) = \Omega_{\mathcal{L},\mathcal{K}}^{(\rho,\sigma)}(cx+\gamma).\end{aligned} \tag{13}$$

Thus, the required derivatives of first and second orders of the approximate solutions are then estimated as

$$\begin{aligned}\mathcal{Z}_{\mathcal{K}}^{'}(x) &= \sum_{j=0}^{\mathcal{K}} \varsigma_j \left(\mathcal{J}_{\mathcal{L},j}^{(\rho,\sigma)}(x)\right)^{'}\\ &= \sum_{j=0}^{\mathcal{K}} \varsigma_j \frac{j+\rho+\sigma+1}{\mathcal{L}} \mathcal{P}_{\mathcal{L},j-1}^{(\rho+1,\sigma+1)}(x)\\ &= \wp_{\mathcal{L},\mathcal{K}}^{(\rho,\sigma)}(x),\end{aligned} \tag{14}$$

$$\begin{aligned}\mathcal{Q}_{\mathcal{K}}^{'}(x) &= \sum_{j=0}^{\mathcal{K}} \varepsilon_j \left(\mathcal{J}_{\mathcal{L},j}^{(\rho,\sigma)}(x)\right)^{'}\\ &= \sum_{j=0}^{\mathcal{K}} \varepsilon_j \frac{j+\rho+\sigma+1}{\mathcal{L}} \mathcal{P}_{\mathcal{L},j-1}^{(\rho+1,\sigma+1)}(x)\\ &= \Im_{\mathcal{L},\mathcal{K}}^{(\rho,\sigma)}(x),\end{aligned} \tag{15}$$

$$\begin{aligned}\mathcal{Z}_{\mathcal{K}}^{'}(bx+\beta) &= \sum_{j=0}^{\mathcal{K}} \varsigma_j \left(\mathcal{J}_{\mathcal{L},j}^{(\rho,\sigma)}(bx+\beta)\right)^{'}\\ &= \sum_{j=0}^{\mathcal{K}} \varsigma_j \frac{b(j+\rho+\sigma+1)}{\mathcal{L}} \mathcal{P}_{\mathcal{L},j-1}^{(\rho+1,\sigma+1)}(bx+\beta)\\ &= \phi_{\mathcal{L},\mathcal{K}}^{(\rho,\sigma)}(x),\end{aligned} \tag{16}$$

and

$$\begin{aligned}\mathcal{Q}_{\mathcal{K}}^{'}(bx+\beta) &= \sum_{j=0}^{\mathcal{K}} \varepsilon_j \left(\mathcal{J}_{\mathcal{L},j}^{(\rho,\sigma)}(bx+\beta)\right)^{'}\\ &= \sum_{j=0}^{\mathcal{K}} \varepsilon_j \frac{b(j+\rho+\sigma+1)}{\mathcal{L}} \mathcal{P}_{\mathcal{L},j-1}^{(\rho+1,\sigma+1)}(bx+\beta)\\ &= \varphi_{\mathcal{L},\mathcal{K}}^{(\rho,\sigma)}(x).\end{aligned} \tag{17}$$

Also, we get

$$
\begin{aligned}
\mathcal{Z}''_{\mathcal{K}}(ax+\alpha) =& \sum_{j=0}^{\mathcal{K}} \varsigma_j \big(\mathcal{J}^{(\rho,\sigma)}_{\mathcal{L},j}(ax+\alpha)\big)'' \\
=& \sum_{j=0}^{\mathcal{K}} \varsigma_j \frac{a^2(j+\rho+\sigma+1)(j+\rho+\sigma+2)}{\mathcal{L}^2} \mathcal{P}^{(\rho+2,\sigma+2)}_{\mathcal{L},j-2}(ax+\alpha) \\
=& \chi^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K}}(x),
\end{aligned}
\tag{18}
$$

and

$$
\begin{aligned}
\mathcal{Q}''_{\mathcal{K}}(ax+\alpha) =& \sum_{j=0}^{\mathcal{K}} \varepsilon_j \big(\mathcal{J}^{(\rho,\sigma)}_{\mathcal{L},j}(ax+\alpha)\big)'' \\
=& \sum_{j=0}^{\mathcal{K}} \varepsilon_j \frac{a^2(j+\rho+\sigma+1)(j+\rho+\sigma+2)}{\mathcal{L}^2} \mathcal{P}^{(\rho+2,\sigma+2)}_{\mathcal{L},j-2}(ax+\alpha) \\
=& \psi^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K}}(x).
\end{aligned}
\tag{19}
$$

