Chiral Effective Field Theory for Nuclear Matter: Technical report

J. A. Oller[a] and A. Lacour[b]

[a] *Departamento de Física, Universidad de Murcia, E-30071 Murcia, Spain*
[b] *Helmholtz-Institut für Strahlen- und Kernphysik (Theorie) and Bethe Center for Theoretical Physics, Universität Bonn, Nußallee 14-16, D-53115 Bonn, Germany*

Abstract

We derive a novel chiral power counting scheme for in-medium chiral perturbation theory with explicit nucleonic and pionic degrees of freedom coupled to external sources. It allows for a systematic expansion taking into account both local as well as pion-mediated inter-nucleon interactions. Based on this power counting, one can identify classes of non-perturbative diagrams that require a resummation. In this work we develop in detail a non-perturbative method based on Unitary Chiral Perturbation Theory (UCHPT) for performing those needed resummations. We have applied this power counting and non-perturbative techniques to the pion self-energy in asymmetric nuclear matter up-to-and-including next-to-leading order (NLO) contributions. We show explicitly the cancellation of the contributions to the pion self-energy with in-medium nucleon-nucleon interactions at NLO employing the non-perturbative techniques presented here. Some N^2LO contributions to the pion self-energy in the nuclear medium are also evaluated for further illustration of the non-perturbative methods. This technical report covers the methodical and technical details of refs. [1, 2].

Contents

1 Introduction

Nuclear physics treats typically systems of many nucleons. This is a non-perturbative problem and for most of the cases of interest the system is bound due to the strong interactions. One of the long standing issues in nuclear physics is the calculation of atom nuclei and nuclear matter properties from microscopic internucleon forces in a systematic and controlled way.

In the last decades Effective Field Theory (EFT) has shown as an inevitable tool to accomplish such aim. It is based on a power counting that establishes a hierarchy between the infinity amount of contributions so that for an order given a finite number of mechanisms have to be considered. In this way, a controlled expansion results that allows to guess the size of the error due to the truncation of the series. In this work we employ Chiral Perturbation Theory (CHPT) [3, 4, 5] to nuclear systems, being our degrees of freedom the nucleons and the pions. CHPT has also the virtue of being connected with QCD since it shares the same symmetries and breaking of them. For the lightest nuclear systems it has been successfully applied to 2, 3 and 4 nucleons [6, 7, 8, 9, 10, 11] and for such systems the previous aim has been accomplished to a large extend. Nonetheless, still some issues are raised concerning the full consistency of the approach and variations of the power counting are from time to time suggested [12, 13, 14, 15]. For heavier nuclei one standard technique is to employ the chiral nucleon-nucleon potential delivered by CHPT in standard many body algorithms, sometimes supplied with regularization group techniques [16]. In present days one issue of interest is the role of multinucleon interactions involving 3 or more nucleons in nuclear matter and nuclei [17, 18, 19].

In ref.[20] many body field theory was derived from quantum field theory by considering nuclear matter as a continuous set of free nucleons at asymptotic times. The generating functional of CHPT in the presence of external sources was deduced, similarly as in the pion and pion-nucleon sectors [21, 22]. These results were applied in ref.[23] to study CHPT in nuclear matter but including only nucleon interactions due to pion exchanges. Thus, the local nucleon-nucleon (and multinucleon) interactions were neglected. In this work we pursue to fill the gap and derive a power counting that takes the latter into account as well. Let us stress that many present applications of CHPT to nuclei and nuclear matter [18, 24, 23, 25, 26, 27, 28, 29] only consider meson-baryon Lagrangians. Short range interactions are included in an ad-hoc way without any relation to free nucleon-nucleon scattering. In addition, as it is well known since the seminal papers of Weinberg [4, 5], the nucleon propagators do not always count as $1/k$ but often they do as the inverse of a nucleon kinetic energy, m/k^2, so that they are much larger than assumed. This, of course, invalidates the straightforward application of the pion-nucleon power counting valid in vacuum as applied e.g. in [23, 30, 24, 25, 26].

Our novel power counting is applied to the problem of calculating up to NLO the pion self-energy in asymmetric nuclear matter. This problem is in tight relation to that of pionic atoms [31, 32] due to the relation between the pion self-energy and the pion-nucleus optical potential. Despite it is an old subject it still lacks a conclusive calculation of the pion self-energy in a systematic and controlled expansion. For recent calculations see [23, 33, 28, 34, 30]. In particular, the issues of the pion-nucleus S-wave missing repulsion, the renormalization of the scattering length a^- in the medium [33, 28] and the energy dependence of the isovector amplitude [32] is not settled yet, despite the recent progresses [23, 35, 32].

After this introduction, we derive in section 2 a novel chiral power counting in the medium that takes into account multinucleon local interactions, pion exchanges and the enhancement of nucleon propagators. In the following sections, up to section 7, we calculate the different pion-nucleon contributions to the pion self-energy in asymmetric nuclear matter. We dedicate section 8 to the calculation of the vacuum and in-medium nucleon-nucleon scattering amplitudes up to next-to-leading order. These new contributions will be applied to the pion self-energy in the sections 9, 10 and 11, where we also give some $\mathrm{N}^2\mathrm{LO}$

contributions. The sections 12, 13 and 14 provide technical details for calculations of the pion-production box diagrams. Results for the pion self-energy are shown in section 15, including the pion mass in the medium discussed in the subsection 15.5. Some conclusions are presented in sec.16. In the Appendices we include various mathematical derivations used in previous sections. Appendix B offers a derivation of the partial wave expansion on nucleon-nucleon scattering. Appendix D provides the one-pion exchange nucleon-nucleon partial waves up to the $F-$wave. The appendices E, G and H develop the calculation of the in-medium integrals needed for the evaluations performed in the earlier sections.

2 Chiral Power Counting

In ref.[20] the effective chiral pion Lagrangian was determined in the nuclear medium with the presence of external sources. For that the Fermi seas of protons and neutrons were integrated out making use of functional techniques. A similar approach was followed in ref.[22] but for the case of only one nucleon. Nonetheless, in ref.[20] only the meson-baryon chiral Lagrangian is employed. That is, if we write a general chiral Lagrangian in terms of an increasing number of baryon fields

$$\mathcal{L}_\chi = \mathcal{L}_{\pi\pi} + \mathcal{L}_{\bar{\psi}\psi} + \mathcal{L}_{\bar{\psi}\bar{\psi}\psi\psi} + \ldots \tag{2.1}$$

with ψ denoting a generic baryon field, ref.[20] only retains $\mathcal{L}_{\pi\pi}$ and $\mathcal{L}_{\bar{\psi}\psi}$. Based on these results ref.[23] derived a chiral power counting in the nuclear medium.

Also in ref.[20] there was established the concept of a "generalized in-medium vertex" (IGV). These vertices result because one can connect several bilinear vacuum vertices through the exchange of baryon propagators with the flow through the loop of one unit of baryon number, contributed by the nucleon Fermi seas. This is schematically shown in fig.1 where the thick arc segment indicates an insertion of one Fermi. At least there is needed one because otherwise we would have a vacuum closed nucleon loop that in a low energy effective field theory is not *explicitly* taken into account. On the other hand, the filled larger circles in fig.1 indicate a bilinear nucleon vertex from $\mathcal{L}_{\pi N}$, while the dots refer to the insertion of any number of them. It was also stressed in ref.[23] that within a nuclear environment a nucleon propagator could have a "standard" or "non-standard" chiral counting. To see this, note that a soft momentum Q, related to pions or external sources attached to the bilinear vertices in the figure, can be associated to any of the vertices. This together with the Dirac delta function of four-momentum conservation implies that the momenta running along the nucleon propagators in fig.1 just differ from each other by quantities of $\mathcal{O}(Q)$. Denoting by k the on-shell four-momenta associated with one Fermi sea insertion in the in-medium generalized vertex, the four-momentum running through the j_{th} nucleon propagator can be written as $p_j = k + Q_j$. In this way,

$$iD_0^{-1}(p_j) = i\frac{\not{k} + \not{Q}_j + m_N}{(k+Q_j)^2 - m_N^2 + i\epsilon} = i\frac{\not{k} + \not{Q}_j + m_N}{Q_j^2 + 2Q_j^0 E(\mathbf{k}) - 2\mathbf{Q}_j\mathbf{k} + i\epsilon} \ . \tag{2.2}$$

and $E(\mathbf{k}) = \frac{\mathbf{k}^2}{2m}$, with m the physical nucleon mass (not the bare one). Two different situations occur depending on the value of Q_j^0. If $Q_j^0 = \mathcal{O}(m_\pi) = \mathcal{O}(p)$ one has the standard counting so that the chiral expansion of the propagator in eq.(2.2) is

$$iD_0^{-1}(p_j) = i\frac{\not{k} + \not{Q}_j + m_N}{2Q_j^0 m_N + i\epsilon}\left(1 - \frac{Q_j^2 - 2\mathbf{Q}_j\cdot\mathbf{k}}{2Q_j^0 m_N} + \mathcal{O}(p^2)\right) \ . \tag{2.3}$$

Thus, iD_0^{-1} counts as a quantity of $\mathcal{O}(p^{-1})$. But it could also occur that Q_j^0 is $\mathcal{O}(E(\mathbf{k}))$, that is, of the order of a kinetic nucleon energy in the nuclear medium, or that it even vanishes. The dominant term in

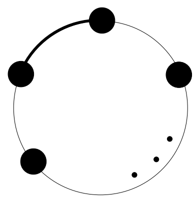

Figure 1: In-medium generalized vertex (IGV). The thick solid line corresponds to a Fermi sea insertion while the filled circles are bilinear nucleon vertices from $\mathcal{L}_{\pi N}$.

eq.(2.2) is then

$$iD_0^{-1} = -i\frac{\not{k} + \not{Q}_j + m}{\mathbf{Q}_j^2 + 2\mathbf{Q}_j \cdot \mathbf{k} - i\varepsilon} \ , \tag{2.4}$$

and then the nucleon propagator should be counted as $\mathcal{O}(p^{-2})$, instead of the previous $\mathcal{O}(p^{-1})$. This is referred as the "non-standard" case in ref.[23]. We should stress that this situation also occurs already at vacuum when considering the two-nucleon reducible diagrams in nucleon-nucleon scattering. This is indeed the reason advocated in ref.[4] for solving a Lippmann-Schwinger equation with the nucleon-nucleon potential given by the two-nucleon irreducible diagrams. The case of nucleon reducible diagrams also occurs in the nuclear medium where there are an infinite number of nucleons.

In the present investigation, we extend the results of refs.[20, 23] in a twofold way. i) We are able to consider chiral Lagrangians with an arbitrary number of baryon fields (bilinear, quartic, etc). First only bilinear vertices like in refs.[20, 23] are considered, but now the additional exchanges of heavy meson fields of any type are allowed. The latter should be considered as merely auxiliary fields that allow one to find a tractable representation of the multi-nucleon interactions that result when the masses of the heavy mesons tend to infinity. ii) We take the non-standard counting from the start and count any nucleon propagator as $\mathcal{O}(p^{-2})$. In this way, no diagram whose chiral order is actually lower than expected if the nucleon propagators were counted assuming the standard rules is lost. This is a novelty in the literature.

Let us denote by H the heavy mesons responsible because of their exchanges between bilinear vertices, of local nucleon interactions, NN, NNN, etc, and by π the pions. From the counting point of view there is a clear similarity between the interactions driven by the exchanges of H and π fields as both emerge from bilinear vertices. The large mass of the former is responsible of the local character of the induced interactions. A heavy meson propagator is counted as p^0.

The chiral order of a given diagram is represented by ν and it is given by

$$\nu = 4L_H + 4L_\pi - 2I_\pi + \sum_{i=1}^{V_\rho}\left[\sum_j d_j - 2m_i\right] + \sum_{i=1}^{V_\pi}\ell_i + \sum_{i=1}^{V_\rho} 3 \ . \tag{2.5}$$

Here, V_ρ is the number of in-medium generalized vertices, m_i is the number of nucleon propagators in the i_{th} in-medium generalized vertex minus one, the one that corresponds to the needed Fermi sea insertion for each in-medium generalized vertex. In addition, d_i is the chiral order of the i_{th} vertex bilinear in the baryonic fields, ℓ_i is the chiral order of a vertex without baryons (only pions and external sources) and V_π is the number of the latter ones. As usual, L_π is the number of pionic loops and I_π is the number of internal pionic lines. L_H is the number of loops due to the internal heavy mesonic lines.

