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THE MEASURE OF SCRAMBLED SETS: A SURVEY

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Abstract. The aim of this note is to survey all known results concerning the (Lebesgue) measure of scrambled sets for continuous chaotic functions on the interval, and to complete some of them. In particular it is shown that neither continuous maps of type 2^∞ (in Šarkovskii's ordering) nor C^1 maps of any type can possess measurable scrambled sets of full measure. "Chaos" means here chaos in the sense of Li and Yorke [LY].

1. Introduction.

In what follows $C^k(I)$, $0 \leq k \leq \infty$, will denote the set of functions $f : I \to I$ of the class C^k, where $I = [0,1]$ is the unit interval (in particular, $C^0(I)$ is the set of continuous functions from I into itself). As usual, the nth-iterate of a function f will be denoted by f^n (when f^0 is the identity map) and a point $p \in I$ will be called *periodic —of period* r— if $f^r(p) = p$ and $f^i(p) \neq p$ for any $i = 1, \dots, r-1$. The *orbit* of a point $x \in I$ is the set $\{f^n(x)\}_{n=0}^{\infty}$; a *periodic orbit* is the orbit of a periodic point.

During the last few decades, there have been many attempts to apprehend the idea of complex dynamical behaviour (so called "chaos") for functions from $C^0(I)$. One of the most successful, and no doubt the most popular one, is due to Li and Yorke and their famous paper "Period three implies chaos" [LY].

Definition 1.1. Let $f \in C^0(I)$ and let $S \subset I$ be a set having at least two elements and such that for any $x, y \in S$, $x \neq y$, the following properties hold:

(i) $\limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0$,

(ii) $\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0$.

Then S is called a *scrambled set (of f)* and f is said to be *chaotic (in the sense of Li and Yorke)*.

Remark 1.2. Initially the set S was also assumed to be uncountable and satisfy

(iii) $\limsup_{n\to\infty} |f^n(x) - f^n(p)| > 0$

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for any $x \in S$ and any periodic point p. However, it was later realized [Ji5, p. 8], [BC, pp. 144–145] (and it is very easy to check) that if S is a scrambled set then (taking off one point at most) it also satisfies (iii). Further, Kuchta and Smítal proved in [KS] (using also [Sm3] and [JS1]) that if f has a two-points scrambled set then it also possesses a Cantor-like scrambled set.

The relevance of Li and Yorke's idea is stressed by the fact that, in a sense, it turns out to be the minimal requirement for a function $f \in C^0(I)$ to be "complex". This was shown by Smítal in the key paper [Sm3] (completed with [JS1]), where it was proved:

Theorem 1.3 [Sm3], [JS1]. *Let $f \in C^0(I)$. Then it must satisfy one of the following properties:*

(i) *f is chaotic;*

(ii) *all trajectories of f are* approximable by cycles, *that is, for any $x \in I$ and any $\epsilon > 0$ there is a periodic point p such that $\limsup_{n\to\infty} |f^n(x) - f^n(p)| < \epsilon$.*

Remark 1.4. As according to [Ji1, pp. 117–118] (cf. also [BC, pp. 144–145]) if two points x, y are approximable by cycles then they cannot satisfy both (i) and (ii) in Definition 1.1, properties (i) and (ii) in Theorem 1.3 are incompatible.

But, as it was emphasized in the literature from the very beginning (see e.g. [CE, pp. 21–22]), from a "practical" point of view it is important to know whether chaos in the sense of Li and Yorke can be observable, that is, whether a chaotic map can have "large" scrambled sets. As always when the word "large" is used, two possible approaches are available: topological and measure-theoretic. The first one makes probably no sense here, as Gedeon proved that if $S \subset I$ is scrambled for a map f then it cannot be residual in any subinterval of I [Ge1]. Instead we shall concentrate on the question whether a continuous function may possess (measurable) scrambled sets of positive Lebesgue measure (which in the sequel will be denoted by λ) and how large this measure can be. The present note aims to survey all results (up to the authors' knowledge) concerning this problem, and to add some new few ones. In what follows we shall use the word *pm-chaotic* (resp. *fm-chaotic*) to refer to any chaotic function possessing a measurable scrambled set of positive (resp. full) measure.