Then, we can estimated the residual of (10) as:

$$
\begin{aligned}
\chi^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K}}(x) + \frac{\eta_1}{x}\phi^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K}}(x) + \Omega^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K}}(x)\Delta^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K}}(cx+\gamma) =& \mathcal{F}(x) \\
\psi^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K}}(x) + \frac{\eta_2}{x}\varphi^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K}}(x) + \Delta^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K}}(x)\Omega^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K}}(cx+\gamma) =& \mathcal{H}(x).
\end{aligned}
\tag{20}
$$

In the shifted Jacobi collocation method, the residual (20) is let to be zero at the $\mathcal{K}-1$ points

$$
\begin{aligned}
\chi^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K}}(x^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},i}) + \frac{\eta_1}{x^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},i}}\phi^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K}}(x^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},i}) + \Omega^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K}}(x^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},i})\Delta^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K}}(cx^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},i}+\gamma) =& \mathcal{F}(x^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},i}), \\
\psi^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K}}(x^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},i}) + \frac{\eta_2}{x^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},i}}\varphi^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K}}(x^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},i}) + \Delta^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K}}(x^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},i})\Omega^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K}}(cx^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},i}+\gamma) =& \mathcal{H}(x^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K},i}),
\end{aligned}
\tag{21}
$$

where $i = 1, 2, 3 \ldots, \mathcal{K}-1$. So, the $2\mathcal{K}-2$ algebraic model for $2\mathcal{K}+2$, the remaining unknown equations can achieved from the conditions (11) as:

$$
\Delta^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K}}(\mathcal{L}) = \zeta_1, \quad \wp^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K}}(0) = \zeta_2, \quad \Omega^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K}}(\mathcal{L}) = \zeta_3, \quad \Im^{(\rho,\sigma)}_{\mathcal{L},\mathcal{K}}(0) = \zeta_4, \tag{22}
$$

Finally, from Eqs. (21) and (22), the ($\mathcal{K}$+1) nonlinear algebraic system can be implemented to the unidentified coefficients ς_j, $j = 0, \ldots, \mathcal{K}$.

4 Numerical results and comparisons

Using the algorithm presented in the last section, the three numerical variants to solve the coupled FD model to show the high accuracy as well as precision of the proposed method.

4.1 Problem I

Consider the following nonlinear singular second order coupled functional differential model of LE type is given as:

$$\begin{aligned} &\mathcal{Z}^{''}(2x-1)+\frac{3}{x}\mathcal{Z}^{'}(3x)+\mathcal{Q}(x)\mathcal{Z}(x+1)=\mathcal{F}(x), \\ &\mathcal{Q}^{''}(2x-1)+\frac{2}{x}\mathcal{Q}^{'}(3x)+\mathcal{Z}(x)\mathcal{Q}(x+1)=\mathcal{H}(x),\ 0\leq x\leq 1, \\ &\mathcal{Z}(1)=2, \quad \mathcal{Z}^{'}(0)=0, \quad \mathcal{Q}(1)=0, \quad \mathcal{Q}^{'}(0)=0, \end{aligned} \tag{23}$$

where, $\mathcal{F}(x)$ and $\mathcal{H}(x)$ are selected as an exact solution as:
$\mathcal{Z}(x)=1+x^3$, $\mathcal{Q}(x)=1-x^3$.
In Table (1), the numerical solutions are ($\mathcal{Z}_{\mathcal{K}}$ and $\mathcal{Q}_{\mathcal{K}}$) of Problem I for different parameter values. The resulting values of Table (1), shows more accurate results. The prefect matching of the obtained and exact solutions is observed in Figs. 1 and 2 . The curves of absolute error (AE) $E_{\mathcal{Z}}$ and $E_{\mathcal{Q}}$ for the Problem I is provided in 3 and 4.