Let us note that associated with the bilinear vertices in an in-medium generalized vertex one has four-momentum conservation delta functions that can be used to fix the momentum of each of the baryonic lines joining them, except one for the running three-momentum due to the Fermi sea insertion. Of course, this cannot be fixed because one four-momentum delta function has to do with the conservation of the total four-momentum. This is the reason why we referred above only to loops attached to mesonic lines and not to baryonic ones.

Let us now introduce another symbol, V_Φ. Here, we take as a whole any set of generalized in-medium vertices that are joined only through *heavy* mesonic lines H. The number of all them is denoted by V_Φ. These clusters are connected among them by pionic lines and associated to every of these sets there is a total four-momentum conservation delta function. In this way,

$$L_\pi = I_\pi - V_\pi - V_\Phi + 1 \ , \tag{2.6}$$

Similarly,

$$L_H = I_H - \sum_{i=1}^{V_\Phi} (V_{\rho,i} - 1) = I_H - V_\rho + V_\Phi \ . \tag{2.7}$$

$V_{\rho,i}$ is the number of in-medium generalized vertices within the i_{th} set of generalized vertices connected by the heavy mesonic lines. In turn,

$$2I_H + 2I_\pi + E = \sum_{i=1}^{V} v_i + \sum_{i=1}^{V_\pi} n_i \ . \tag{2.8}$$

Where V is the total number of bilinear vertices, v_i is the number of mesonic lines attached to the i_{th} bilinear vertex and n_i is the number of pions in the i_{th} mesonic vertex. E is the number of external pionic lines.

Taking into account eqs.(2.6) and (2.7) one has,

$$4L_H + 4L_\pi - 2I_\pi = 4I_H + 2I_\pi - 4V_\rho - 4V_\pi + 4 \ . \tag{2.9}$$

Now considering eq.(2.8) as well,

$$4L_H + 4L_\pi - 2I_\pi = 2I_H - E + \sum_{i=1}^{V} v_i + \sum_{i=1}^{V_\pi} n_i - 4V_\rho - 4V_\pi + 4 \ . \tag{2.10}$$

Substituting the previous line in eq.(2.5) ,

$$\nu = 2I_H - E + 4 - 4V_\pi + \sum_{i=1}^{V_\pi}(\ell_i + n_i) + \sum_{i=1}^{V}(d_i + v_i) - 2m - V_\rho \ . \tag{2.11}$$

with $m = \sum_{i=1}^{V_\rho} m_i$. We now employ that $V_\rho + m = V$, and $2I_H = \sum_{i=1}^{V} \omega_i$, where ω_i is the number of heavy meson internal lines for the i_{th} bilinear vertex. Then, we arrive to our final equation,

$$\nu = 4 - E + \sum_{i=1}^{V_\pi}(n_i + \ell_i - 4) + \sum_{i=1}^{V}(d_i + \omega_i - 1) + \sum_{i=1}^{m}(v_i - 1) + \sum_{i=1}^{V_\rho} v_i \ . \tag{2.12}$$

Note that ν given in eq.(2.12) is bounded from below because

$$n_i + \ell_i - 4 \geq 0 \ , \tag{2.13}$$

as $\ell_i \geq 2$ and $n_i \geq 2$, except for a finite number of terms that could contain only one pion line but always having external sources attached to them. Similarly

$$d_i + \omega_i - 1 \geq 0 \ . \tag{2.14}$$

For pion-nucleon Lagrangians this is always true as $d_i \geq 1$. For those bilinear vertices mediated by heavy lines $d_i \geq 0$ but then $w_i \geq 1$. For the term before the last one $v_i - 1 \geq 0$, except for the finite number of terms which would not have pionic lines but only external sources from $\mathcal{L}_{\pi N}$. For the last term in eq.(2.12) $v_i \geq 0$ and then positive. It is specially important to note that adding a new in-medium generalized vertex to a connected diagram increases the counting at least by one unit because then $\nu_i \geq 1$.

The number ν given in eq.(2.12) represents a lower bound for the actual chiral power of a diagram, let us call this by μ, and then $\mu \geq \nu$. The reason why μ might be different from ν is because the nucleon propagators are counted always as $\mathcal{O}(p^{-2})$, while for some diagrams there could be propagators following the standard counting. The point of eq.(2.12) is that it allows to ensure that no other contributions to those already considered would have a lower chiral order. As a result, one can handle systematically the so called anomalous chiral counting.

Another form of eq.(2.12) that is also useful for practical applications stems

$$\nu \;=\; 4 - E + \sum_{i=1}^{V_\pi}(n_i + \ell_i - 4) + \sum_{i=1}^{V}(d_i + \omega_i - 1) + \sum_{i=1}^{V} v_i - m \tag{2.15}$$

From these equations one can augment the number of lines in a diagram without increasing the power counting by:

1. Adding pion lines attached to mesonic vertices, $\ell_i = n_i = 2$.
2. Adding pion lines attached to meson-baryon vertices, $d_i = v_i = 1$.
3. Adding heavy mesonic lines attached to bilinear vertices, $d_i = 0$, $w_i = 1$.

There is no way to decrease the order.[#1] We apply eq.(2.12) by increasing step by step V_ρ up to the order pursued. For each V_ρ then we look for those diagrams that do not further increase the order according to the previous list. Some of these diagrams are indeed of higher order and one can refrain from calculating them by establishing which of the propagators scales as $\mathcal{O}(p^{-1})$. In this way, the actual chiral order of the diagrams is determined and one can select those diagrams that correspond to the precision required. For higher orders one should consider the other possibilities for a fixed V_ρ.

3 In-medium pion self-energy diagram Σ_1

Here we start the application of the chiral counting in eq.(2.12) to calculate the pion self-energy in the nuclear medium up to NLO or $\mathcal{O}(p^5)$. The different contributions are denoted by Σ_i and are calculated in the following sections. In terms of the pion self energy Σ the dressed pion propagators reads

$$\frac{1}{q^2 - m_\pi^2 + \Sigma} \ . \tag{3.1}$$

[#1]Only by adding vertices with $\ell_i = 2$ and $n_i < 2$ or $d_i = 1$ and $v_i = 0$. However, its number is bounded from above by the necessarily finite number of external sources.

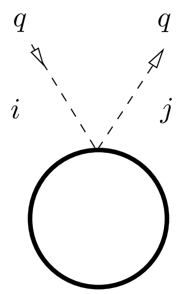

Figure 2: The diagram Σ_1 is obtained by closing the Weinberg-Tomozawa pion-nucleon interaction.

The nucleon propagator contains both the free and the in-medium piece [36],

$$\frac{\theta(\xi_{i_3}-|\mathbf{k}|)}{k^0-E(\mathbf{k})-i\epsilon}+\frac{\theta(|\mathbf{k}|-\xi_{i_3})}{k^0-E(\mathbf{k})+i\epsilon}=\frac{1}{k^0-E(\mathbf{k})+i\epsilon}+2\pi i\,\theta(k_F^{i_3}-|\mathbf{k}|)\delta(k^0-E(\mathbf{k}))\ . \tag{3.2}$$

In this equation the superscript i_3 refers to the third component of isospin of the nucleon, so that, $i_3=+1/2$ corresponds to the proton and $-1/2$ to the neutron, and the symbol ξ_{i_3} is the Fermi momentum of the Fermi sea for the corresponding nucleon. We consider that isospin symmetry is conserved so that all the nucleon and pion masses are equal. One can use a common expression for the proton and neutron propagators,

$$\begin{aligned}&\left(\frac{1+\tau_3}{2}\theta(\xi_p-|\mathbf{k}|)+\frac{1-\tau_3}{2}\theta(\xi_n-|\mathbf{k}|)\right)\frac{1}{k^0-E(\mathbf{k})-i\epsilon}+\\&\left(\frac{1+\tau_3}{2}\theta(|\mathbf{k}|-\xi_p)+\frac{1-\tau_3}{2}\theta(|\mathbf{k}|-\xi_n)\right)\frac{1}{k^0-E(\mathbf{k})+i\epsilon}\end{aligned} \tag{3.3}$$

or the equivalent one,

$$\frac{1}{k^0-E(\mathbf{k})+i\epsilon}+2\pi i\,\delta(k^0-E(\mathbf{k}))\left(\frac{1+\tau_3}{2}\theta(\xi_p-|\mathbf{k}|)+\frac{1-\tau_3}{2}\theta(\xi_n-|\mathbf{k}|)\right)\ . \tag{3.4}$$

In the previous equations τ^i corresponds to the Pauli matrices in the isospin space. In the same way, σ^i will correspond to the same matrices but in the spin space.

Σ_1 results by closing the Weinberg-Tomozawa pion-nucleon interaction (WT), eq.(A.3),

$$\begin{aligned}\Sigma_1&=\frac{-iq^0}{2f^2}\varepsilon_{ijk}\int\frac{d^3k}{(2\pi)^3}2\ \mathrm{tr}\left[\tau^k\left\{\frac{1+\tau_3}{2}\theta(\xi_p-|\mathbf{k}|)+\frac{1-\tau_3}{2}\theta(\xi_n-|\mathbf{k}|)\right\}\right]\\&=\frac{-iq^0}{2f^2}\varepsilon_{ij3}(\rho_p-\rho_n)\ .\end{aligned} \tag{3.5}$$

In the previous equation, and this will be the case in the following, we denote by tr (with lower case t) the trace including only the isospin space. Instead, Tr (with capital T) will denote the trace both including the isospin and spin spaces. Indeed, the factor 2 inside the integral corresponds to the sum over spins. On the other hand, the proton and neutron densities are indicated by ρ_p and ρ_n, respectively, and are given by

$$\begin{aligned}\rho_p&=2\int\frac{d^3k}{(2\pi)^3}\theta(\xi_p-|\mathbf{k}|)=\frac{\xi_p^3}{3\pi^2}\ ,\\\rho_n&=2\int\frac{d^3k}{(2\pi)^3}\theta(\xi_n-|\mathbf{k}|)=\frac{\xi_n^3}{3\pi^2}\ .\end{aligned} \tag{3.6}$$

Eq.(3.5) is a S-wave isovector self-energy.

4 In-medium pion self-energy diagram Σ_2

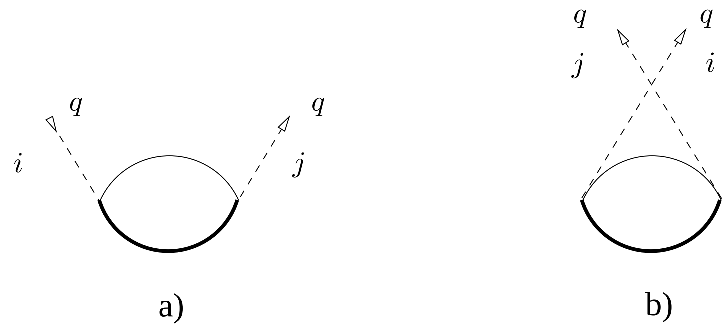

Figure 3: Σ_2 results by closing the nucleon pole terms in the pion-nucleon scattering.