As we shall see later, the type of the map f may play a prominent role in this regard. Let us recall this notion. Order the set $\mathbb{N} \cup \{2^\infty\}$ as follows:

$$\begin{gathered} 3 \succ 5 \succ 7 \succ \ldots \succ \\ 2\cdot 3 \succ 2\cdot 5 \succ 2\cdot 7 \succ \ldots \succ \\ \ldots \succ \\ 2^n\cdot 3 \succ 2^n\cdot 5 \succ 2^n\cdot 7 \succ \ldots \succ \\ \ldots \succ \\ 2^\infty \succ \ldots \succ 2^n \succ \ldots \succ 2^3 \succ 2^2 \succ 2 \succ 1. \end{gathered}$$

We shall use the symbol $\succeq$ in the natural way. For any $f \in C^0(I)$, let $\mathrm{Per}(f)$ denote the set of periods of periodic points of f. If $n \in \mathbb{N} \cup 2^\infty$ has the property that $\mathrm{Per}(f) = \mathbb{N} \cap \{m : n \succeq m\}$ then we shall say that f is of *type* n. As is well known, Šarkovskii's Theorem [Sa1], [Sa2] states that any $f \in C^0(I)$ is of type n for some $n \in \mathbb{N} \cup \{2^\infty\}$ and, conversely, that for any $n \in \mathbb{N} \cup 2^\infty$ there is a map $f \in C^0(I)$ of type n. For instance the paradigmatic logistic family $\{f_\alpha\}_{0 \leq \alpha \leq 4}$, $f_\alpha(x) = \alpha x(1-x)$, includes maps of all possible types (cf e.g. [Sm4, pp. 73–77]).

It is worth emphasizing that the word "chaotic" is often used to refer exclusively to maps of type larger than 2^∞, as only in this setting complicated dynamics are guaranteed: for instance they can be characterized by the property of having positive topological entropy. Concerning Li and Yorke chaos, we conclude this introductory section recalling that:

(a) if f has type larger than 2^∞ then it is chaotic [LY], [Gr];

(b) if f has type 2^∞ then it may be both chaotic or not [Sm3] (see also [Xi], [MS], [Du5] and the next section);

(c) if f has type less than 2^∞ then all points from I are attracted by periodic orbits, that is, for any $x \in I$ there is a periodic point p such that $\lim_{n\to\infty} |f^n(x) - f^n(p)| = 0$ [Le] (cf. also [Ot], [Co], [ML], [Sa2] or [CH]). Hence f cannot be chaotic.

2. Results.

For the first chaotic function for which the problem of pm-chaoticity was elucidated the answer was negative (Nathanson [Na]). Additional examples were provided by Guckenheimer [Gu], Preston [Pr, Chapter 9], Nusse [Nu], Du [Du1], [Du2], [Du3], [Du4], Sivak [Si] and Jiménez López [Ji5, pp. 40–41] in different and/or progressively more general settings. All these functions share a common feature: almost all their points x are attracted by periodic orbits (the corresponding periodic orbit may depend on the point x).

An example of a radically different nature is the "tent" map $f : I \to I$ defined by $f(x) = 1 - |2x - 1|$. As shown in [Sm1], it possesses a scrambled set of full *outer* Lebesgue measure; still it cannot have *measurable* scrambled sets of positive measure. More general examples in this same vein can be found in [Ge2], [BJ], [Ji5, Section 4.2] and [BKT]. In [Ji7] (which among other papers is strongly based on [Gu] and [BKT]) both trends are merged to produce a rather general result (Theorem 2.1 below). Before stating it we need to introduce some additional notation.

We shall deal with the class $S(I)$ of *unimodal maps with negative Schwarzian derivative*. More precisely, $f \in S(I)$ provided that

(a) $f \in C^3(I)$;

(b) there is exactly one point $c \in I$ such that $f'(c) = 0$; further, $f''(c) \neq 0$;

(c) $S(f)(x) := \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 < 0$ for any $x \neq c$.

Recall that the *ω-limit set* of a point $x \in I$ for a map $f \in C^0(I)$, $\omega_f(x)$, is the set of limit points of the sequence $(f^n(x))_{n=0}^\infty$. Let μ be a probability measure defined on the σ-algebra of Borel subsets of I. The measure μ is said to be *invariant* (with respect to f) if $\mu(f^{-1}(A)) = \mu(A)$ for any Borel set A. We say that μ is *absolutely continuous (with respect to the Lebesgue measure)* if $\lambda(A) = 0$ implies $\mu(A) = 0$ for any Borel set A.