4.2 Problem II

The following nonlinear singular second order coupled functional differential model of Lane-Emden type is written as:

$$\begin{aligned} &\mathcal{Z}^{''}(2x-1)+\frac{3}{x}\mathcal{Z}^{'}(3x)+\mathcal{Q}(x)\mathcal{Z}(x+1)=\mathcal{F}(x) \\ &\mathcal{Q}^{''}(2x-1)+\frac{2}{x}\mathcal{Q}^{'}(3x)+\mathcal{Z}(x)\mathcal{Q}(x+1)=\mathcal{H}(x),\ 0\leq x\leq 1, \\ &\mathcal{Z}(1)=1+\cos(1), \quad \mathcal{Z}^{'}(0)=0, \quad \mathcal{Q}(1)=1-\cos(1), \quad \mathcal{Q}^{'}(0)=0, \end{aligned} \tag{24}$$

where $\mathcal{F}(x)$ and $\mathcal{H}(x)$ are selected for the exact solutions as: $\mathcal{Z}(x)=1+\cos(x)$, $\mathcal{Q}(x)=1-\cos(x)$. Table 2 highlights the accurate obtained results for the $\mathcal{M}_{E_{\mathcal{Z}}}$ and $\mathcal{M}_{E_{\mathcal{Q}}}$ using

Table 1: Numerical solutions of Problem I.

$\mathcal{K}$	2	3
	$\rho=\sigma=0$	
$\mathcal{Z}_{\mathcal{K}}$	$1.92174x^2-2.2204\times10^{-16}x+0.0782624$	$x^3+2.2204\times10^{-16}x^2+1.11022\times10^{-16}x+1$
$\mathcal{Q}_{\mathcal{K}}$	$-1.87192x^2+2.2204\times10^{-16}x+1.87192$	$1-1.11022\times10^{-16}x-4.44089\times10^{-16}x^2-x^3$
	$\rho=-0.5,\ \sigma=0$	
$\mathcal{Z}_{\mathcal{K}}$	$-0.463357+2.46336x^2$	$1+2.22045\times10^{-16}x^2+x^3$
$\mathcal{Q}_{\mathcal{K}}$	$2.48434-2.48434x^2$	$1-4.44089\times10^{-16}x-x^3$
	$\rho=\sigma=0.5$	
$\mathcal{Z}_{\mathcal{K}}$	$0.0782624+1.92174x^2$	$1-1.11022\times10^{-16}x+x^3$
$\mathcal{Q}_{\mathcal{K}}$	$1.87192-1.87192x^2$	$1+2.22045\times10^{-16}x^2-x^3$
	$\rho=\sigma=-0.5$	
$\mathcal{Z}_{\mathcal{K}}$	$0.0782624+1.92174x^2$	$1+2.22045\times10^{-16}x+x^3$
$\mathcal{Q}_{\mathcal{K}}$	$1.87192+2.22045\times10^{-16}x-1.87192x^2$	$1+1.11022\times10^{-16}x-x^3$

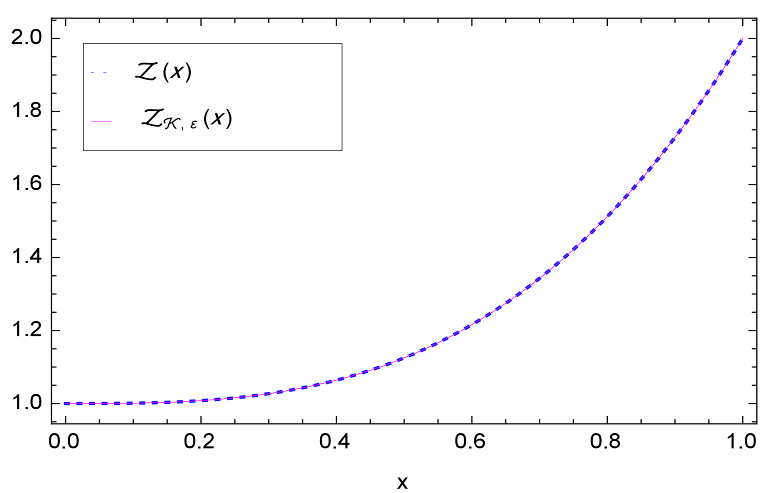

Figure 1: Plots of the exact and numerical results ($\mathcal{Z}$ and $\mathcal{Z}_{\mathcal{K}}$) of Problem I with $\rho=\sigma=0$, and $\mathcal{K}=3$.