The diagrams in fig.3 involve the one-pion vertex from the lowest order meson-baryon chiral Lagrangian $\mathcal{L}_{\pi N}$ which is given in eq.(A.1). The expression for the diagram in fig.3 is

$$\Sigma_2^a = -\frac{g_A^2}{4f^2}\int \frac{d^3k}{(2\pi)^3}\,\mathrm{Tr}\left[\left(\frac{1+\tau_3}{2}\theta(\xi_p-|\mathbf{k}|)+\frac{1-\tau_3}{2}\theta(\xi_n-|\mathbf{k}|)\right)\frac{\tau^i\tau^j\,\vec{\sigma}\cdot\mathbf{q}\vec{\sigma}\cdot\mathbf{q}}{E(\mathbf{k})-q^0-E(\mathbf{k}-\mathbf{q})+i\epsilon}\;.\right] \tag{4.1}$$

In the previous equation we have not included the in-medium part of the intermediate nucleon propagator because $q^0 \simeq m_\pi \gg E(\mathbf{k})-E(\mathbf{k}-\mathbf{q})$, since the latter corresponds to nucleon kinetic energies. In this way, the argument in the in-medium Dirac delta function in eq.(3.4) cannot be fulfilled. By the same token

$$\frac{1}{E(\mathbf{k})-E(\mathbf{k}-\mathbf{q})-q^0} = \frac{1}{-q^0}-\frac{E(\mathbf{k})-E(\mathbf{k}-\mathbf{q})}{{q^0}^2}+\mathcal{O}(q)\;, \tag{4.2}$$

and the $\mathcal{O}(q)$ terms contribute one order higher to NLO. On the other hand

$$E(\mathbf{k})-E(\mathbf{k}-\mathbf{q}) = -\frac{\mathbf{q}^2-2\mathbf{k}\cdot\mathbf{q}}{2m}\;. \tag{4.3}$$

The $\mathbf{k}\cdot\mathbf{q}$ term in this equation, when included in eq.(4.1), does not contribute because of the angular integration. Then,

$$\Sigma_2^a = \frac{g_A^2}{4f^2q^0}\left(1-\frac{\mathbf{q}^2}{2mq^0}\right)\int\frac{d^3k}{(2\pi)^3}\,\mathrm{Tr}\left[\left(\frac{1+\tau_3}{2}\theta(\xi_p-|\mathbf{k}|)+\frac{1-\tau_3}{2}\theta(\xi_n-|\mathbf{k}|)\right)\tau^i\tau^j\,\vec{\sigma}\cdot\mathbf{q}\vec{\sigma}\cdot\mathbf{q}\right]\;. \tag{4.4}$$

Proceeding in the same way for Σ_2^b (which corresponds to the same expression as for Σ_2^a but with $q^0 \to -q^0$ and $i \leftrightarrow j$), and summing both, one has

$$\begin{aligned}\Sigma_2^{iv} &= \frac{ig_A^2\,\mathbf{q}^2}{2f^2q^0}\varepsilon_{ij3}(\rho_p-\rho_n)\;,\\ \Sigma_2^{is} &= \frac{-g_A^2}{4f^2}\frac{(\mathbf{q}^2)^2}{m{q^0}^2}\delta_{ij}(\rho_p+\rho_n)\;.\end{aligned} \tag{4.5}$$

The superscript iv refers to the isovector part and it is leading order, while is refers to the isoscalar part, which is of next-to-leading order. Both are P-wave self-energies but Σ_2^{is} is a relativistic correction of Σ_2^{iv}, rising by the expansion of the free nucleon propagator, and it is suppressed by the inverse of the nucleon mass.

5 In-medium pion self-energy diagram Σ_3

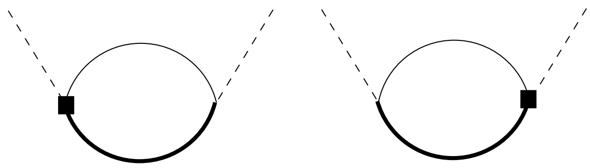

Figure 4: The Σ_3 contribution stems from diagrams similar to those of fig.3 but with one vertex from the NLO meson-baryon Lagrangian $\mathcal{L}_{\pi N}$. This vertex is indicated by the square on the figure. Every diagram actually represents two diagrams by the exchange of the initial and final pion lines.

We now consider the diagrams shown in fig.4. It should be understood that the pion lines can be leaving or entering the diagram, similarly as explicitly shown in fig.3. In the figure the square indicates a NLO one pion vertex from $\mathcal{L}_{\pi N}$, given in eq.(A.2).

We also employ the expansion of eq.(4.2) for the nucleon propagator but for this case it is only necessary to keep the term $\pm 1/q^0$ because the diagram is already a NLO contribution. The calculation is straightforward with the result,

$$\Sigma_3 = \frac{g_A^2 \mathbf{q}^2}{2mf^2}(\rho_p + \rho_n)\delta_{ij} \ . \tag{5.1}$$

This is a P-wave isoscalar contribution. In this case the NLO vertex is a relativistic correction to the LO one and this is why Σ_3 is suppressed by the inverse of the nucleon mass.

6 In-medium pion self-energy diagram Σ_4

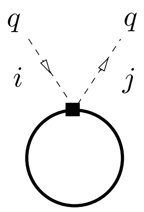

Figure 5: Σ_4 is similar to Σ_1, but instead of the WT vertex it has a NLO vertex, indicated by the square.

We now consider the diagram in fig.5 with the NLO vertex given in eq.(A.4). When summing over the nucleons in the proton and neutron Fermi seas one has,

$$\Sigma_4 = \frac{-2\delta_{ij}}{f^2}\left(2c_1\, m_\pi^2 - {q^0}^2(c_2 + c_3 - \frac{g_A^2}{8m}) + c_3\,\mathbf{q}^2\right)(\rho_p + \rho_n) \ . \tag{6.1}$$

In this equation the term $-2\delta_{ij}c_3\mathbf{q}^2(\rho_p+\rho_n)/f^2$ is a P-wave contribution and the rest is S-wave.

7 Pion loop nucleon self-energy

Let us consider the contributions to the pion self-energy due to the nucleon self-energy from the one-pion loop, as represented by the diagrams in fig.6. These diagrams originate by the dressing of the in-medium

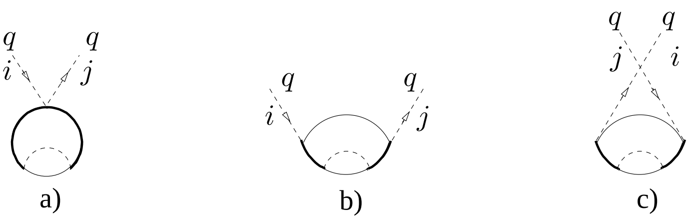

Figure 6: The pion self-energy due to the pion loop contribution to the nucleon self-energy in the nuclear medium that dresses the diagrams in figs.2 and 3.

nucleon propagator of figs.2 and 3 due to the one-pion loop nucleon self-energy. As a preliminary result we first evaluate the nucleon self-energy in the nuclear medium corresponding to fig.7.

7.1 Pion loop nucleon self-energy

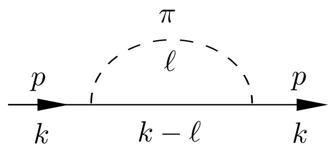

Figure 7: Pion loop contribution to the nucleon self-energy in the nuclear medium. The four-momenta are indicated below the corresponding line in the figure.

First we consider the case of a π^0. The results for the charged pions follow immediately from the π^0 case. In Heavy Baryon CHPT (HBCHPT) the proton self-energy due to a π^0 loop, see fig.7, is given by,

$$\begin{aligned}\Sigma_p^{\pi^0} &= -i\frac{g_A^2}{f^2}S_\mu S_\nu \int \frac{d^D\ell}{(2\pi)^D}\frac{\ell^\mu \ell^\nu}{(\ell^2-m_\pi^2+i\epsilon)(v(k-\ell)+i\epsilon)} \\ &+ 2\pi\frac{g_A^2}{f^2}S_\mu S_\nu \int \frac{d^4\ell}{(2\pi)^4}\frac{\ell^\mu\ell^\nu}{\ell^2-m_\pi^2+i\epsilon}\delta(v(k-\ell))\theta(k_F^p-|\mathbf{k}-\vec{\ell}|)~. \end{aligned} \tag{7.1}$$

Here v is a four-vector normalized to unity, $v^2=1$, such that the four-momentum of a nucleon is given by $p=mv+k$, where k is a residual small momentum, $v\cdot k \ll m$. In practical calculations we will take $v=(1,\mathbf{0})$. Instead of the full non-relativistic nucleon propagator eq.(3.2), HBCHPT implies the so called extreme non-relativistic limit in which $E(\mathbf{k})\to 0$. In addition, instead of k^0 one uses the scalar $v\cdot k$. The covariant spin operator S_μ fulfills

$$\{S_\mu,S_\nu\}=\frac{1}{2}(v_\mu v_\nu - g_{\mu\nu})~,~~[S_\mu,S_\nu]=i\epsilon_{\mu\nu\gamma\delta}\,v^\gamma S^\delta~. \tag{7.2}$$

Thus, the combination $S_\mu S_\nu \ell^\mu \ell^\nu$ that enters in the integrals of eq.(7.1) corresponds to

$$S_\mu S_\nu \ell^\mu \ell^\mu = \frac{1}{2}\{S_\mu,S_\nu\}\ell^\mu\ell^\nu = \frac{1}{4}\left((v\cdot\ell)^2-\ell^2\right)~. \tag{7.3}$$

7.1.1 Free part

For the free part we have the integral,

$$\Sigma_{p,f}^{\pi^0} = -i\frac{g_A^2}{4f^2}\int\frac{d^D\ell}{(2\pi)^D}\frac{(v\ell)^2-\ell^2}{(\ell^2-m_\pi^2+i\epsilon)(v(k-\ell)+i\epsilon)}~, \tag{7.4}$$

as follows from eqs.(7.1) and (7.3). This can be evaluated straightforwardly in dimensional regularization with the result [37, 38],

$$\Sigma^{\pi^0}_{p,f} = \frac{g_A^2 m_\pi^2 \omega}{64\pi^2 f^2}\left(\frac{1}{\hat{\epsilon}} + \log\frac{m_\pi^2}{\lambda^2}\right) + \frac{g_A^2 b\omega}{32\pi^2 f^2}\left\{\frac{1}{\hat{\epsilon}} - 1 + \log\frac{m_\pi^2}{\lambda^2} + \frac{\sqrt{b}}{\omega}\left(i\log\frac{\omega + i\sqrt{b}}{-\omega + i\sqrt{b}} + \pi\right)\right\} . \tag{7.5}$$

where $\omega = v\cdot k$ and $1/\hat{\epsilon} = 1/\epsilon + \gamma_E - 1 - \log 4\pi$, with γ_E the Euler constant and $\epsilon = (D-4)/2$. We use here $b = m_\pi^2 - \omega^2 - i\epsilon$, because $\omega = v\cdot k = k^0$ for our final choice of $v = (1,\mathbf{0})$. The divergent pieces cancel with the appropriate $\mathcal{O}(p^3)$ counterterms of the Heavy-Baryon meson-baryon Lagrangian [39].[#2] In addition, there is also the contribution of the charged pions in the intermediate pion loop. This contribution is a factor 2 larger than for the π^0 case. Adding both one has

$$\Sigma^{\pi}_{p,f} = \Sigma^{\pi}_{n,f} = \frac{3g_A^2 b}{32\pi^2 f^2}\left\{-\omega + \sqrt{b}\left(i\log\frac{\omega + i\sqrt{b}}{-\omega + i\sqrt{b}} + \pi\right)\right\} . \tag{7.6}$$

As indicated in the equation above, the same expression is also valid for the neutron case because of isospin symmetry, that we assume is conserved.

We also need below the derivative

$$\frac{\partial \Sigma^{\pi}_{p(n),f}}{\partial w} = \frac{3g_A^2}{32\pi^2 f^2}\left[m_\pi^2 + \omega^2 - 3\omega\sqrt{b}\left(i\log\frac{\omega + i\sqrt{b}}{-\omega + i\sqrt{b}} + \pi\right)\right] . \tag{7.7}$$

7.1.2 In-medium part

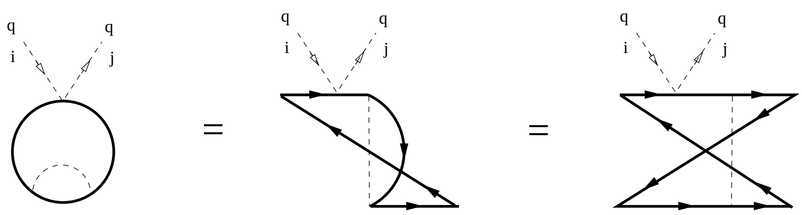

Figure 8: The equivalence between the diagram of fig.6a and the one-pion exchange reduction of the diagram on the right hand side of fig.18. We show the second diagram from the left as an intermediate step in the continuous transformation of the diagram to the far left to that on the far right.

The contribution to the diagrams of fig.6 from the second integral on the right hand side of eq.(7.1) is accounted for by the one-pion exchange reduction of the nucleon-nucleon scattering amplitudes in the crossed exchange diagrams (see diagrams in figs.18 and 20). In fig.8 we depict such equivalence for the diagram a) of fig.6 and the second diagram of fig.18. An analogous result holds for the diagrams b)–c) of fig.6 and the second diagram of fig.20. For the latter it should be understood that any pion line can leave or enter the diagram. Since all these contributions will be calculated in sections 10 and 11 we skip them by now.[#3]

[#2] The counterterms are $\widetilde{d}_{24}$ and $\widetilde{d}_{28}$ of ref.[39]. They do not have finite counterpart because they are proportional to the nucleon equation of motion.