Theorem 2.1 [Gu], [BKT], [Ji7]**.** *Let $f \in S(I)$.*

(i) *Suppose that $\omega_f(x)$ has empty interior for any $x \in I$. Then almost all points* *from I are approximable by cycles.*

(ii) *Suppose that $\omega_f(x)$ has nonempty interior for some $x \in I$ and f has an absolutely continuous invariant measure. Then f is not pm-chaotic but possesses a scrambled set of outer positive Lebesgue measure.*

Remark 2.2. It is easy to show that if c is attracted by a hyperbolic attracting periodic orbit then f satisfies (i), and from [Mi1] it follows that if some iterate of c belongs to a hyperbolic repelling periodic orbit then f satisfies (ii). [Recall that the orbit of a periodic point p of period r is said to be *hyperbolic attracting* (resp. *hyperbolic repelling*) if $|f^r(p)| < 1$ (resp. $|f^r(p)| > 1$).]

Remark 2.3. Although a map f satisfying (i) may be chaotic, according to Remark 1.4 it cannot be pm-chaotic either. In fact, using some specific properties of maps from $S(I)$ it is possible to prove (under the additional assumption that all its periodic points are hyperbolic) that if f satisfies (i) then there is a measure zero set including all its scrambled sets.

It is interesting to apply Theorem 2.1 to the logistic family $\{f_\alpha\}_{0\leq\alpha\leq 4}$ [which is included in $S(I)$]. For instance, if $\alpha = 3.83187...$ then $c = 1/2$ is a periodic point of period three and according to Remark 2.2 f_α is a (chaotic) map satisfying (i). On the other hand f_4 satisfies (ii) and, indeed, it possesses a scrambled set of full outer Lebesgue measure. It is worth noticing that from the union of works by Lyubich [Ly1], [Ly2] and Martens and Nowicki [MN] it follows that for almost all $\alpha \in [0, 4]$ a map f_α must satisfy either (i) or (ii).

The first examples of pm-chaotic continuous functions were simultaneously provided by Kan [Ka] and Smítal [Sm2]; the corresponding scrambled sets S hold in both cases the property $\lambda(S) < 1$. Later it was realized that, since any chaotic function possesses a Cantor-like scrambled set (cf. Remark 1.2), it is always topologically conjugate to a pm-chaotic function possessing a scrambled set with measure as close to 1 as required [here $f, g \in C^0(I)$ are said to be *topologically conjugate* if there is a homeomorphism $h \in C^0(I)$ satisfying $f \circ h = h \circ g$]. In particular, this implies that there are pm-chaotic maps of all types greater than or equal to 2^∞. Finally Misiurewicz [Mi2] and then Bruckner and Hu [BH] and Iwanik [Iw] constructed examples of (topologically conjugate to the tent map) fm-chaotic functions, and one can improve slightly a recent result by Babilonová [Ba] to show that if f is *transitive* (that is, it has a dense orbit) then it is topologically conjugate to a fm-chaotic function. Since transitive maps have always type greater than 2^∞ ([Sa3], cf. also [JS2]), it is natural to wonder whether there are fm-chaotic functions of type 2^∞. The answer is negative:

Proposition 2.4. *Let $f \in C^0(I)$ be of type 2^∞. Then it cannot be fm-chaotic.*

Proposition 2.4 is a direct consequence of two results from [Sm3] and a simple fact. To state them we need some notation.

Let $f \in C^0(I)$ and let J be a subinterval of I. We say that J is *periodic —of period r—* if $f^r(J) = J$ and $J, f(J), \dots, f^{r-1}(J)$ are pairwise disjoint. Also, we say that $u, v \in I$ are *separable* if there are disjoint periodic intervals J_u, J_v such that $u \in J_u$, $v \in J_v$. If this is not the case the points u, v are called *non-separable.* A subinterval J of I is called *wandering* if its iterates are pairwise disjoint and neither of its points is attracted by any periodic orbit.