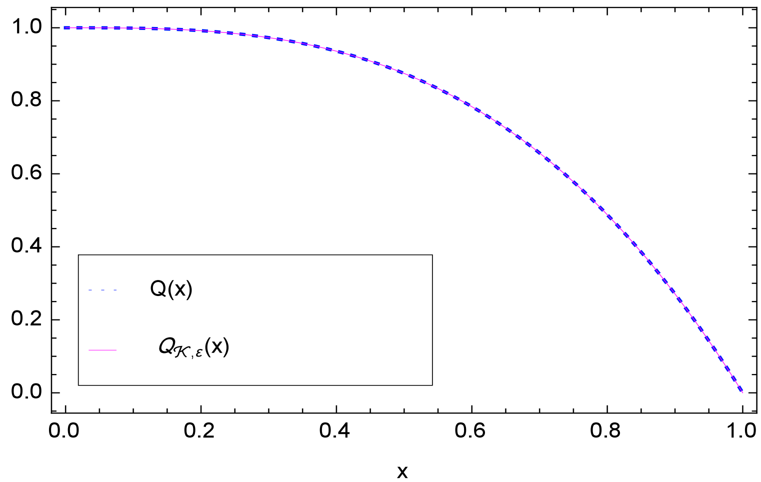

Figure 2: Plots of the exact and numerical results ($\mathcal{Q}$ and $\mathcal{Q}_{\mathcal{K}}$)of Problem I with $\rho = \sigma = 0$, and $\mathcal{K} = 3$.

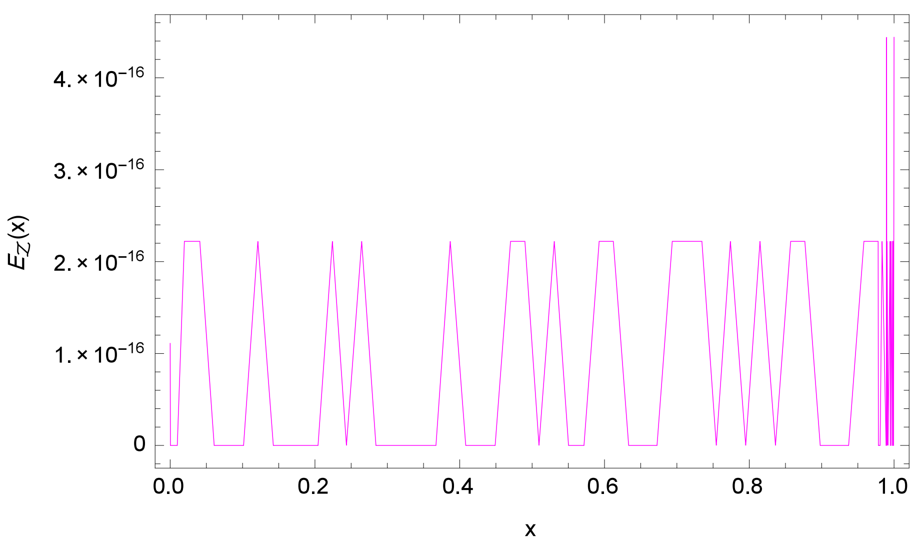

Figure 3: Plots of the AE ($E_{\mathcal{Z}}$) of Problem I with $\rho = \sigma = 0$, and $\mathcal{K} = 3$.

