ON THE PERIODIC STRUCTURE OF THE ANISOTROPIC MANEV PROBLEM

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ABSTRACT. The aim of the present work is to provide sufficient conditions for the existence of periodic orbits of first and second kind in the sense of Poincaré for the Anisotropic Manev problem. Moreover, we are able to provide too information on the stability of the orbits obtained. The main tool that we use is the Averaging theory of dynamical systems.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

In the *Methodes nouvelles de la Mecanique Celeste*, seminal work of the modern qualitative theory of dynamical systems (1899), Poincaré considered the investigation of periodic solutions as a principal topic of interest and a key point for classifying the solutions of a dynamical system. In his investigations on the Restricted Three Body Problem, classified the periodic orbits in three kinds, see [1]–[4] for instance for more details. The *first kind* are those that are generated by the planar circular orbits of the unperturbed Kepler Problem. The *second kind* are generated by the planar elliptical orbits of the Kepler Problem and the *third kind* are generated by the spatial circular orbits of the keplerian system with no null inclination. In this paper, following this classification, we carry out an investigation of the periodic orbits of first and second kind of the *Anisotropic Manev Problem.* This problem is formulated by means of the following Hamiltonian

$$\mathcal{H} = \frac{1}{2}\left(P_1^2 + P_2^2\right) - \frac{1}{\sqrt{\mu Q_1^2 + Q_2^2}} - \frac{\varepsilon b}{\mu Q_1^2 + Q_2^2} \tag{1}$$

with three distinguised parameters μ, b and ε. In what follows, we will make the following hypothesis on the above parameters. The parameter μ is near 1, $b \neq 0$ and ε is small.

The Hamiltonian (1) for $b = 0$ and $\mu = 1$ provides the formulation of the classical Kepler problem, see [8] or [12] for more details. If $b = 0$ and $\mu \neq 1$ we have the anisotropic Kepler problem, studied for instance in [10] in the early 1970s or in [5] and [7]. On the other hand, if $\mu = 1$ then we have the so called Manev Problem, that explains the perihelion advances of Mercury with the same accuracy as relativity see [15] and references therein.

For $\mu \neq 1$ and $b \neq 0$, the problem is called the *Anisotropic Manev Problem (AMP).* This problem is introduced in [6] in the early 1990s. The work on the AMP was inspired by the Anisotropic Kepler Problem. One of the main reason for considering the AMP is to further analyze similarities between classical mechanics and quantum theory. In [14], some

2010 *Mathematics Subject Classification.* Primary: 70F07, 70F15.

Key words and phrases. Celestial mechanics, Anisotropic Manev problem, Averaging theory of dynamical systems, Periodic orbits.

qualitative propierties on the dynamics of (1) are considered. In that paper, among other results, conditions for the existence of periodic orbits of second kind are obtained by means of the continuation method.

In this work, a perturbation approach of (1) is considered. If μ is near 1 and ε is small, we take $\mu = 1 - \varepsilon$, and doing Taylor series in ε at $\varepsilon = 0$ of the Hamiltonian (Problem), we obtain

$$\mathcal{H} = \frac{1}{2}\left(P_1^2 + P_2^2\right) - \frac{1}{\sqrt{Q_1^2 + Q_2^2}} - \varepsilon \mathcal{P}_1(Q_1, Q_2) + O(\varepsilon^2). \tag{2}$$

with perturbing function

$$\mathcal{P}_1(Q_1, Q_2) = -\left(\frac{Q_1^2}{2\left(Q_1^2 + Q_2^2\right)^{3/2}} + \frac{b}{Q_1^2 + Q_2^2}\right). \tag{3}$$

The statement of our main results related with the periodic structure of the AMP are the following:

Theorem 1 (Second kind orbits of the AMP)**.** *If $L = (-2h)^{-1/2}$ with $h < 0$ is a fixed value of the energy in the unperturbed Keplerian problem associated to the AMP then:*