Theorem 2.5 [Sm3]. *Let $f \in C^0(I)$ be of type 2^∞ and let $x \in I$ be such that $\omega_f(x)$ is infinite. Then there is a sequence $(J_s)_{s=1}^{\infty}$ of periodic intervals such that for any $s \geq 1$ the following properties hold:*

(i) *J_s has period 2^s;*
(ii) *$J_{s+1} \subset J_s$;*
(iii) *$\omega_f(x) \subset \bigcup_{i=0}^{2^s-1} f^i(J_s)$.*

Theorem 2.6 [Sm3]. *Let $f \in C^0(I)$ be of type 2^∞. Then f is chaotic if and only if there is an $x \in I$ such that $\omega_f(x)$ is infinite and contains two non-separable points.*

Lemma 2.7. *Let $f \in C^0(I)$ be a chaotic function of type 2^∞. Then it has a wandering interval.*

Proof. Use Theorem 2.6 to find a point $x \in I$ for which $\omega_f(x)$ is infinite and contains two non-separable points $u \neq v$, and then Theorem 2.5 to construct the corresponding sequence $(J_s)_{s=1}^{\infty}$ of periodic intervals. Without loss of generality we can assume that $u \in J_s$ for any s. Since u, v are non-separable we get $v \in J_s$ for any s as well. Then $J = \bigcap_{s=1}^{\infty} J_s$ is a non-degenerate interval. We shall show that J is wandering.

Obviously $f^n(J) \cap f^m(J) = \emptyset$ for any $n \neq m$ [which in particular implies that $\lambda(f^n(J)) \to 0$ as $n \to \infty$]. Now assume that there is a periodic point p of period 2^{s-1} for some $s \geq 1$ such that $\lim_{n\to\infty} |f^n(y) - f^n(p)| = 0$ for any $y \in J$. Since J_s is periodic of period 2^s, $\{p, f(p), \ldots, f^{2^{s-1}-1}(p)\} \cap J_s = \emptyset$. On the other hand it is clear that there exists $0 \leq i \leq 2^{s+2} - 1$ such that the closure of $f^i(J_{s+2})$ is included in the interior of J_s. Note that $f^{i+2^{s+2}\cdot n}(J) \subset f^i(J_{s+2})$ for any n. This is impossible since $f^{i+2^{s+2}\cdot n}(p) = f^i(p) \notin J_s$.

□

Proof of Proposition 2.4. It immediately follows from Lemma 2.7 and the fact that a wandering interval can intersect a scrambled set in at most one point.

□

Remark 2.8. Beginning with [Gu] and culminating in [MMS] an impressive string of papers has established the non-existence of wandering intervals for a large number of "natural maps" including all analytic ones and all maps from $S(I)$. According to Lemma 2.7 neither of these maps can simultaneously be chaotic and of type 2^∞.

All pm-chaotic maps previously referred to are just continuous. However, if we are speaking from a "practical" point of view it is reasonable to expect for our functions some additional smoothness properties. The existence of C^1 pm-chaotic functions was stated without proof in [JS1]; a first explicit example can be found in [Ji1, pp. 198-229]. This result was later improved in [Ji3], where it was shown that for any $n \succeq 2^\infty$ and any $0 \leq \delta < 1$ there is a function $f \in C^\infty(I)$ of type n and possessing a Cantor-type scrambled set of measure δ. Analogous examples to those of [Ji3], but now of the class C^1 and analytic on a neighbourhood of their (only) critical point, were given in [Ji2] (compare with Remark 2.8).

Notice that the case $\delta = 1$ has been excluded from the discussion above. As we are next going to show, this cannot be helped.

Proposition 2.9. *Let $f \in C^1(I)$. Then it cannot be fm-chaotic.*

Proposition 2.9 (which is stated without proof in [Sm2]) is based on the following simple lemma.

Lemma 2.10. *Let $f \in C^1(I)$ be non-monotone. Then there are interior points $a \neq b$ such that $f(a) = f(b)$ and $f'(a) \neq 0 \neq f'(b)$.*

Proof. Since f is not monotone one can find points $u < v$ satisfying $f'(u)f'(v) < 0$. Say e.g. $f'(u) > 0$, $f'(v) < 0$ (the other case is similar).