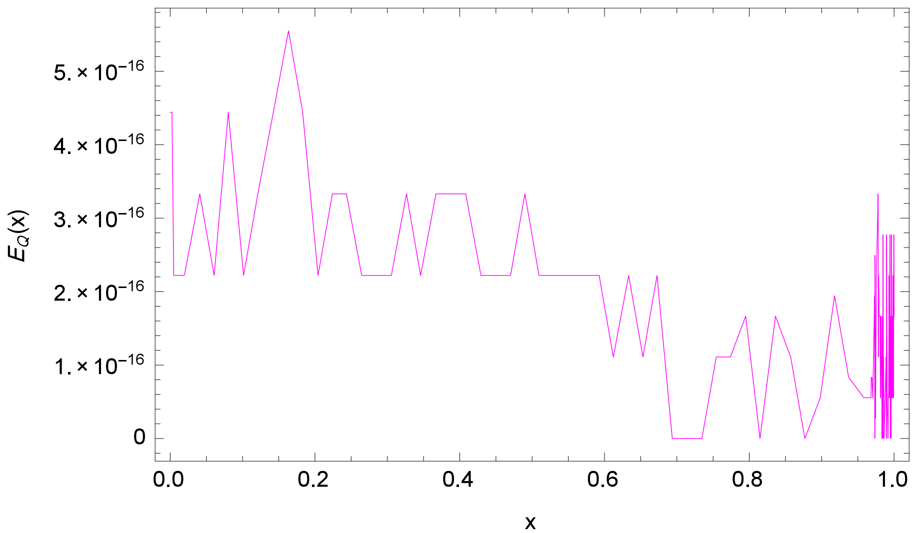

Figure 4: Plots of the AE ($E_{\mathcal{Q}}$) of Problem I with $\rho = \sigma = 0$, and $\mathcal{K} = 3$.

Table 2: $\mathcal{M}_{\mathcal{E}_{\mathcal{Z}}}$ and $\mathcal{M}_{\mathcal{E}_{\mathcal{Q}}}$ of Problem II.

	$\rho = 0,\ \sigma = \frac{1}{2}$		$\rho = \sigma = \frac{1}{2}$		$\rho = \sigma = 0$	
$\mathcal{K}$	$\mathcal{M}_{\mathcal{E}_{\mathcal{Z}}}$	$\mathcal{M}_{\mathcal{E}_{\mathcal{Q}}}$	$\mathcal{M}_{\mathcal{E}_{\mathcal{Z}}}$	$\mathcal{M}_{\mathcal{E}_{\mathcal{Q}}}$	$\mathcal{M}_{\mathcal{E}_{\mathcal{Z}}}$	$\mathcal{M}_{\mathcal{E}_{\mathcal{Q}}}$
2	1.34×10^{-1}	1.21×10^{-1}	1.04×10^{-1}	9.47×10^{-2}	1.04×10^{-1}	9.47×10^{-2}
6	3.09×10^{-4}	9.56×10^{-4}	3.43×10^{-4}	9.73×10^{-4}	4.61×10^{-4}	1.14×10^{-3}
10	7.16×10^{-6}	3.04×10^{-5}	7.28×10^{-6}	2.91×10^{-5}	7.88×10^{-6}	2.97×10^{-5}
14	6.57×10^{-8}	2.94×10^{-7}	5.35×10^{-8}	2.82×10^{-7}	4.72×10^{-8}	2.86×10^{-7}
18	3.93×10^{-9}	1.28×10^{-8}	6.46×10^{-9}	1.00×10^{-8}	1.16×10^{-8}	1.22×10^{-9}

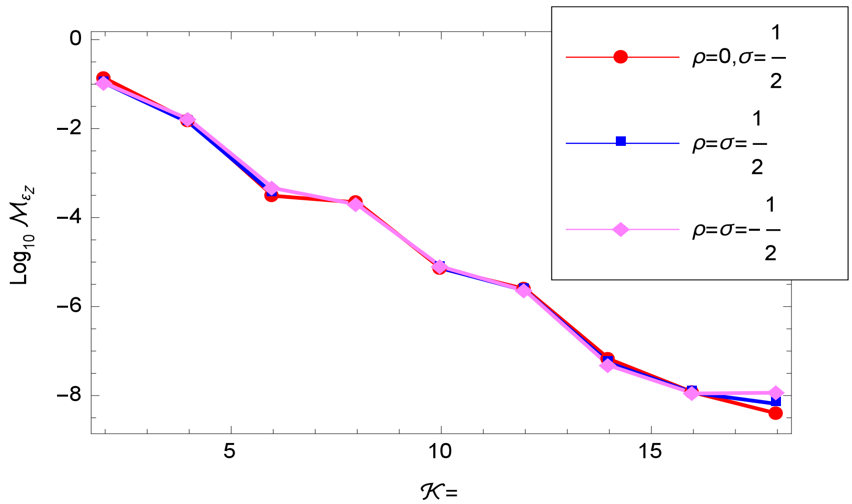

Figure 5: $\mathcal{M}_{E_{\mathcal{Z}}}$ convergence of Problem II.