(1) *If $L = \sqrt{2\,|b|}$ the AMP has, at least, two $2\pi-$periodic orbits of second kind. The two orbits are retrograde and unstable.*
(2) *If $0 < L < 2\sqrt{2\,|b|}$ with $L \neq \sqrt{2\,|b|}$ the AMP has, at least, two $2\pi-$periodic orbits of second kind. The two orbits are retrograde and unstable.*
(3) *If $L > 2\sqrt{2\,|b|}$ the AMP has, at least, four $2\pi-$periodic orbits of second kind, two prograde orbits linearly stable and two retrograde orbits linearly unstable.*

Theorem 2 (First kind orbits of the AMP)**.** *If $L = (-2h)^{-1/2}$ with $h < 0$ is a fixed value of the energy in the unperturbed Keplerian problem associated to the AMP then:*

(1) *The AMP has, at least, one $2\pi-$periodic orbit of first kind with are prograde. If $0 < L < 2\sqrt{2\,|b|}$ the orbit is linearly stable and if $L > 2\sqrt{2\,|b|}$ the orbit is unstable.*
(2) *If $L = 2\sqrt{2\,|b|}$ exist a Hamiltonian pitchfork bifurcation of the periodic orbit.*

2. Theoretical background

In the following we use the Delaunay variables for studying the periodic orbits of the Hamiltonian system associated to the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\left(P_1^2 + P_2^2\right) - \frac{1}{\sqrt{Q_1^2 + Q_2^2}} + \varepsilon \mathcal{P}_1(Q_1, Q_2) \tag{4}$$

see [8, 12] for more details on the Delaunay variables. Thus, in Delaunay variables the Hamiltonian (4) has the form

$$\mathcal{H} = -\frac{1}{2L^2} + \varepsilon \mathcal{P}(l, g, L, G) \tag{5}$$

where l is the *mean anomaly*, g is the *argument of the perigee* of the unperturbed elliptic orbit, L is the *square root of the semi-major axis* a of the unperturbed elliptic orbit and G is the *modulus of the total angular momentum*. Moreover, $\mathcal{P}$ is the perturbation obtained from the perturbation $\mathcal{P}_1$ using the transformation to Delaunay variables, namely $Q_1 = r\cos(f+g)$ and $Q_2 = r\sin(f+g)$. The *true anomaly* f and the *eccentric anomaly* E are auxiliary quantities defined by the relations

$$\sqrt{1-e^2} = \frac{G}{L}, \qquad r = a(1-e\cos E), \qquad l = E - e\sin E$$

with

$$\sin f = \frac{a\sqrt{1-e^2}\sin E}{r}, \qquad \cos f = \frac{a(\cos E - e)}{r}$$

and e is the eccentricity of the unperturbed elliptic orbit.

As usual the *Poisson bracket* of two smooth functions f_1 and f_2 in the Delaunay variables is given by

$$\{f_1, f_2\} = \left(\frac{\partial f_1}{\partial l}\frac{\partial f_2}{\partial L} - \frac{\partial f_1}{\partial L}\frac{\partial f_2}{\partial l}\right) + \left(\frac{\partial f_1}{\partial g}\frac{\partial f_2}{\partial G} - \frac{\partial f_1}{\partial G}\frac{\partial f_2}{\partial g}\right).$$

We denote by $\mathcal{K}$ the averaged map of the smooth function $\mathcal{P}$ with respect to the mean anomaly l,

$$\mathcal{K}(g, L, G) = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{P}(l, g, L, G)dl.$$

The averaged Hamiltonian system is given by

$$\mathcal{H}(g, L, G) = -\frac{1}{2L^2} + \varepsilon\mathcal{K}(g, L, G).$$

In practical applications, $\mathcal{K}$ is computed by mean of a change of variable in the eccentric anomaly E or the true anomaly f. The election of the variable E or f is very related with the form of the perturbed function $\mathcal{P}$. In this work we shall average using the true anomaly.