Let $c \in (u, v)$ be an absolute maximum of f in $[u, v]$. Say for example $f(u) \geq f(v)$. Take $d \in (u, c)$ such that $f'(x) > 0$ for any $x \in [u, d]$ and notice that $f(u) < f(d) < f(c)$. Write $G = (c, v) \cap f^{-1}((f(u), f(d)))$. Then $f(G) = (f(u), f(d))$. Since G is open and its image by f is uncountable, it has a component J with the property that f is not constant on it. Take $b \in J$ satisfying $f'(b) \neq 0$ and use the property of intermediate values to find $a \in (u, d)$ such that $f(a) = f(b)$ and $f'(a) \neq 0$.

□

Proof of Proposition 2.9. We shall assume that $f \in C^1(I)$ has a full measure scrambled set S to arrive to a contradiction. It is well known (and easy to prove) that a monotone function must be of type 1 or 2 and hence it cannot be chaotic. Then we can apply Lemma 2.10 to find interior points $a \neq b$ satisfying $f(a) = f(b)$ and $f'(a) \neq 0 \neq f'(b)$.

Let $w = f(a) = f(b)$ and find open intervals L, R, J such that $a \in L$, $b \in R$, $w \in J$, $L \cap R = \emptyset$, and the maps $g = f|_L : L \to J$ and $h = f|_R : R \to J$ are diffeomorphisms. Write $A = S \cap L$, $B = S \cap R$. As S is a full measure scrambled set we have $\lambda((h^{-1} \circ g)(A)) = 0$ because $\lambda(B) = \lambda(R)$, and $(h^{-1} \circ g)(A) \cap B = \emptyset$. But this is not possible since $\lambda(A) > 0$ and $h^{-1} \circ g$ is a diffeomorphism.

□

3. Final remarks.

1. Although this paper is devoted to continuous maps on the interval let us recall here some related results in the more general setting of metric spaces (where distance plays the role of absolute value in Definition 1.1). The existence of continuous maps (even homeomorphisms) $f : X \to X$ having measurable scrambled sets of full Lebesgue measure has been proved in the cases $X = [0, 1]^n$ for any $n \geq 2$ [Mz1], [Iw], [Ka]. If $X = (0, 1)^n$ and $n \geq 2$ one can even construct a homeomorphism for which the whole space is a scrambled set [Ma2]; analogous examples [replacing "$(0, 1)$" by "$(0, \infty)$" or "$(-\infty, \infty)$" and "homeomorphism" by "C^∞ diffeomorphism", now also including the case $n = 1$] are given in [Ma1]. Moreover Huan and Ye have shown that there are many compacta X admitting homeomorphisms having X as a scrambled set, including some countable compacta, the Cantor set and continua with arbitrary dimensions [HY]. By the way, Ceder also constructed in [Ce] a non-continuous map $f : I \to I$ having the whole interval as a scrambled set.

2. In sight of Theorem 2.1 and the comments below Remark 2.3 it is reasonable to conjecture that there are no pm-chaotic analytic maps, and it is worth noticing that according to [Mz2] the set of chaotic but not pm-chaotic maps is residual in $C^0(I)$. These and some other similar results clearly emphasize the fact that pm-chaoticity is a too strong property to characterize "physically observable" chaos.

Inspired by [Pi], Jiménez López proposed in [Ji6] the following weaker definition: chaos is *observable* for a function $f \in C^0(I)$ if the set $\mathrm{Ch}(f) \subset I \times I$ of points (x, y) satisfying

(a) $\limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0$,
(b) $\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0$,
(c) $\limsup_{n\to\infty} |f^n(x) - f^n(p)| > 0$, $\limsup_{n\to\infty} |f^n(y) - f^n(p)| > 0$ for any periodic point p of f,

(which is a Borel set [Ji5] and then measurable), has positive two-dimensional Lebesgue measure. For instance the tent map is not pm-chaotic but chaos is observable for it; in fact the corresponding set $\mathrm{Ch}(f)$ has full measure (as it is implicitly shown in [Sm1]). The inclusion of (c) as in the original Li and Yorke's framework may seem strange, but in [Ji6] (which also includes a list of relevant bibliography on the subject) it is shown that there are chaotic maps for which $\mathrm{Ch}(f)$ has zero measure but the set of points just satisfying (a) and (b) has positive measure. In [Ji7] it is proved that if f is in the conditions of Theorem 2.1(ii) then $\mathrm{Ch}(f)$ has positive measure.

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