[#3] In ref.[1] these contributions were finally neglected at NLO because they are of higher order.

7.2 In-medium pion self-energy diagram Σ_5

The expression for the diagram in fig.6a is

$$\Sigma_5 = \frac{q^0}{2f^2}\varepsilon_{ijk}\int \frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\left\{\tau^k\left(\frac{1+\tau_3}{2}\frac{\theta(\xi_p-|\mathbf{k}|)}{k^0-E(\mathbf{k})-i\epsilon}+\frac{1-\tau_3}{2}\frac{\theta(\xi_n-|\mathbf{k}|)}{k^0-E(\mathbf{k})-i\epsilon}\right)\Sigma^\pi \right.$$
$$\left.\times\left(\frac{1+\tau_3}{2}\frac{\theta(\xi_p-|\mathbf{k}|)}{k^0-E(\mathbf{k})-i\epsilon}+\frac{1-\tau_3}{2}\frac{\theta(\xi_n-|\mathbf{k}|)}{k^0-E(\mathbf{k})-i\epsilon}\right)\right\}\ . \tag{7.8}$$

Here,

$$\Sigma^\pi = \frac{1+\tau_3}{2}\Sigma_p^\pi + \frac{1-\tau_3}{2}\Sigma_n^\pi\ . \tag{7.9}$$

with Σ_p^π and Σ_n^π given in eq.(7.6). Once the trace in the isospin and spin spaces is taken, the previous expression simplifies to

$$\Sigma_5 = \frac{q^0}{f^2}\varepsilon_{ij3}\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^0-E(\mathbf{k})-i\epsilon)^2}(\theta_p^- - \theta_n^-)\Sigma_f^\pi\ , \tag{7.10}$$

where Σ_f^π is given in eq.(7.6). We have also introduced the shorter notation, to be kept in mind, that $\theta(\xi_p-|\mathbf{k}|)\equiv\theta_p^-$ and $\theta(\xi_n-|\mathbf{k}|)\equiv\theta_n^-$. The k^0 integration is done by applying the Cauchy theorem closing the integration contour with a circle at infinity on the upper half k^0-complex plane, so that the contribution from the nucleon Fermi seas is picked up [36]. Then, we have:

$$\Sigma_5 = \frac{iq^0}{f^2}\varepsilon_{ij3}\int\frac{d^3k}{(2\pi)^3}\,(\theta_p^- - \theta_n^-)\frac{\partial\Sigma_f^\pi}{\partial k^0}\bigg|_{k^0=E(\mathbf{k})}\ . \tag{7.11}$$

Σ_5 is an isovector S-wave pion self-energy contribution. We have evaluated Σ_5 considering it as a contribution of $\mathcal{O}(p^5)$, however it is actually $\mathcal{O}(p^6)$. This is due to the fact that $\partial\Sigma_f^\pi/\partial k^0$ is $\mathcal{O}(p^2)$, as follows directly from eq.(7.7). We originally booked Σ_5 as $\mathcal{O}(p^5)$ because $\partial\Sigma_f^\pi\partial k^0$ was taken as $\mathcal{O}(p)$, due to the fact that Σ_f^π is $\mathcal{O}(p^3)$ and $k^0=\mathcal{O}(p^2)$. However, this dimensional evaluation of the order of a derivative represents indeed a lower bound and its actual order can be higher. This is indeed what occurs in this case.

7.3 In-medium pion self-energy diagram Σ_6

Let us consider now the diagrams corresponding to fig.6b and c. The diagram of fig.6b is given by the expression

$$\Sigma_6^a = -i\frac{g_A^2}{4f^2}\int\frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\left\{\left(\frac{1+\tau_3}{2}\theta_p^- + \frac{1-\tau_3}{2}\theta_n^-\right)\tau^i\tau^j\vec{\sigma}\cdot\mathbf{q}\vec{\sigma}\cdot\mathbf{q}\left(\frac{1+\tau_3}{2}\theta_p^- + \frac{1-\tau_3}{2}\theta_n^-\right)\right.$$
$$\left.\times\,\Sigma^\pi\ \frac{1}{(k^0-E(\mathbf{k})-i\epsilon)^2}\frac{1}{k^0-q^0+i\epsilon}\right\}\ . \tag{7.12}$$

The integration over k^0 is done as usual, picking up the pole at $k^0=E(\mathbf{k})$, and we also employ the expansion of the nucleon propagator, eq.(4.2). Then,

$$\Sigma_6^a = -\frac{g_A^2\mathbf{q}^2}{4f^2q^0}\int\frac{d^3k}{(2\pi)^3}\,\mathrm{Tr}\left\{\left(\frac{1+\tau_3}{2}\theta_p^- + \frac{1-\tau_3}{2}\theta_n^-\right)\tau^i\tau^j\left(\frac{1+\tau_3}{2}\theta_p^- + \frac{1-\tau_3}{2}\theta_n^-\right)\left[\frac{\partial\Sigma^\pi}{\partial k^0}+\frac{\Sigma^\pi}{q^0}\right]\right\}\bigg|_{k^0=E(\mathbf{k})} \tag{7.13}$$

with Σ^π defined in eq.(7.9). The analogous expression for Σ_6^b can be obtained just by exchanging $q^0 \to -q^0$ in Σ_6^a. Summing both and taking the trace over isospin and spin, one has

$$\Sigma_6 = \frac{-ig_A^2}{f^2}\frac{\mathbf{q}^2}{q^0}\varepsilon_{ij3}\int \frac{d^3k}{(2\pi)^3}\,(\theta_p^- - \theta_n^-)\frac{\partial \Sigma_f}{\partial k^0}\bigg|_{k^0=E(\mathbf{k})} - \frac{g_A^2}{f^2}\frac{\mathbf{q}^2}{q^{0\,2}}\delta_{ij}\int \frac{d^3k}{(2\pi)^3}(\theta_p^- + \theta_n^-)\Sigma_f\bigg|_{k^0=E(\mathbf{k})} \;. \tag{7.14}$$

Σ_6 is a P-wave self-energy contribution but while the first line is of isovector character, denoted by Σ_6^{iv}, the latter one is isoscalar and denoted by Σ_6^{is}. Notice that the integral in Σ_6^{iv} is the same as the one for Σ_5 in eq.(7.11). It is also the case that Σ_6^{iv} is actually one order higher than expected, as follows from the same reasons given above for Σ_5. Then, both Σ_6^{iv} and Σ_6^{is} are $\mathcal{O}(p^6)$. For the latter this is obvious from its expressions in the last line of eq.(7.14) as the nucleon self-energies are $\mathcal{O}(p^3)$.

8 Nucleon-nucleon interactions

The inclusion of the nucleon-nucleon interactions for the calculation of the pion self-energy takes place at NLO, because they require at least $V_\rho = 2$. As a result, for our purposes, it is only necessary to work out them at the lowest chiral order, $\mathcal{O}(p^0)$. First, we discuss the nucleon-nucleon interactions in the vacuum and then consider their extension to the nuclear medium. For the vacuum case we also discuss the nucleon-nucleon interactions calculated at $\mathcal{O}(p)$.

8.1 Free nucleon-nucleon interactions

We follow the standard chiral counting [4, 5] where the lowest order amplitudes for the two-nucleon irreducible diagrams, $\mathcal{O}(p^0)$ are given by the quartic nucleon Lagrangian without quark masses or derivatives and by the one-pion exchange with the lowest order pion-nucleon coupling, eq.(A.1). The $\mathcal{O}(p^0)$ lowest order four nucleon Lagrangian [5] is

$$\mathcal{L}_{NN}^{(0)} = -\frac{1}{2}C_S(\overline{N}N)(\overline{N}N) - \frac{1}{2}C_T(\overline{N}\vec{\sigma}N)(\overline{N}\vec{\sigma}N) \;. \tag{8.1}$$

Of course, this Lagrangian only contributes to the S-wave nucleon-nucleon scattering. The scattering amplitude for the process $N_{s_1,i_1}(\mathbf{p}_1)N_{s_2,i_2}(\mathbf{p}_2) \to N_{s_3,i_3}(\mathbf{p}_3)N_{s_4,i_4}(\mathbf{p}_4)$, with s_m a spin label and i_m an isospin one, that follows from the previous Lagrangian is

$$\begin{aligned} T_{NN}^c = & -C_S\left(\delta_{s_3s_1}\delta_{s_4s_2}\,\delta_{i_3i_1}\delta_{i_4i_2} - \delta_{s_3s_2}\delta_{s_4s_1}\,\delta_{i_3i_2}\delta_{i_4i_1}\right) \\ & - C_T\left(\vec{\sigma}_{s_3s_1}\cdot\vec{\sigma}_{s_4s_2}\,\delta_{i_3i_1}\delta_{i_4i_2} - \vec{\sigma}_{s_3s_2}\cdot\vec{\sigma}_{s_4s_1}\,\delta_{i_3i_2}\delta_{i_4i_1}\right) \;. \end{aligned} \tag{8.2}$$

Because of the selection rule $S+\ell+I=$odd (with S the total spin of the system and ℓ its orbital angular momentum) that holds for any possible nucleon-nucleon partial wave due to the Fermi statistics, the only partial waves from eq.(8.2) are

$$\begin{aligned} N^c(^1S_0) &= -2(C_S - 3C_T) \;, \\ N^c(^3S_1) &= -2(C_S + C_T) \;. \end{aligned} \tag{8.3}$$

In addition one also has the one-pion exchange amplitudes depicted in fig.9. Which are given by the expression,

$$T_{NN}^{1\pi} = \frac{g_A^2}{4f^2}\left[\frac{(\vec{\tau}_{i_3i_1}\cdot\vec{\tau}_{i_4i_2})(\vec{\sigma}\cdot\mathbf{q})_{s_3s_1}(\vec{\sigma}\cdot\mathbf{q})_{s_4s_2}}{\mathbf{q}^2+m_\pi^2-i\epsilon} - \frac{(\vec{\tau}_{i_4i_1}\cdot\vec{\tau}_{i_3i_2})(\vec{\sigma}\cdot\mathbf{q}')_{s_4s_1}(\vec{\sigma}\cdot\mathbf{q}')_{s_3s_2}}{\mathbf{q}'^2+m_\pi^2-i\epsilon}\right] \;,$$

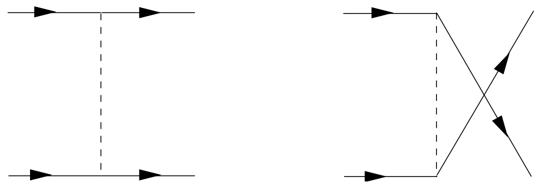

Figure 9: One-pion exchange diagrams for the nucleon-nucleon scattering amplitude. The digram on the left corresponds to the direct contribution and the one on the right to the exchange one.

with $\mathbf{q} = \mathbf{p}_3 - \mathbf{p}_1$ and $\mathbf{q}' = \mathbf{p}_4 - \mathbf{p}_1$. For the singlet case ($S = 0$) and $I = 0\ ,1$ one has,

$$\begin{aligned} T^{1\pi}(S=0, I=0) &= \frac{3g_A^2}{4f^2}\left[\frac{\mathbf{q}^2}{\mathbf{q}^2 + m_\pi^2 - i\epsilon} - \frac{\mathbf{q}'^2}{\mathbf{q}'^2 + m_\pi^2 - i\epsilon}\right] ~, \\ T^{1\pi}(S=0, I=1) &= \frac{-g_A^2}{4f^2}\left[\frac{\mathbf{q}^2}{\mathbf{q}^2 + m_\pi^2 - i\epsilon} + \frac{\mathbf{q}'^2}{\mathbf{q}'^2 + m_\pi^2 - i\epsilon}\right] ~. \end{aligned} \tag{8.4}$$