2.1. **Periodic orbits of second kind.** Fixed the level of energy $h < 0$ of the Kepler problem the averaged system are given by

$$\frac{dg}{dl} = \varepsilon(L)^3\frac{\partial\mathcal{K}}{\partial G}; \quad \frac{dG}{dl} = -\varepsilon(L)^3\frac{\partial\mathcal{K}}{\partial g}$$

with $L = \sqrt{-2h}$. The averaged differential system is a Hamiltonian system with Hamiltonian function $\mathcal{K}$. If $\varepsilon \neq 0$ is sufficiently small then for every solution $\mathbf{p}_0 = (g_0, G_0)$ of the system of equations $\mathbf{f}(\mathbf{p}) = \mathbf{0}$ with component functions $f_1 = \varepsilon(L)^3\frac{\partial\mathcal{K}}{\partial G}$, $f_2 = -\varepsilon(L)^3\frac{\partial\mathcal{K}}{\partial g}$ satisfying that $\det(d\mathbf{f}(\mathbf{p}_0)) \neq 0$ there exists a 2π-periodic solution $\gamma_\varepsilon(l) = (L(l,\varepsilon), g(l,\varepsilon), G(l,\varepsilon))$ such that $\gamma_\varepsilon(0) \to (L, g_0, G_0)$ when $\varepsilon \to 0$, see [16] for more details.

2.2. **Periodic orbits of first kind.** We recall that the domain of definition of the Delaunay variables are $\Omega \times T^2$ with Ω the open set of $\mathbb{R}^2$ given by $\Omega = \{(L, G) \in \mathbb{R}^2 \ / \ 0 < |G| < L\}$ and T a torus. Some families of periodic orbits of the unpertubed Keplerian problem are in the boundary of Ω. We denote by $\mathcal{C}_L = \{(L, G) \in \mathbb{R}^2 \ / \ |G| = L\}$ the set of circular orbits of the unperturbed Keplerian problem. The bounded rectilinear orbits of (1) are given by $\mathcal{R} = \{(L, G) \in \mathbb{R}^2 \ / \ G = 0\}$. These orbits, in the following, are denominated *critical periodics orbits of the Kepler problem.*

To study the critical orbits that survive under a perturbation, is necessary to introduce a series of different symplectic changes of variables to give sufficient conditions of existence by the Averaging Method. A necessary condition for the existence of critical orbits using the method of Averaging is that the function $\langle \mathcal{P} \rangle$ is smooth in the boundary of $\Omega_L \times T^2$. The following simplectic map $(l, g, L, G) \overset{\Phi_1}{\leftrightarrow} (l, X, L, Y)$ with $X = -\sqrt{2(L-G)} \sin g$ and $Y_1 = \sqrt{2(L-G)} \cos g$ transforms the averaged Hamiltonian in

$$\mathcal{H}_{\Phi_1}(X, L, Y) = -\frac{1}{2L^2} + \varepsilon \mathcal{K}_1(X, Y) \tag{6}$$

with $\mathcal{K}_1(X, Y)$ obtained using Φ_1 in $\langle \mathcal{P} \rangle$. The domain of definition of $\mathcal{H}_{\gamma_1}$ is $\mathbb{R}^+ \times T \times D$ with $D = \{(X, Y) \in \mathbb{R}^2 \ / \ X^2 + Y^2 < 2L\}$. Similarly the simplectic map $(l, g, L, G) \overset{\Phi_2}{\leftrightarrow} (l, X, L, Y)$ with $X = -\sqrt{2(L+G)} \sin g$ and $Y_1 = \sqrt{2(L+G)} \cos g$ transforms the averaged Hamiltonian in

$$\mathcal{H}_{\Phi_2}(X, L, Y) = -\frac{1}{2L^2} + \varepsilon \mathcal{K}_2(X, Y) \tag{7}$$

with $\mathcal{K}_2(X, Y)$ obtained using Φ_2 in $\langle \mathcal{P} \rangle$.

If $\varepsilon \neq 0$ is sufficiently small then for every solution $\mathbf{p}_0 = (X_0, Y_0)$ of the system of equations $\mathbf{f}(\mathbf{p}) = \mathbf{0}$ with component functions $f_1 = \varepsilon \dfrac{\partial \mathcal{K}_i}{\partial Y}$, $f_2 = -\varepsilon \dfrac{\partial \mathcal{K}_i}{\partial X}$ satisfying that $\det(d\mathbf{f}(\mathbf{p}_0)) \neq 0$ there exists a 2π-periodic solution $\gamma_\varepsilon(l) = (L(l, \varepsilon), X(l, \varepsilon), Y(l, \varepsilon))$ such that $\gamma_\varepsilon(0) \to (L, X_0, Y_0)$ when $\varepsilon \to 0$. The point $\mathbf{p} = (0, 0)$ is solution of the system $\mathbf{f}(\mathbf{p}) = \mathbf{0}$ if $\mathcal{K}_i$ is analytic in the domain of definition, see [16] for more details.