the spectral collocation method. Moreover, the logarithmic graphs of $\mathcal{M}_{E_{\mathcal{Z}}}$ and $\mathcal{M}_{E_{\mathcal{Q}}}$ are plotted using the current scheme for different values of ρ, σ and ($\mathcal{K} = 2, 4, \cdots, 18$) in Figs. 4 and 5. Take $\rho = \sigma = -\frac{1}{2}$, the obtained form of the numerical solution becomes as:

$$\begin{aligned}\mathcal{Z}_{18}(x) =& 2 + 2.64225 \times 10^{-17}x - 0.5x^2 + 2.93062 \times 10^{-8}x^3 + 0.0416666x^4 + \\ & 2.36602 \times 10^{-9}x^5 - 0.00138888x^6 - 5.80661 \times 10^{-9}x^7 + 0.000024802x^8 + \\ & 7.2645 \times 10^{-10}x^9 - 2.75948 \times 10^{-7}x^{10} + 7.96816 \times 10^{-11}x^{11} + 2.09201 \times 10^{-9}x^{12} - \\ & 9.37722 \times 10^{-12}x^{13} - 7.83276 \times 10^{-12}x^{14} - 8.39506 \times 10^{-13}x^{15} + 1.72737 \times 10^{-13}x^{16} - \\ & 1.12777 \times 10^{-14}x^{17} + 3.31034 \times 10^{-16}x^{18},\end{aligned}$$

$$\begin{aligned}\mathcal{Q}_{18}(x) =& 1.21754 \times 10^{-9} - 1.89035 \times 10^{-17}x + 0.5x^2 + 1.22541 \times 10^{-8}x^3 - 0.0416667x^4 - \\ & 1.16041 \times 10^{-8}x^5 + 0.00138889x^6 + 2.43626 \times 10^{-9}x^7 - 0.0000248026x^8 + \\ & 1.25299 \times 10^{-11}x^9 + 2.75684 \times 10^{-7}x^{10} - 4.29394 \times 10^{-11}x^{11} - 2.08151 \times 10^{-9}x^{12} + \\ & 1.24306 \times 10^{-12}x^{13} + 1.05259 \times 10^{-11}x^{14} + 2.72614 \times 10^{-13}x^{15} - 9.49649 \times 10^{-14}x^{16} + \\ & 4.89297 \times 10^{-15}x^{17} - 9.21167 \times 10^{-17}x^{18}.\end{aligned}$$

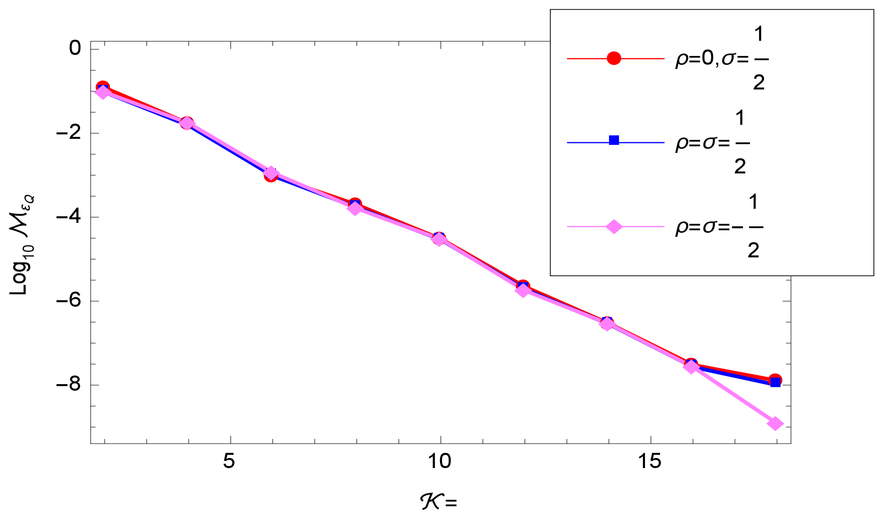

Figure 6: $\mathcal{M}_{E_{\mathcal{Q}}}$ convergence of Problem II.