For the triplet case ($S = 1$) a 3×3 matrix results with labels given by the third component of the total spin, σ_f, σ_i, with the subscripts f (rows) and i (columns) referring to the final and initial third components, respectively:

$$||B_{s_3' s_3}|| = \left(\begin{array}{c|lll} & -1 & 0 & +1 \\ \hline -1 & q_3^2 & -\sqrt{2}(q_1 + iq_2)q_3 & (q_1 + iq_2)^2 \\ 0 & -\sqrt{2}(q_1 - iq_2)q_3 & q_1^2 + q_2^2 - q_3^2 & \sqrt{2}(q_1 + iq_2)q_3 \\ +1 & (q_1 - iq_2)^2 & \sqrt{2}(q_1 - iq_2)q_3 & q_3^2 \end{array}\right) \tag{8.5}$$

The Cartesian coordinates of $\mathbf{q}$ are indicated as subscripts. For $I = 0, 1$ one has

$$\begin{aligned} T^{1\pi}_{s_3' s_3}(S=1, I=0) &= \frac{-3g_A^2}{4f^2}\frac{B_{s_3' s_3}}{\mathbf{q}^2 + m_\pi^2 - i\epsilon} + (\mathbf{q} \leftrightarrow \mathbf{q}')~. \\ T^{1\pi}_{s_3' s_3}(S=1, I=1) &= \frac{g_A^2}{4f^2}\frac{B_{s_3' s_3}}{\mathbf{q}^2 + m_\pi^2 - i\epsilon} - (\mathbf{q} \leftrightarrow \mathbf{q}')~. \end{aligned} \tag{8.6}$$

Considering the eqs.(8.4) and (8.6), one can calculate the corresponding nucleon-nucleon partial wave due to one-pion exchange using eq.(B.31). Since these amplitudes are already calculated in terms of nucleon-nucleon states with definite spin and isospin, the latter equation simplifies to

$$N^{1\pi}_{JI}(\bar{\ell}, \ell, S) = \frac{Y_\ell^0(\hat{\mathbf{z}})}{2J+1}\sum_{\sigma_i, \sigma_f = -S}^{S} (0\sigma_i\sigma_i|\ell S J)(\bar{m}\sigma_f\sigma_i|\bar{\ell} S J)\int d\hat{p}\, T^{1\pi}_{\sigma_f\sigma_i}(S, I) Y_{\bar{\ell}}^{\bar{m}}(\hat{p})^*~. \tag{8.7}$$

In practical calculations we shall keep all the partial waves up to and including $\ell = 3$. Explicit expression for the resulting one-pion exchange nucleon-nucleon partial waves $N^{1\pi}_{JI}(\bar{\ell}, \ell, S)$ are given in Appendix D.

The sum of the local contributions, eq.(8.3), and the one-pion exchange partial wave amplitudes, eq.(8.7), is represented diagrammatically in the following by the exchange of a wiggly line as in fig.10.

It is well known [4, 5] that due to the large nucleon mass one has to resum the two-nucleon reducible diagrams, as it is schematically depicted in fig.11. For these diagrams the nucleon propagators follow the non-standard counting and each of them is $\mathcal{O}(p^{-2})$. The two nucleon propagators altogether are $\mathcal{O}(p^{-4})$

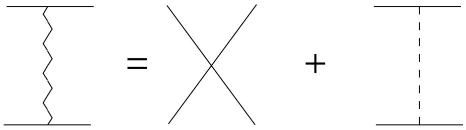

Figure 10: The exchange of a wiggly line between two nucleons indicate in the following the sum of the local and one-pion exchange contributions.

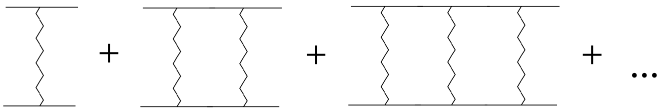

Figure 11: Resummation of the two-nucleon reducible diagrams. This is referred in the text also as a resummation of the right hand cut or unitarity cut.

that multiplied by the $\mathcal{O}(p^4)$ contribution from the measure of the loop integral produces an $\mathcal{O}(p^0)$ contribution which does not rise the chiral order and the series of diagrams in fig.11 must be resummed. The resummation of the two-nucleon reducible diagrams makes the resulting amplitude to fulfill unitarity. For this resummation we follow the techniques of the so called Unitary Chiral Perturbation Theory (UCHPT) [41, 42, 43]. This allows to resum the right hand cut or unitarity cut partial wave by partial wave (this cut is the one generated by the infinite string of diagrams in fig.11). This is different to solving a Lippmann-Schwinger equation, as performed in many recent nucleon-nucleon scattering analyses using CHPT [6, 7, 8, 9] following refs.[4, 5]. In this case, the scattering amplitude is calculated and afterwards the different partial waves are obtained. UCHPT has been applied with great success in meson-meson [41, 44, 45], meson-baryon scattering [46, 42, 47, 48, 49] and we now use it for the nucleon-nucleon case.

The master equation for UCHPT is the same independently of whether we have fermions, mesons or both in the scattering process and can be written as [42]

$$T_{JI}(\bar{\ell},\ell,S) = \left[I + N_{JI}(\bar{\ell},\ell,S)\cdot g\right]^{-1}\cdot N_{JI}(\bar{\ell},\ell,S)~. \tag{8.8}$$

This equation, derived in detail in refs.[41, 42, 50], results by performing a once subtracted dispersion relation of the inverse of a partial wave amplitude. The latter fulfills, because of unitarity,

$$\text{Im}T_{JI}(\bar{\ell},\ell,S)^{-1} = -\frac{m|\mathbf{q}|}{4\pi}\delta_{\ell\bar{\ell}}~, \tag{8.9}$$

in the CM frame and above the elastic threshold. A dispersion relation along the physical energy axis from threshold up to infinity is written. One subtraction is needed because $|\mathbf{q}| = \sqrt{2m\omega}$, with ω the energy of one nucleon in the CM. As a result of this dispersion relation one ends with the integral,

$$g(A) = g(B) - \frac{m(A-B)}{4\pi^2}\int_0^\infty dk^2 \frac{k}{(k^2-A-i\epsilon)(k^2-B-i\epsilon)}~, \tag{8.10}$$

with $B<0$ so that $g(B)$ must be real because there is imaginary part only above threshold. This integral can be done straightforwardly with the result,

$$g(A) = g(B) - \frac{im}{4\pi}\left(\sqrt{A} - i\sqrt{|B|}\right) \equiv g_0 - i\frac{m\sqrt{A}}{4\pi}~. \tag{8.11}$$

Note that g_0 is the value of $g(A)$ at threshold, $A=0$. This function corresponds to the divergent integral

$$g(A) = -m\int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2-A-i\epsilon} ~. \tag{8.12}$$

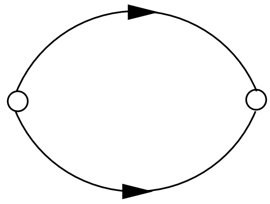

Figure 12: Unitarity loop corresponding to the function g in eq.(8.12).

The previous integral, depicted in fig.12, is linearly divergent although it shares the same analytical properties as eq.(8.11). In dimensional regularization with $D\to 3$ one has,

$$g(A) = -m\lim_{D\to 3}\int \frac{d^Dk}{(2\pi)^D}\frac{1}{k^2-A-i\epsilon} = -i\frac{m\sqrt{A}}{4\pi} ~. \tag{8.13}$$

This result is purely imaginary above threshold, $A>0$, and it corresponds to the imaginary part of eq.(8.11). However, this is just a specific characteristic of the regularization method employed since, as it is explicitly shown in the dispersion relation, eq.(8.10), there is always the freedom to choice any value of the real subtraction constant $g(B)$. On more physical grounds, we can calculate the function $g(A)$ of eq.(8.12) in dimensional regularization but also preserving the purely linear divergence contribution,

$$g(A) = -m\int^\Lambda \frac{d^3k}{(2\pi)^3}\frac{1}{k^2} - m\int \frac{d^Dk}{(2\pi)^D}\frac{1}{k^2-A-i\epsilon} = -\frac{m\Lambda}{2\pi^2} - i\frac{m\sqrt{A}}{4\pi} ~, \tag{8.14}$$

with Λ a cut-off in the modulus of the three-momentum appearing only in the first integral. Comparing with eq.(8.11) it follows that

$$g_0 = -\frac{m\Lambda}{2\pi^2} ~. \tag{8.15}$$

This result is the same as the one obtained by calculating $g(A)$ in terms of a three-momentum cut-off, eq.(E.3). In the following we take this expression in terms of Λ for g_0 and fix the former by comparing with the nucleon-nucleon scattering data.

Next, we consider how to fix $N_{JI}(\bar{\ell},\ell,S)$ in eq.(8.8). The effects of the large nucleon mass associated with the two-nucleon reducible diagrams corresponding to the unitarity loop in fig.12 are taken into account by eq.(8.8), since the latter results by integrating over the two-nucleon intermediate states at the level of the inverse of a partial wave, eq.(8.10). The imaginary part in eq.(8.9) gives rise to the right hand or unitarity cut and this is resummed by the dispersion integral eq.(8.8). In a plain perturbative chiral calculation of a nucleon-nucleon partial wave the previous effects are not resummed. However, the perturbative result can be matched with the expansion in powers of g of eq.(8.8) up to the same number of two-nucleon reducible loops. The aforementioned expansion corresponds to the geometric series

$$\begin{aligned} T_{JI}(\bar{\ell},\ell,S) &= N_{JI}(\bar{\ell},\ell,S) - N_{JI}(\bar{\ell},\ell,S)\cdot g\cdot N_{JI}(\bar{\ell},\ell,S) \\ &\quad + N_{JI}(\bar{\ell},\ell,S)\cdot g\cdot N_{JI}(\bar{\ell},\ell,S)\cdot g\cdot N_{JI}(\bar{\ell},\ell,S) + \ldots \end{aligned} \tag{8.16}$$

Together with this expansion one also has the standard chiral one. In this way, for the calculation of $N_{JI}^{(n)}$, one has to match the $\mathcal{O}(p^n)$ CHPT calculation of a nucleon-nucleon partial wave with at most n two-nucleon reducible loops with eq.(8.16), where N_{JI} is also expanded up to the considered order

$$N_{JI} = \sum_{m=0}^{n} N_{JI}^{(m)} \ . \tag{8.17}$$

Here, the chiral order is indicated by the superscript. Thus, at lowest order $N_{JI}^{(0)}(\bar{\ell},\ell,S)$ is given by the calculation in CHPT at $\mathcal{O}(p^0)$ without any two-nucleon reducible loop (the first diagram of the right hand side of fig.11). At $\mathcal{O}(p)$ the new contribution is the two-nucleon reducible part of the second diagram in the same figure, that for a given partial wave is denoted by $L_{JI}^{(1)}(\bar{\ell},\ell,S)$. It corresponds to the reducible part of the first iteration of the one-pion exchange plus local vertices. Writing $N_{JI} = N_{JI}^{(0)} + N_{JI}^{(1)} + \mathcal{O}(p^2)$, and matching eq.(8.16) up to order g with the sum of the first two diagrams on the right hand side of fig.11 one has

$$N_{JI}^{(0)} + N_{JI}^{(1)} - N_{JI}^{(0)} \cdot g \cdot N_{JI}^{(0)} + \mathcal{O}(p^2) = N_{JI}^{(0)} + L_{JI}^{(1)} + \mathcal{O}(p^2) \ , \tag{8.18}$$

with the result

$$N_{JI}^{(1)} = L_{JI}^{(1)} + N_{JI}^{(0)} \cdot g \cdot N_{JI}^{(0)} \ . \tag{8.19}$$

Notice that in the expansion of eq.(8.16) each factor of the kernel $N_{JI}(\bar{\ell},\ell,S)$ multiplies the loop function g with its value on-shell. This is why in eq.(8.18) we have $-N_{JI}^{(0)} \cdot g \cdot N_{JI}^{(0)}$ for one iteration of g. This result is then subtracted to the function $L_{JI}^{(1)}$ in eq.(8.19). In this way, it is clear that the previous expansion in the number n of two-nucleon reducible loops for fixing $N_{JI}^{(n)}$ implies that UCHPT takes as $\mathcal{O}(p)$ the difference between a full calculation of a two-nucleon reducible loop and the result obtained by factorizing on shell the vertices, eq.(8.16). The difference is incorporated in the interaction kernel $N_{JI}(\bar{\ell},\ell,S)$, which is then improved order by order.