2.3. **Stability and Kam Theorem.** The stability or instability of the periodic solution γ_ε obtained by the Averaging Method is given by the stability or instability of the equilibrium point $\mathbf{p}$ of the averaged system. In fact, the equilibrium point $\mathbf{p}$ has the stability behavior of the Poincaré map associated to the periodic solution. The averaged system is a $1D$ canonical system with Hamiltonian $\mathcal{K}$ or $\mathcal{K}_i$ with $i = 1, 2$. The linear stability of the periodic orbit is determined by mean of the characteristic polynomial of the infinitely symplectic matrix $d\mathbf{f}(\mathbf{p}_0)$. The polynomial p_{γ_ε} is a quadratic of the form $p_{\gamma_\varepsilon}(\lambda) = \lambda^2 + \omega^2$ with ω a function of the parameters of the problem. If p_{γ_ε} has purely imaginary roots then the periodic solution γ_ε is linearly stable. A necessary and sufficient condition for the existence of roots $i\omega$, $-i\omega$ with $\omega > 0$. The multipliers of the orbit γ_ε are 1, 1, $1 - \dfrac{2\pi\varepsilon}{\omega} + O(\varepsilon^2)$, $1 + \dfrac{2\pi\varepsilon}{\omega} + O(\varepsilon^2)$. On the other hand, $d\mathbf{f}(\mathbf{p}) = \mathbb{I}A$ with $\mathbb{I} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$, $A = \text{Hess}(\mathcal{K})|_{(g,\, G) = (g_0, G_0)}$ or $A = \text{Hess}(\mathcal{K}_i)|_{(X,\, Y) = (X_0, Y_0)}$. By mean of the Lejeune-Dirichlet Theorem, γ_ε is Lyapunov stable if the symmetric matrix A is definite positive. This conditions is achieved if $\omega > 0$.

3. Proof of Theorems 1 and 2

Proof of Theorem 1. The perturbation (3) in Delaunay mixed variables has the form

$$\mathcal{P}_1(l,g,L,G) = -\frac{\left(1-e^2\right)^{3/2}\left(\frac{a\eta^2}{1+e\cos(f)}\right)^{-2}\left(2ab\eta^2+(1+e\cos(f))\cos^2(f+g)\left(\frac{a\eta^2}{(1+e\cos(f))}\right)^2\right)}{2a\eta(1+e\cos(f))^2}.$$

Computing $\mathcal{K}$, the averaged map of $\mathcal{P}_1$ respect to the mean anomaly l, we obtain

$$\mathcal{K}(g,G) = \frac{GL\cos(2g)(G-L)-(G+L)(4b+GL)}{4GL^3(G+L)}.$$

The averaged system generated by the Hamiltonian $\mathcal{K}$ is given by the following differential equations

$$\frac{dg}{dl} = \frac{b}{G^2}+\frac{L^2\cos(2g)}{2(G+L)^2},\ \frac{dG}{dl} = \frac{L(G-L)\sin g\cos g}{G+L}. \tag{8}$$

In the following we define by G_i the possible real solutions of the equation

$$\left|2\left|b\right|-L^2\right|G^2+4bLG+2bL^2 = 0 \tag{9}$$

in the interval $(-L,L)$. We note that if $L=\sqrt{2\left|b\right|}$ then $G_1=-\dfrac{L}{2}$. If $b<0$ then $(g,G)=\left(0,-\frac{L}{2}\right)$ and $(g,G)=\left(\pi,-\frac{L}{2}\right)$ are equilibria solutions of (8). Similarly, if $b>0$ then $(g,G)=\left(\frac{\pi}{2},-\frac{L}{2}\right)$ and $(g,G)=\left(\frac{3\pi}{2},-\frac{L}{2}\right)$ are equilibria solutions of (8). If $L\neq\sqrt{2\left|b\right|}$ and $0<L<2\sqrt{2\left|b\right|}$ then (9) has an only real solution G_1 in the interval $(-L,0)$. If $L>2\sqrt{2\left|b\right|}$ the equation (9) has a real solution G_1 in the interval $(-L,0)$ and a real solution G_2 in the interval $(0,L)$.