4.3 Problem III

Consider the nonlinear singular second kind of coupled functional differential LE system model is given as:

$$\begin{aligned}
&\mathcal{Z}''(2x-1) + \frac{3}{x}\mathcal{Z}'(3x) + \mathcal{Q}(x)\mathcal{Z}(x+1) = \mathcal{F}(x) \\
&\mathcal{Q}''(2x-1) + \frac{2}{x}\mathcal{Q}'(3x) + \mathcal{Z}(x)\mathcal{Q}(x+1) = \mathcal{H}(x),\ 0 \le x \le 1, \\
&\mathcal{Z}(1) = 1 + \cos(1), \quad \mathcal{Z}'(0) = 0, \quad \mathcal{Q}(1) = 1 - \cos(1), \quad \mathcal{Q}'(0) = 0,
\end{aligned} \tag{25}$$

where $\mathcal{F}(x)$ and $\mathcal{H}(x)$ are selected as the exact solution, i.e., $\mathcal{Z}(x) = x + e^{-x}$, $\mathcal{Q}(x) = x - e^{-x}$. Table 3 provides the accurate results for the $\mathcal{M}_{E_{\mathcal{Z}}}$ and $\mathcal{M}_{E_{\mathcal{Q}}}$ using the Spectral collection method. Moreover, the sketches in Figs. 7 and 8 shows the logarithmic graphs of $\mathcal{M}_{E_{\mathcal{Z}}}$ and $\mathcal{M}_{E_{\mathcal{Q}}}$, that are obtained from the present scheme for different values of ρ, σ and ($\mathcal{K} = 2, 4, \cdots, 18$). Taking $\rho = \sigma = 0$, the numerical solutions of Problem III are

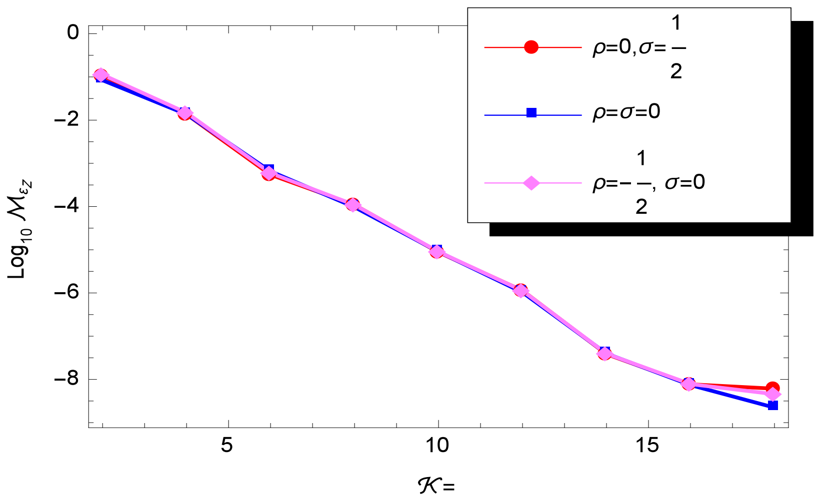

Figure 7: $\mathcal{M}_{E_{\mathcal{Z}}}$ convergence of Problem III.

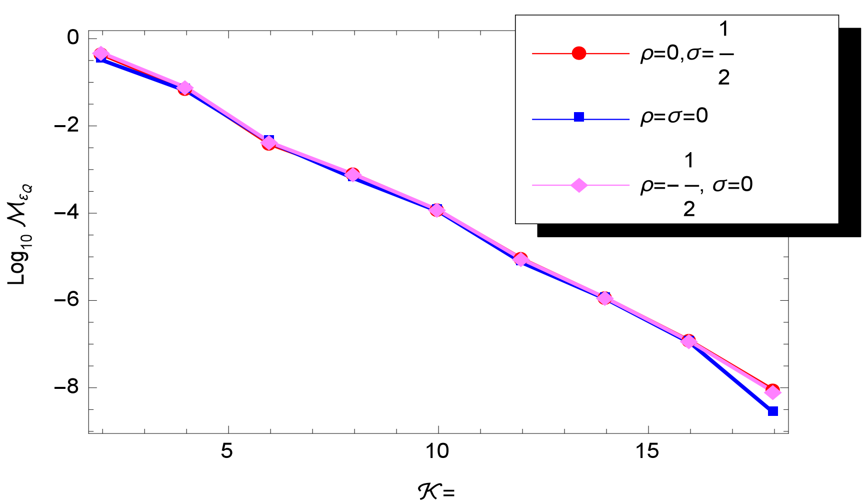

Figure 8: $\mathcal{M}_{E_{\mathcal{Q}}}$ convergence of Problem III.

Table 3: $\mathcal{M}_{\mathcal{E}_\mathcal{Z}}$ and $\mathcal{M}_{\mathcal{E}_\mathcal{Q}}$ of Problem III.