At $\mathcal{O}(p^2)$ new contributions arise which require the calculation of the irreducible part of the box in fig.11 and the reducible part of the second iteration of the wiggly line, last diagram of fig.11. In addition there are also chiral counterterms from the quartic nucleon Lagrangian and two-nucleon irreducible pion loops [18, 9, 8]. If we denote all these new contributions by $L_{JI}^{(2)}(\bar{\ell},\ell,S)$, projected in the corresponding partial wave, one ends with

$$N_{JI}^{(2)} = L_{JI}^{(2)} + N_{JI}^{(1)} \cdot g \cdot N_{JI}^{(0)} + N_{JI}^{(0)} \cdot g \cdot N_{JI}^{(1)} - N_{JI}^{(0)} \cdot g \cdot N_{JI}^{(0)} \cdot g \cdot N_{JI}^{(0)} \ . \tag{8.20}$$

That is, we are just subtracting to $L_{JI}^{(2)}$ the two-nucleon reducible contributions obtained from eq.(8.16) up to $\mathcal{O}(p^2)$, in the UCHPT expansion of the interaction kernel $N_{JI}(\bar{\ell},\ell,S)$. In the previous equations we have been using the notation $N_{JI} \cdot g$ as if g were a matrix for the case of coupled channels. However, since g is given by the same expression for all the partial waves it just corresponds to the identity matrix times eq.(8.11).

The resulting N_{JI}, eq.(8.17), is then substituted in eq.(8.8). The latter gives rise to contributions of all chiral orders to the full partial wave amplitude $T_{JI}(\bar{\ell},\ell,S)$, which is then calculated non-perturbatively. A comparison with experimental data of a perturbative calculation of the latter, particularly for the partial waves $\ell \leq 2$ [18], would be not realistic because the already discussed necessity to resum the two-nucleon reducible diagrams, fig.11.

Regarding the issue of the large size of the S-wave nucleon-nucleon scattering lengths [12], one can match formally eq.(8.8) with a perturbative chiral calculation of $T_{JI}(\bar{\ell},\ell,S)$ because the latter enter parametrically in the calculation. This procedure gives rise to values of the chiral counterterms C_S and C_T, consistent with their ascribed $\mathcal{O}(p^0)$ scaling, see eq.(8.24) below. Finally, we also want to stress that eq.(8.8) is an algebraic one, which simplifies tremendously the numerical burden for in-medium calculations.

We now concentrate on fixing the constants C_S and C_T from the local quartic nucleon Lagrangian, eq.(8.1). These constants and g_0 are the only free parameters that enter in the evaluation of the nucleon-nucleon scattering amplitudes from eq.(8.8) up to $\mathcal{O}(p)$. We first discuss the LO result and then the NLO one, fixing C_S and C_T by considering the S-wave nucleon-nucleon scattering lengths a_t and a_s for the triplet and singlet S-waves, respectively. At $\mathcal{O}(p^0)$ we have

$$T_{JI}(\bar{\ell},\ell,S)\big|_{LO} = \left[I + N_{JI}^{(0)}\cdot g\right]^{-1}\cdot N_{JI}^{(0)}. \tag{8.21}$$

Note that the one-pion exchange, eq.(8.4), vanishes at the nucleon-nucleon threshold because it depends quadratically on the nucleon three-momentum. For the amplitudes[#4] ${}^1S_0$ and ${}^3S_1$ at threshold one has from eq.(8.21),

$$\begin{aligned} T({}^1S_0) &= \frac{-(C_S-3C_T)}{1-g_0(C_S-3C_T)}~, \\ T({}^3S_1) &= \frac{-(C_S+C_T)}{1-g_0(C_S+C_T)}~. \end{aligned} \tag{8.22}$$

The factor $-(C_S-3C_T)$ for $N_{01}(0,0,0)$ is a factor 2 smaller than $N({}^1S_0)$ in eq.(8.3), and similarly also for the triplet case, because N_{JI} is given by the direct term. The resulting expressions for the scattering lengths from eq.(8.22) are

$$\begin{aligned} \frac{1}{a_s} &= \frac{2\Lambda}{\pi}+\frac{4\pi/m}{C_S-3C_T}~, \\ \frac{1}{a_t} &= \frac{2\Lambda}{\pi}+\frac{4\pi/m}{C_S+C_T}~. \end{aligned} \tag{8.23}$$

So that

$$\begin{aligned} C_S &= \frac{\pi}{m}\frac{3/a_s+1/a_t-8\Lambda/\pi}{(1/a_s-2\Lambda/\pi)(1/a_t-2\Lambda/\pi)}~, \\ C_T &= \frac{\pi}{m}\frac{1/a_s-1/a_t}{(1/a_s-2\Lambda/\pi)(1/a_t-2\Lambda/\pi)}~. \end{aligned} \tag{8.24}$$

One of the characteristics of nucleon-nucleon scattering are the large absolute values of the S-wave scattering lengths $a_s = (-23.758 \pm 0.04)$ fm and $a_t = (5.424 \pm 0.004)$ fm. For typical values of Λ, $\Lambda \gg |1/a_s|$, $1/a_t$, and then $|C_S| \simeq 2\pi^2/m\Lambda \gg |C_T| = \mathcal{O}(\pi^3/a_t/m\Lambda^2)$. Because of the introduction of the subtractions constant g_0, the low energy constants C_S and C_T do not diverge for a_s, $a_t \to \infty$. In this way, Λ is a new scale that adds to the inverse of the scattering lengths so that their sum, the one that appears for determining the values of C_S and C_T, eq.(8.24), has a natural size. Indeed, taking into account that: i) Eq.(8.24) is the same as in the pionless nucleon-nucleon EFT, because the pion-exchange contribution

[#4] For general considerations the already introduced notation $T_{JI}(\bar{\ell},\ell,S)$ is employed. Specific partial waves are denoted by the standard spectroscopic notation $T({}^{2S+1}\ell_J)$.

to $N_{JI}^{(0)}$ vanishes at threshold. ii) Naive dimensional analysis implies that C_S, $C_T \sim 4\pi/mQ$, with Q the expansion scale for the pionless EFT because of i). It follows then that Λ should be comparable to Q so that $Q, \Lambda = \mathcal{O}(m_\pi)$. Note also that the order of C_S in eq.(8.24) is fixed by the product $m\Lambda$, which is $\mathcal{O}(p^0)$ for $\Lambda = \mathcal{O}(p)$ because of the largeness of the nucleon mass. In the same way that before g was counted as $\mathcal{O}(p^0)$ with its imaginary part linear in mp, see eq.(8.11). One has to stress that only local terms and one-pion exchange contributions enter in the calculation of $N_{JI}^{(0)}(\bar{\ell},\ell,S)$. This is certainly too simplistic in order to properly describe the nucleon-nucleon interactions as a function of energy.

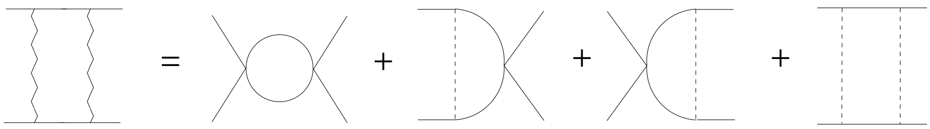

Figure 13: Box diagram, $L_{JI}^{(1)}$, from the first iteration of a wiggly line. It consists of the diagrams shown on the right hand side of the figure with two, one or no one local or one-pion exchange amplitudes.

Let us now consider eq.(8.8) with N_{JI} evaluated up to $\mathcal{O}(p)$. Thus,

$$T_{JI}(\bar{\ell},\ell,S)\big|_{NLO} = \left[I + (N_{JI}^{(0)} + N_{JI}^{(1)}) \cdot g\right]^{-1} \cdot (N_{JI}^{(0)} + N_{JI}^{(1)}) \ . \tag{8.25}$$

with $N_{JI}^{(1)}$ given in eq.(8.19). For the evaluation at this order of C_S and C_T we consider eq.(8.25) at threshold for the $^1S_0$ and $^3S_1$ partial waves, as in the LO case. The last partial wave is elastic at this energy, without mixing with the $^3D_1$ partial wave because of the vanishing of the three-momentum. Then, if we denote by a the scattering length of any of the S-waves, we have from eq.(8.8)

$$a = -\frac{1}{k}\frac{\mathrm{Im}T_{JI}}{\mathrm{Re}T_{JI}} = -\frac{m}{4\pi}\frac{N_{JI}}{1+g_0 N_{JI}} \ . \tag{8.26}$$

Particularizing eqs.(8.18) and (8.19) at threshold we rewrite $N_{JI}^{(0)}$ as $-C$, with the latter given by C_S+C_T for the triplet case and C_S-3C_T for the singlet one. In addition, we express $L_{JI}^{(1)} \equiv -C^2 g_0 + C\ell_1 + \ell_2$. This rewriting is based on the fact that the box diagram, $L_{JI}^{(1)}$, as shown in fig.13, consists of four contributions with two, one and no one local vertices. The first contribution is given by $-C^2 g_0$, the second by $C\ell_1$ and the last one by ℓ_2, respectively. The coefficients ℓ_1 and ℓ_2 are given in terms of g_0 and the known parameters m, g_A and m_π. ℓ_1 is the same for the partial waves $^3S_1$ and $^3D_1$ while ℓ_2 is different. In fig.14 we show the values of ℓ_1 and ℓ_2 as a function of Λ in units of m_π. Substituting these expressions in eq.(8.26) one obtains

$$C = \frac{C^{(0)} + \ell_2}{1-\ell_1} \ , \tag{8.27}$$

with

$$C^{(0)} = \frac{1}{\frac{m}{4\pi a} + g_0} \ , \tag{8.28}$$

the $\mathcal{O}(p^0)$ result, compare with eq.(8.23).

In figs. 15, 16 and 17 we show the LO and NLO results for the nucleon-nucleon scattering data by employing eq. (8.8) up to $|\mathbf{p}_{cm}| = 300$ MeV, with $\mathbf{p}_{cm}$ the CM three-momentum. The values of Λ employed

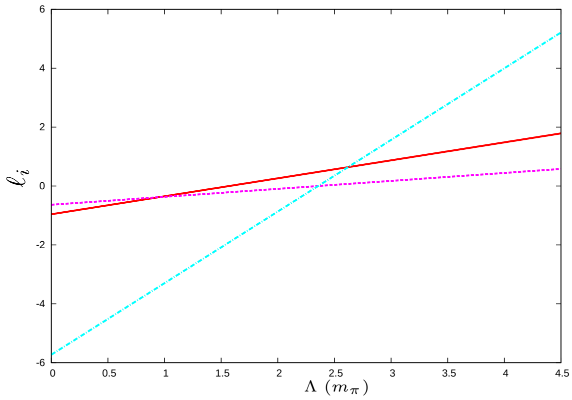

Figure 14: Values for ℓ_1 (solid line), $\ell_2(^1S_0)$ (dashed line) and $\ell_2(^3S_1)$ (dashed dotted line) as a function of Λ. ℓ_1 is expressed in units of m_π^{-2}.

are 90 MeV for the LO results (dashed lines) and of 90 MeV (dot-dashed lines) and 50 MeV (dotted lines) at NLO. For the latter value the mixing parameter ϵ_1 is better reproduced, while the other observables are very similar for both Λ values employed at NLO. These values for Λ are consistent with being $\mathcal{O}(p)$. For $|\mathbf{p}| \simeq 360$ MeV the pion production threshold opens and it does not make sense to compare with data above this point with the simple inputs employed for N_{JI}. In the same way, since the Fermi momentum for nuclear matter saturation density is around $2m_\pi$, close to the upper limit of $|\mathbf{p}|$ shown in the figures, an accurate description of nuclear matter requires a better description of the nucleon-nucleon S-waves. At least an $\mathcal{O}(p^2)$ calculation, which includes important new physical mechanisms, as non-reducible two-pion exchanges between others as indicated above before eq.(8.20), is presumably needed. Once this is done, a discussion on the issue of the convergence of the UCHPT expansion of T_{JI} for nucleon-nucleon scattering will be in order.

8.2 Nucleon-nucleon scattering in the nuclear medium

When calculating a loop function in the nuclear medium we typically use the notation L_{ij}, where i indicates the number of two-nucleon states in the diagram (0 or 1) and j the number of pion exchanges (0, 1 or 2). In addition, we also use $L_{ij,f}$, $L_{ij,m}$ and $L_{ij,d}$, with the subscripts m and d indicating one or two Fermi sea insertions from the nucleon propagators in the medium, in that order. The subscript f refers to the "free" part and it does not involve any Fermi sea insertion. The subscripts f, m and d originate because the nucleon propagator in the nuclear medium contains both a free and an in-medium part, the last proportional to the Dirac delta function in eq.(3.2). In this way, the function $g = L_{10,f}$ and its in-medium counterpart is L_{10}. The former function is calculated in the Appendix E.