From the relations $L=\sqrt{a}$ and $G=\pm\sqrt{a\left(1-e^2\right)}$ we obtain

$$a_1 = \frac{2\left|b\right|\left(2-e^2-2\sqrt{1-e^2}\right)}{1-e^2},\ a_2 = \frac{2\left|b\right|\left(2-e^2+2\sqrt{1-e^2}\right)}{1-e^2}$$

and the roots G_i are given by

$$G_1 = -\sqrt{2\left|b\right|\left(2-e^2-2\sqrt{1-e^2}\right)},\ G_2 = \sqrt{2\left|b\right|\left(2-e^2+2\sqrt{1-e^2}\right)}.$$

The equilibria of (8) are:

(1) If $L=\sqrt{2\left|b\right|}$ and $b<0$ then $(g,G)=\left(m\pi,-\frac{L}{2}\right)$ for $m=0,1$. If $b>0$ then $(g,G)=\left(\frac{(2m-1)\pi}{2},-\frac{L}{2}\right)$ for $m=1,2$.
(2) If $L\neq\sqrt{2\left|b\right|}$, $0<L\leq 2\sqrt{2\left|b\right|}$ then $(g,G)=(m\pi,G_1)$ for $m=0,1$ if $b<0$. Similarly $(g,G)=\left(\frac{(2m-1)\pi}{2},G_1\right)$ for $m=1,2$ if $b>0$.
(3) If $L>2\sqrt{2\left|b\right|}$, then if $b<0$

$$(g,G)\in\left\{(m\pi,G_n)\ /\ (m,n)\in\{0,1\}\times\{1,2\}\right\}.$$

On the other hand, if $b>0$

$$(g,G)\in\left\{\left(\tfrac{(2m-1)\pi}{2},G_n\right)\ /\ (m,n)\in\{1,2\}\times\{1,2\}\right\}.$$

We consider $\mathbf{f}$ the vectorial function whose component functions are the right hand side of (8). We obtain

$$d\mathbf{f}\left(g,G\right)=\begin{pmatrix} 0 & \frac{sign(b)L^2}{(G+L)^3}-\frac{2b}{G^3} \\ \frac{sign(b)L(L-G)}{G+L} & 0 \end{pmatrix}$$

and

$$\det\left(d\mathbf{f}(\,(g,G)\right)=\frac{sign(b)L(G-L)\left(sign(b)G^3L^2-2b(G+L)^3\right)}{G^3(G+L)^4}.$$

Then:

- If $b<0$ then $\det\left(d\mathbf{f}\left(\left(m\pi,-\frac{L}{2}\right)\right)\right)=-48$ and $\det\left(d\mathbf{f}\left(\left(\frac{(2m-1)\pi}{2}\pi,-\frac{L}{2}\right)\right)\right)=-48$.
- If $g_k=0,\frac{\pi}{2},\pi,\frac{3\pi}{2}$ then

$$\det\left(d\mathbf{f}(\,(g_k,G_1)\right)=\mathcal{RE}(e)=-\frac{e^2}{e^6+\left(5\sqrt{1-e^2}-13\right)e^4+4\left(7-5\sqrt{1-e^2}\right)e^2+16\left(\sqrt{1-e^2}-1\right)}.$$

The function $\mathcal{RE}$ verifies the following propierties: a) $\lim_{e\to 0^+}\mathcal{RE}(e)=-\infty$, b) $\lim_{e\to 1^-}\mathcal{RE}(e)=-\infty$, c) $\mathcal{RE}$ has a local maximun in $e_{\max}=\frac{\sqrt{35}}{6}$ with $\mathcal{RE}(e_{\max})=-\frac{9072}{625}$. Then $\mathcal{RE}(e)<0$ for $e\in(0,1)$.