	$\rho=\sigma=0$		$\rho=0,\ \sigma=\frac{1}{2}$		$\rho=-\frac{1}{2},\ \sigma=0$	
$\mathcal{K}$	$\mathcal{M}_{\mathcal{E}_\mathcal{Z}}$	$\mathcal{M}_{\mathcal{E}_\mathcal{Q}}$	$\mathcal{M}_{\mathcal{E}_\mathcal{Z}}$	$\mathcal{M}_{\mathcal{E}_\mathcal{Q}}$	$\mathcal{M}_{\mathcal{E}_\mathcal{Z}}$	$\mathcal{M}_{\mathcal{E}_\mathcal{Q}}$
2	8.36×10^{-2}	3.18×10^{-1}	1.06×10^{-1}	4.28×10^{-1}	1.12×10^{-1}	4.63×10^{-1}
6	6.52×10^{-4}	4.17×10^{-3}	5.44×10^{-4}	3.78×10^{-3}	5.94×10^{-4}	4.15×10^{-3}
10	8.98×10^{-6}	1.10×10^{-4}	8.74×10^{-6}	1.15×10^{-4}	8.96×10^{-6}	1.17×10^{-4}
14	3.96×10^{-8}	1.06×10^{-6}	3.83×10^{-8}	1.107×10^{-6}	3.92×10^{-8}	1.12×10^{-6}
18	2.23×10^{-9}	2.59×10^{-9}	6.10×10^{-9}	8.50×10^{-9}	4.55×10^{-9}	7.78×10^{-9}

given as:

$$
\begin{aligned}
\mathcal{Z}_{18}(x) =&1+2.74967\times10^{-17}x+0.5x^2-0.16765x^3+0.0523669x^4-0.00433457x^5+\\
&0.00238683x^6-0.000198414x^7+0.0000248017x^8-2.75562\times10^{-7}x^9+\\
&2.75502\times10^{-7}x^{10}-2.50341\times10^{-8}x^{11}+3.08738\times10^{-9}x^{12}-\\
&1.62179\times10^{-10}x^{13}+1.21612\times10^{-11}x^{14}-9.28464\times10^{-13}x^{15}+\\
&7.09953\times10^{-14}x^{16}-4.45469\times10^{-15}x^{17}+1.51078\times10^{-16}x^{18},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{Q}_{18}(x) =&-1-1.58323\times10^{-17}x-0.5x^2-0.168767x^3-0.0415439x^4-0.00833331x^5-\\
&0.00267886x^6-0.000198417x^7-0.0000247997x^8-2.75582\times10^{-6}x^9-\\
&2.75762\times10^{-7}x^{10}-2.49717\times10^{-8}x^{11}-2.10074\times10^{-9}x^{12}-1.62453\times10^{-10}x^{13}-\\
&9.7797\times10^{-12}x^{14}-1.24893\times10^{-12}x^{15}+2.79433\times10^{-14}x^{16}-\\
&8.70301\times10^{-15}x^{17}-7.23672\times10^{-17}x^{18}.
\end{aligned}
$$

5 Conclusion

To model the nonlinear Lane-Emden system of functional differential equations and numerical presentations is not easy to handle. However, the solutions of the model are numerically presented by taking three different variants and compared with the true results, which depicts the competency of the designed form of the system. The numerical spectral collocation method is the best choice to handle such complicated, singular, coupled nonlinear functional differential form models, whereas, the traditional/conventional schemes do not work. Consequentially, the adopted numerical approach is an effective and suitable form to solve such systems. Spectral collocation method is a fast track of convergent approach, that can implemented effectively with many types of linear/nonlinear, fractional/integer, singular/non-singular and functional differential models. The present investigations show that the spectral collocation method is an effective and suitable scheme for solving the nonlinear Lane-Emden second order system of functional differential equations.

In future, the designed method is an alternate promising solver to be exploited in order to examine the computational models of fluid dynamics, wire coating model, thin film flow, squeezing flow systems, Jeffery Hamel type of systems, stretching flow problems, calendaring models, food processing systems and related research areas [50-54]

6 Funding Statement

The first author is partially supported by Ministerio de Ciencia, Innovación y Universidades grant number PGC2018-097198-B-I00 and Fundación Séneca de la Región de Murcia grant number 20783/PI/18.

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