We use the same eq.(8.8) but now the function g is substituted by L_{10}. The same process as previously discussed is then used to fix N_{JI} in the medium. At lowest order they can be easily obtained from our previous result in the vacuum since the only modification without increasing the chiral order is by using the corresponding nucleon propagator in the medium, which is directly accomplished by replacing $g(A)$ by L_{10}. Note that any in-medium contribution requires $V_\rho = 1$ at least, which then increases the order at least by one more unit, eq.(2.12). This new in-medium generalized vertex must be associated with the nucleon-nucleon scattering diagrams of LO. The modification of the meson propagators (both heavy or pionic ones) by the inclusion of an in-medium generalized vertex increases by two units the chiral order. However, the modification of the enhanced nucleon propagators with one in-medium generalized

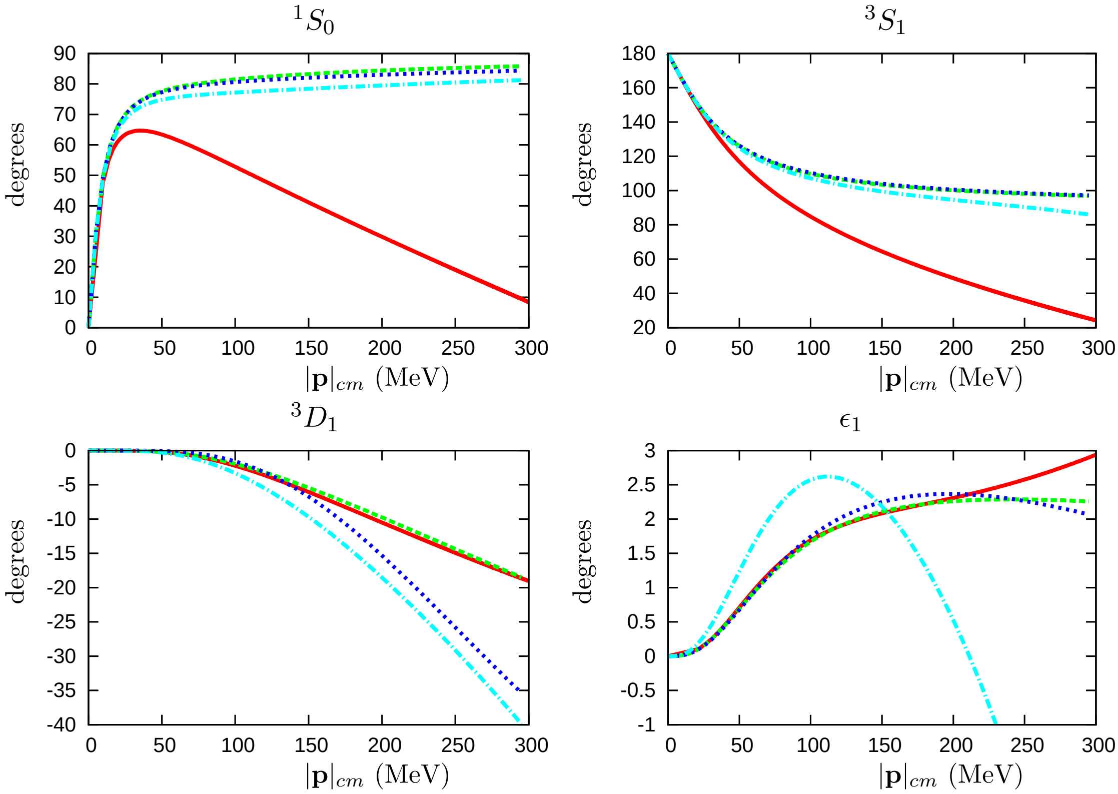

Figure 15: $^1S_0$, $^3S_1$, $^3D_1$ phase shifts and the mixing angle ϵ_1 as a function of $|\mathbf{p}|_{cm}$. The solid lines correspond to the Nijmegen data [51, 52]. The LO results are given by the dashed lines. The NLO ones with $\Lambda = 90$ MeV are the dot-dashed lines and the dotted lines are the NLO results with $\Lambda = 50$ MeV.

vertex only increases the order by one unit and these contributions must be kept at NLO. This is the same reason why Σ_7 and Σ_8 above were considered for the calculation of the pion-self energy at NLO. It is beyond the present research to offer a complete study of the in-medium pion self-energy at N^{2}LO where the full in-medium NLO nucleon-nucleon interactions are needed. What we do here, for illustrative purposes only, is to exchange the free nucleon propagators by the in-medium ones in the calculation of the box diagram $L_{JI}^{(1)}$ as well as in g that enter in fixing $N_{JI}^{(1)}$, eq.(8.19).

Then, eq.(8.8) becomes

$$T_{JI}^{i_3}(\bar{\ell},\ell,S) = \left[I + N_{JI}^{i_3}(\bar{\ell},\ell,S)\cdot L_{10}^{i_3}\right]^{-1}\cdot N_{JI}^{i_3}(\bar{\ell},\ell,S)~. \tag{8.29}$$

We have included the superscript i_3, which corresponds to the third component of the total isospin of the two nucleons involved in the scattering process, both in the partial wave $T_{JI}(\bar{\ell},\ell,S)$ and in L_{10}. This is due to the fact that in the nuclear medium the Fermi momentum of the nucleon and proton Fermi seas are different for asymmetric nuclear matter. In this way, $L_{10,m}$ and $L_{10,d}$ depend on weather one has two protons, neutrons or a proton and a neutron as intermediate states. As a result, a nucleon-nucleon partial wave in the nuclear medium depends on the total charge of the intermediate state. Of course, this is not the case for the nucleon-nucleon interactions in vacuum where they only depend on the total isospin, but not on its third component. Let us also stress that the total isospin of the nucleon-nucleon state is a good quantum number and does not mix because of the nucleon-nucleon interactions. The function L_{10} conserves I, because it is symmetric under the exchange of the two nucleons, though it

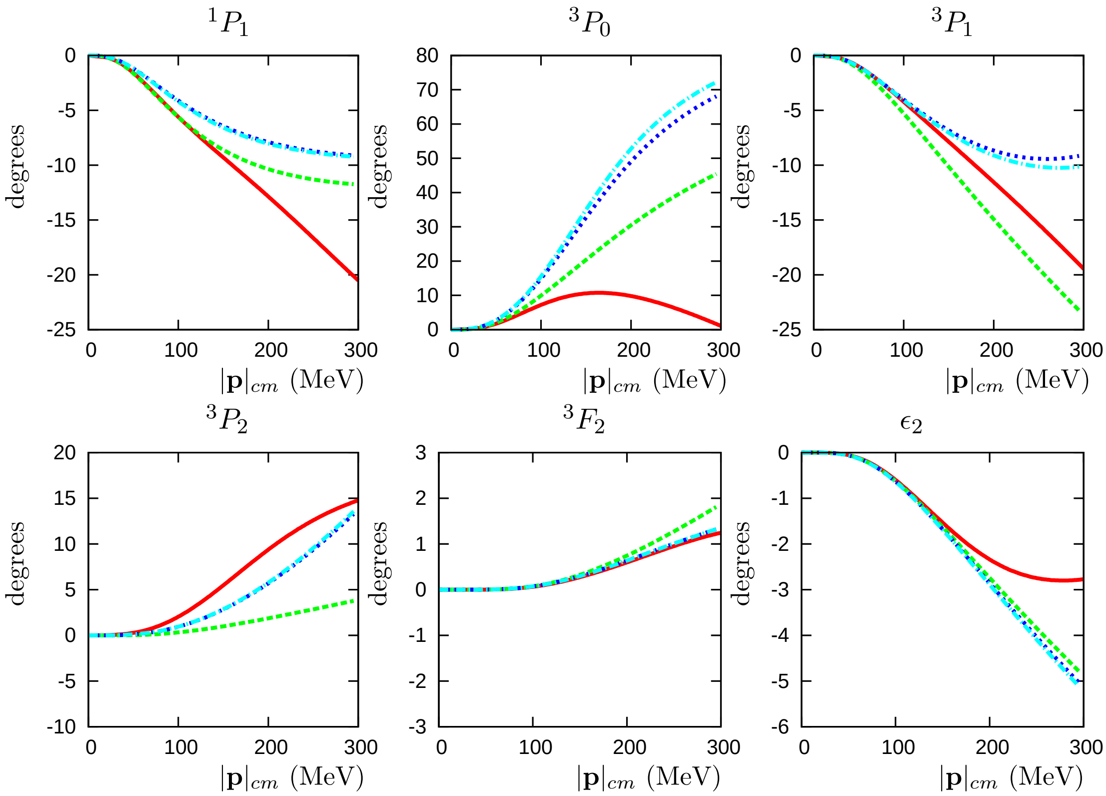

Figure 16: $^1P_1$, $^3P_0$, $^3P_1$, $^3P_2$, $^3F_2$ phase shifts and the mixing angle ϵ_2 as a function of $|\mathbf{p}|_{cm}$. For notation, see fig. 15.

depends on the charge (or third component of the total isospin) of the intermediate state. This is a general rule, all the $i_3 = 0$ operators are symmetric under the exchange $p \leftrightarrow n$, so that they do not mix isospin representations with different exchange symmetry properties.

9 In-medium pion self-energy diagram Σ_7

We now consider the evaluation of the diagrams shown in fig.18. The one on the left corresponds to the direct nucleon-nucleon interactions while that on the right corresponds to the exchange part. We call the sum of both contributions Σ_7. It is given by similar expressions to those used for Σ_5 and Σ_6, eqs.(7.8) and (7.12), respectively.

$$\Sigma_7 = \frac{q^0}{2f^2}\varepsilon_{ijk}\int \frac{d^4k_1}{(2\pi)^4}\,\mathrm{Tr}\left\{\tau^k\left(\frac{1+\tau_3}{2}\theta_p^- + \frac{1-\tau_3}{2}\theta_n^-\right)\Sigma_{NN}\left(\frac{1+\tau_3}{2}\theta_p^- + \frac{1-\tau_3}{2}\theta_n^-\right)\right\} \times \frac{1}{(k_1^0 - E(\mathbf{k}) - i\epsilon)^2} \;. \tag{9.1}$$

On the other hand,

$$\Sigma_{NN} = \frac{1+\tau_3}{2}\Sigma_{p,NN} + \frac{1-\tau_3}{2}\Sigma_{n,NN} \;. \tag{9.2}$$

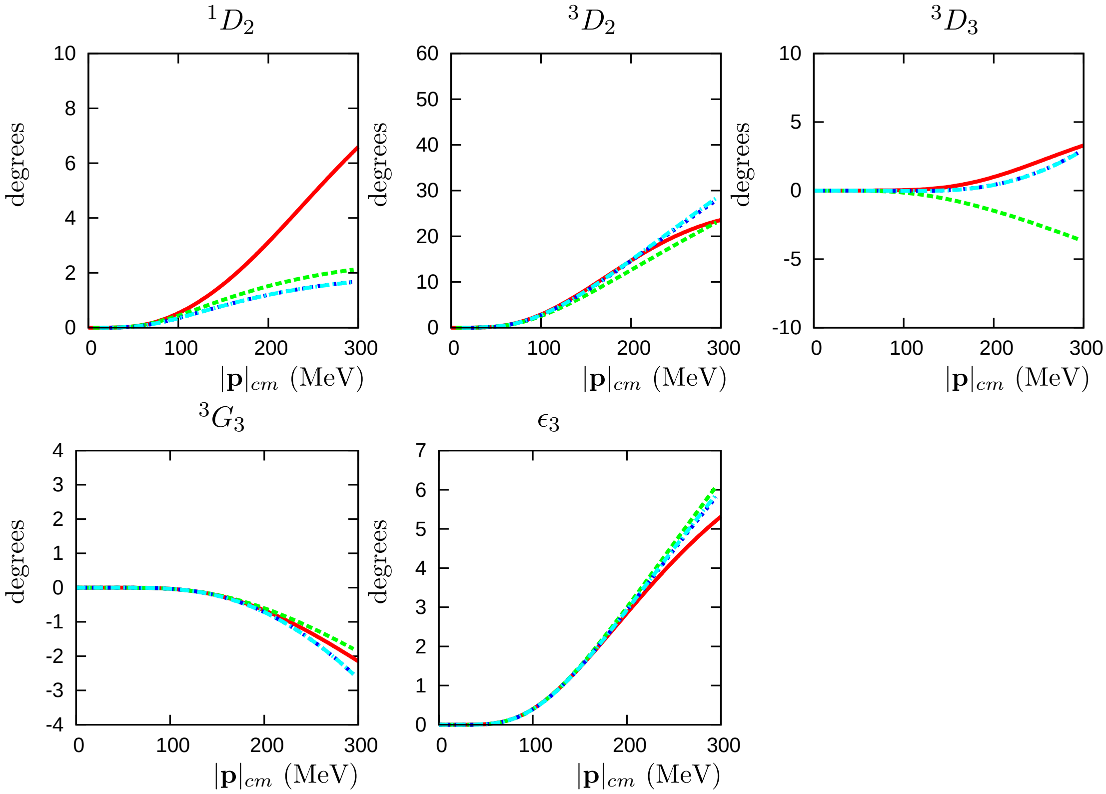

Figure 17: $^1D_2$, $^3D_2$, $^3D_3$, $^3G_3$ phase shifts and the mixing angle ϵ_3 as a function of $|\mathbf{p}|_{cm}$. For notation, see fig. 15.

