Algunas sorpresas en sistemas dinámicos discretos no autónomos

Armengol Gasull

Universitat Autònoma de Barcelona

The study of periodic discrete dynamical systems is a classical topic that has attracted the researcher's interest in the last years, among other reasons, because they are good models for describing the dynamics of biological systems under periodic fluctuations whether due to external disturbances or effects of seasonality.

These k-periodic systems can be written as $x_{n+1} = f_{n+1}(x_n)$, with initial condition x_0, and a set of maps $\{f_m\}_{m\in\mathbb{N}}$ such that $f_m = f_\ell$ if $m \equiv \ell \pmod{k}$ and can be studied via the *composition map* $f_{k,k-1,\ldots,1} = f_k \circ f_{k-1} \circ \cdots \circ f_1$.

The aim of this talk is to present some surprising phenomena, of local or global nature, appearing when we study them. For instance:

Theorem A. *(a) For all $n \geq 1$ there exist $k \geq 3$ polynomial maps $f_i : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^n$, for $i \in \{1,\ldots,k\}$, sharing a common fixed point p which is locally asymptotically for each map, and such that p is repeller for the composition map $f_{k,k-1,\ldots,1}$.*

(b) For all $n = 2m \geq 2$ there exist 2 polynomial maps $f_1, f_2 : \mathcal{U} \subseteq \mathbb{R}^{2m} \to \mathbb{R}^{2m}$, sharing a common fixed point p which is locally asymptotically for both maps, and such that p is repeller for the composition map $f_{2,1}$.

Recall, that the so called *Parrondo's paradox* is a paradox in game theory, that essentially says that *a combination of losing strategies becomes a winning strategy.* Theorem A can be interpreted as a kind of dynamical Parrondo's paradox.

Theorem B. *There exist two rational planar maps f_1 and f_2, well-defined in the first quadrant, such that f_1 has "complicated" dynamics, f_2 is rationally integrable and the composition maps $f_{2,1}$ and $f_{1,2}$ are rationally integrable.*

These maps are constructed studying the non-autonomous periodic second order Lyness difference equations $x_{n+2} = (a_n + x_{n+1})/x_n$, where $\{a_n\}$ is a cycle of k positive numbers, i.e. $a_{n+k} = a_n$.

The above results appear in the joint papers with Anna Cima and Víctor Mañosa, [1, 2].

References

[1] A. Cima, A. Gasull, V. Mañosa. *Parrondo's dynamic paradox for the stability of non-hyperbolic fixed points.* Discrete Contin. Dyn. Syst. **38** (2018) 889–904.

[2] A. Cima, A. Gasull, V. Mañosa. *Integrability and non-integrability of periodic non-autonomous Lyness recurrences.* Dyn. Syst. **28** (2013) 518–538.

The Sard conjecture on Martinet surfaces

André Belotto

Université Paul Sabatier, Institut de Mathématiques de Toulouse

Given a totally nonholonomic distribution of rank two Δ on a three-dimensional manifold M, it is natural to investigate the size of the set of points $\mathcal{X}^x$ that can be reached by singular horizontal paths starting from a same point $x \in M$. In this setting, the Sard conjecture states that $\mathcal{X}^x$ should be a subset of the so-called Martinet surface of 2-dimensional Hausdorff measure zero.

In this seminar, I present a reformulation of the conjecture in terms of the singular behavior of a vector field. Next, I present a recent work in collaboration with Ludovic Rifford where we show that the conjecture holds whenever the Martinet surface is smooth. Moreover, we address the case of singular real-analytic Martinet surfaces and show that the result holds true under an assumption of non-transversality of the distribution on the singular set of the Martinet surface. Our methods rely on the control of the divergence of vector fields generating the trace of the distribution on the Martinet surface and some techniques of resolution of singularities.