- If $g_k=0,\frac{\pi}{2},\pi,\frac{3\pi}{2}$ then

$$\det\left(d\mathbf{f}(\,(g_k,G_2)\right)=\mathcal{PR}(e)=\frac{e^2}{-e^6+\left(13+5\sqrt{1-e^2}\right)e^4-4\left(7+5\sqrt{1-e^2}\right)e^2+16\left(\sqrt{1-e^2}+1\right)}.$$

The function $\mathcal{PR}$ verifies the following propierties: a) $\lim_{e\to 0^+}\mathcal{PR}(e)=0$, b) $\lim_{e\to 1^-}\mathcal{PR}(e)=+\infty$, c) $\mathcal{PR}$ is strictly increasing in $(0,1)$. Then $\mathcal{R}(e)>0$ for $e\in(0,1)$.

In short, we have:

(1) If $b<0$ and $L=\sqrt{2\,|b|}$, at least, two second kind periodic orbits, unstable and retrograde, defined by $\gamma_m^-(l,\varepsilon)$ with $m=1,2$ and $\gamma_m^-(0,\varepsilon)\to\left(\sqrt{2\,|b|},(m-1)\pi,-\frac{\sqrt{2|b|}}{2}\right)$ when $\varepsilon\to 0$. If $b>0$ and $L=\sqrt{2\,|b|}$, at least, two second kind periodic periodic orbits, unstable and retrograde, defined by $\gamma_m^+(l,\varepsilon)$ with $m=1,2$ and $\gamma_m^+(0,\varepsilon)\to\left(\sqrt{2\,|b|},\frac{(2m-1)\pi}{2},-\frac{\sqrt{2|b|}}{2}\right)$ when $\varepsilon\to 0$.

(2) If $b<0$ and $L\neq\sqrt{2\,|b|}$, $0<L\leq 2\sqrt{2\,|b|}$, at least, two second kind periodic orbits, unstable and retrograde, defined by $\psi_m^-(l,\varepsilon)$ with $m=1,2$ and

$$\psi_m^-(0,\varepsilon)\to\left(\sqrt{\frac{2\,|b|\left(2-e^2-2\sqrt{1-e^2}\right)}{1-e^2}},(m-1)\pi,-\sqrt{2\,|b|\left(2-e^2-2\sqrt{1-e^2}\right)}\right)$$

when $\varepsilon \to 0$. If $b > 0$ and $L = \sqrt{2\,|b|}$, at least, the two second kind periodic periodic orbits, unstable and retrograde, defined by $\psi_m^-(l,\varepsilon)$ with $m = 1,2$ and $\psi_m^+(l,\varepsilon)$ with $m = 1,2$

$$\psi_m^+(0,\varepsilon) \to \left(\sqrt{\frac{2\,|b|\left(2-e^2+2\sqrt{1-e^2}\right)}{1-e^2}}, \frac{(2m-1)\pi}{2}, \sqrt{2\,|b|\left(2-e^2+2\sqrt{1-e^2}\right)}\right)$$

when $\varepsilon \to 0$ with are prograde and linearly stable.

□

Proof of Theorem 2. By mean of the symplectic map Φ_1 the Hamiltonian $\mathcal{H}_{\Phi_1}$ are

$$\mathcal{H}_{\Phi_1}(X,L,Y) = -\frac{1}{2L^2} + \varepsilon\mathcal{K}_1(X,Y).$$

with

$$\mathcal{K}_1(X,Y) = \frac{X^2-2L}{2L^2\left(4L-(X^2+Y^2)\right)} - \frac{2b}{L^3\left(2L-(X^2+Y^2)\right)}.$$

The averaged system given by $\frac{dY}{dl} = -\varepsilon\frac{\partial\mathcal{K}_1}{\partial X}$, $\frac{dX}{dl} = \varepsilon\frac{\partial\mathcal{K}_1}{\partial Y}$ has the equilibrium $(X,Y) = (0,0)$. The Taylor expansion of second order of $\mathcal{K}_1$ about $(X,Y) = (0,0)$ are given by

$$\mathcal{K}_1(X,Y) = -\frac{4b+L^2}{4L^4} + a_{02}Y^2 + a_{20}X^2 + o(X^2+Y^2)$$

with

$$a_{02} = -\frac{8b+L^2}{16L^5}, \quad a_{20} = -\frac{8b-L^2}{16L^5}.$$

The coefficients a_{02} and a_{20} has the same sign in the region of parameters $\mathcal{R} = \{(L,b) \in \mathbb{R}^2 \;/\; L < 2\sqrt{2\,|b|}\}$. If the parameters belong this region the first kind periodic orbit is linearly stable.

If the parameters belong to the boundary of $\mathcal{R}$ some of the coefficients a_{02} and b_{20} are null and non exist first kind periodic orbit. If the parameters belong $\mathcal{R}^c$ the first kind periodic orbit is unstable.

We shall prove the existence of a Hamiltonian pitchfork bifurcation if $L = 2\sqrt{2\,|b|}$. For this, will be necessary to consider the Taylor expansion of fourth order of $\mathcal{K}_1$. This expansion are given by

$$\mathcal{K}_1(X,Y) = -\frac{4b+L^2}{4L^4} + a_{02}Y^2 + a_{20}X^2 + a_{04}Y^4 + a_{22}X^2Y^2 + a_{04}X^4 + o(X^4+Y^4)$$

with

$$a_{04} = -\frac{16b+L^2}{64L^6}, \quad a_{22} = -\frac{b}{2L^6}, \quad a_{04} = -\frac{16b-L^2}{64L^6}.$$

The averaged differential equations of motion in a small neighborhood of $(X,Y) = (0,0)$ are

$$\frac{dY}{dl} = -\varepsilon\left(2a_{20}X + 2a_{22}XY^2\right) + h.o.t, \;\; \frac{dX}{dl} = -\varepsilon\left(2a_{02}Y + 2a_{22}X^2Y\right) + h.o.t.$$

We introduce the following relation

$$L^2 - 8b = \sigma^2 \eta$$

with $\sigma \ll 1$and $\eta \in \mathbb{R}$. By mean of the change $X \to \sigma Q$, $Y \to \sigma^3 P$ and $l = \varepsilon^{-1}\sigma^{-3} l_1$ we obtain

$$\frac{dP}{dl_1} = \frac{1}{1024}\left(-\frac{\sqrt{2}\eta Q}{b^5} + \frac{Q^3}{b^2}\right) + O(\sigma), \quad \frac{dQ}{dl_1} = -\frac{\sqrt{2}P}{64b^3} + O(\sigma). \tag{10}$$

The truncated system formed by the rhs of (10) is Hamiltonian with Hamiltonian function $\mathcal{W}$ given by

$$\mathcal{W}(P,Q) = -\frac{\sqrt{2}}{128b^3}P^2 + \frac{1}{1024}\left(\frac{\sqrt{2}}{2b^5}\eta Q^2 - \frac{Q^4}{4b^2}\right).$$

If $\eta < 0$, $\mathcal{W}$ has only a center in $(Q,P) = (0,0)$. If $\eta > 0$, the equilibria $(Q,P) = (0,0)$ is a saddle and exist two symmetrical centers given by

$$(Q,P) = \left(-\frac{\sqrt{\sqrt{2}b\eta}}{b^2}, 0\right), \quad (Q,P) = \left(\frac{\sqrt{\sqrt{2}b\eta}}{b^2}, 0\right),$$

Similarly, the Hamiltonian $\mathcal{H}_{\Phi_2}$ are given by

$$\mathcal{H}_{\Phi_2}(X,L,Y) = -\frac{1}{2L^2} + \varepsilon\left(\frac{(Y^2 - X^2)}{L(X^2+Y^2)^2} - \frac{X^2}{2L^2(X^2+Y^2)} - \frac{2b}{L^3(2L - (X^2+Y^2))}\right)$$

with is singular in $(X,Y) = (0,0)$, so it follows that there are no first kind retrograde periodic orbits. □

Acknowledgements

The first author of this work were partially supported by MINECO grant number MTM2014-51891-P , Fundación Séneca de la Región de Murcia grant number 19219/PI/14 and National Nature Science Foundation of China (Western Region Funds) grant number 11761083. The third author of this work were partially supported by MINECO grant number MTM2014-51891-P and Fundación Séneca de la Región de Murcia grant number 19219/PI/14.

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