Performing the k^0 integration as usual one then has,

$$\Sigma_7 = \frac{iq^0}{2f^2}\varepsilon_{ij3}\sum_{\sigma_1}\int\frac{d^3k_1}{(2\pi)^3}\left(\frac{\partial\Sigma_{p,NN}}{\partial k_1^0}\theta_p^- - \frac{\partial\Sigma_{n,NN}}{\partial k_1^0}\theta_n^-\right)_{k_1^0=E(\mathbf{k}_1)} . \tag{9.3}$$

Here σ_1 corresponds to the spin of the incident nucleon. The nucleon self-energy due to the nucleon-nucleon interactions corresponds to fig.19 and is given by the expression,

$$\Sigma_{\alpha_1,NN} = \sum_{\alpha_2,\sigma_2}\int\frac{d^3k_2}{(2\pi)^3}\theta(\xi_{\alpha_2}-|\mathbf{k}_2|)_A\langle\mathbf{k}_1\sigma_1\alpha_1,\mathbf{k}_2\sigma_2\alpha_2|T_{NN}|\mathbf{k}_1\sigma_1\alpha_1,\mathbf{k}_2\sigma_2\alpha_2\rangle_A . \tag{9.4}$$

In this expression T_{NN} is the nucleon-nucleon scattering amplitude between the indicated initial and final states. These are characterized by three labels. The first label corresponds to the three-momentum, the second to the spin and the third to the isospin. Note that in this equation there is a sum over all the quantum numbers of the second nucleon. The subscript A in the scattering amplitude indicates that the nucleon-nucleon amplitude contains both the direct and exchange contributions.

It is convenient to decompose the nucleon-nucleon interactions in a partial wave expansion as given in eq.(B.8). In the center of mass frame (CM) one has,

$$\begin{aligned}{}_A\langle\mathbf{p}',\sigma_1'\alpha_1'\sigma_2'\alpha_2'|T_{NN}|\mathbf{p},\sigma_1\alpha_1\sigma_2\alpha_2\rangle_A &= 4\pi\sum(\sigma_1'\sigma_2's_3'|s_1s_2S)(\sigma_1\sigma_2s_3|s_1s_2S)(m's_3'\mu|\ell'SJ)(ms_3\mu|\ell SJ)\\ &\times Y_{\ell'}^{m'}(\hat{\mathbf{p}}')Y_\ell^m(\hat{\mathbf{p}})^*(\alpha_1'\alpha_2'i_3|\tau_1\tau_2I')(\alpha_1\alpha_2i_3|\tau_1\tau_2I)\chi(S\ell'I)\chi(S\ell I)T_{JI}^{i_3}(\ell',\ell,S) .\end{aligned} \tag{9.5}$$

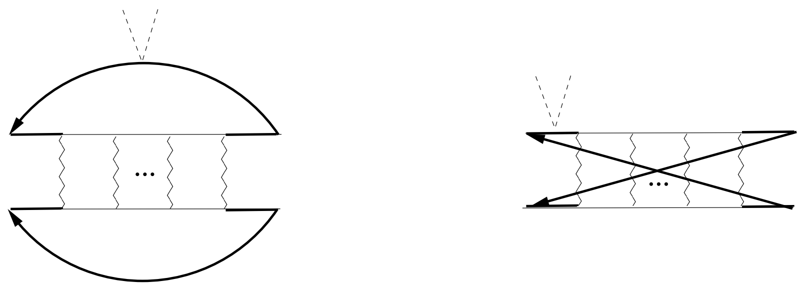

Figure 18: Contribution to the pion self-energy by dressing the nucleon propagator in fig.2 due to the in-medium nucleon-nucleon interactions. This is called Σ_7.

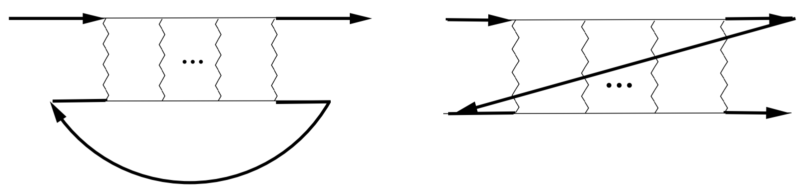

Figure 19: In-medium nucleon self-energy due to the nucleon-nucleon interactions with the Fermi seas.

In this expression the repeated indices must be summed. The Clebsch-Gordan coefficients for spin and isospin follow the notation $(c_1 c_2 c_3|C_1 C_2 C_3)$, where c_i is the third component and C_i is its highest value. In our case, $s_1 = s_2 = 1/2$ for spin and $\tau_1 = \tau_2 = 1/2$ for isospin. $Y_\ell^m(\hat{\mathbf{n}})$ is a standard spherical harmonics of angular momentum ℓ and third component m. In addition, ℓ' and ℓ are the final and initial orbital angular momenta, respectively, with m' and m their third components, in order. The coefficient

$$\chi(S\ell I) = \begin{cases} \sqrt{2} & \ell + S + I = \text{ odd} \\ 0 & \ell + S + I = \text{ even} \end{cases} \tag{9.6}$$

The nucleon-nucleon scattering partial wave amplitude is denoted by $T_{JI}^{i_3}(\ell', \ell, S)$, with $\vec{J} = \vec{\ell} + \vec{S}$ the total angular momentum and μ its third component. I is the total isospin and i_3 is its third component. Although the partial wave amplitude do not depend on i_3 in the vacuum they do in the case of asymmetric nuclear matter because the Fermi momenta are different for protons and nucleons. Let us recall that because of parity the total spin S is conserved. The partial wave decomposition of the nucleon-nucleon amplitudes is derived in detail in Appendix B.

While eq.(9.5) is given in the CM of the two nucleons involved in the scattering, eqs.(9.1) and (9.4) are given in the nuclear matter rest frame. This implies that one must take the boost from the former frame to the latter in order to use eq.(9.5). However, as we show in Appendix C, up to two chiral orders higher in the counting one can use the nucleon-nucleon scattering amplitude as a Lorentz invariant, similarly as for the meson-meson ones. Thus, we can directly use eq.(9.5) in eq.(9.4). Let us recall that our calculation is up to NLO $\mathcal{O}(p^5)$ and these relativistic corrections are of $\mathcal{O}(p^7)$.

From eqs.(9.3) and (9.4) one has to sum over the spins σ_1 and σ_2. The fact that both the initial and final nucleon-nucleon states are the same implies an important simplification in the equations, as we show now. First, if we place $\sigma_1 = \sigma_1'$ and $\sigma_2 = \sigma_2'$ and sum,

$$\sum_{\sigma_1, \sigma_2} (\sigma_1 \sigma_2 s_3'|s_1 s_2 S')(\sigma_1 \sigma_2 s_3|s_1 s_2 S) = \delta_{s_3' s_3} \delta_{S' S} \ . \tag{9.7}$$

Thus, independently of whether S is conserved in the nucleon-nucleon interactions, when summing over spins for the evaluation of the pion self-energy the equality of both the initial and final nucleon-nucleon total spins results. The sum over the third components of orbital angular momentum and s_3 in eq.(9.4) results,

$$\sum_{m',m,s_3}(m's_3\mu|\ell'SJ)(ms_3\mu|\ell SJ)Y_{\ell'}^{m'}(\hat{\mathbf{p}})Y_\ell^m(\hat{\mathbf{p}})^* = \frac{2J+1}{2\ell+1}\delta_{\ell'\ell}\sum_m |Y_\ell^m(\mathbf{p})|^2 \ . \tag{9.8}$$

Here we have made use of the symmetry properties of the Clebsch-Gordan coefficients [40],

$$(m's_3\mu|\ell'SJ) = (-1)^{s_3+S}\left(\frac{2J+1}{2\ell'+1}\right)^{1/2}(-s_3\mu m'|SJ\ell') \ , \tag{9.9}$$

and similarly for $(ms_3\mu|\ell SJ)$. Because of the addition theorem for the spherical harmonics one has,

$$\frac{1}{2\ell+1}\sum_m |Y_\ell^m(\hat{\mathbf{p}})|^2 = \frac{1}{4\pi} \ . \tag{9.10}$$

Then, eq.(9.8) can be written as

$$\sum_{m',m,s_3}(m's_3\mu|\ell'SJ)(ms_3\mu|\ell SJ)Y_{\ell'}^{m'}(\hat{\mathbf{p}})Y_\ell^m(\hat{\mathbf{p}})^* = \delta_{\ell'\ell}\frac{2J+1}{4\pi} \ . \tag{9.11}$$

The sum of partial waves that matters for eq.(9.1) can be expressed as,

$$\sum_{\sigma_1,\sigma_2} {}_A\langle \mathbf{p},\sigma_1\alpha_1\sigma_2\alpha_2|T_{NN}|\mathbf{p},\sigma_1\alpha_1\sigma_2\alpha_2\rangle_A = \sum_{I,J,\ell,S}(2J+1)T_{JI}^{i_3}(\ell,\ell,S)\chi(S\ell I)^2(\alpha_1\alpha_2 i_3|I_1I_2I)^2 \ . \tag{9.12}$$

Here we have taken one step more, since we have equalized also $I'=I$ of eq.(9.5). This can be done because after summing over the third components of total spin and orbital angular momentum the labels ℓ, S diagonalize. Hence, because of the rule $\ell+S+I=$odd, I must be the same as well for the initial and final nucleon-nucleon states. Let us remark that in an asymmetric nuclear medium $I=0$ and 1 could mix for $i_3=0$, though in our present problem this does not occur.#5

Inserting the result of eq.(9.12) in eq.(9.3) we are left with

$$\Sigma_7 = \frac{iq^0}{2f^2}\varepsilon_{ij3}\int\frac{d^3k_1}{(2\pi)^3}\int\frac{d^3k_2}{(2\pi)^3}\sum_{J,\ell,S}(2J+1)\chi(S\ell 1)^2\left\{\theta(\xi_p-|\mathbf{k}_1|)\theta(\xi_p-|\mathbf{k}_2|)\frac{\partial T_{J1}^{+1}(\ell,\ell,S)}{\partial k_1^0}\right.$$
$$\left. -\,\theta(\xi_n-|\mathbf{k}_1|)\theta(\xi_n-|\mathbf{k}_2|)\frac{\partial T_{J1}^{-1}(\ell,\ell,S)}{\partial k_1^0}\right\}_{k_1^0=E(\mathbf{k}_1)} \ . \tag{9.13}$$

This is a S-wave isoscalar self-energy contribution. The integration over two Fermi seas is discussed in Appendix F.

10 In-medium pion self-energy diagram Σ_8

We now consider the diagrams in fig.20. They are similar to those of fig.6, but now the nucleon self-energy is dressed due to the in-medium nucleon-nucleon interactions. An equation analogous to eq.(7.14) for Σ_6 can be used here but now with $\Sigma_{p,NN}$ and $\Sigma_{n,NN}$ instead of Σ_π (of course, Σ_8 has no a free term),

#5In all our calculations the $i_3=0$ operators are symmetric under the exchange $p\leftrightarrow n$, so that they do not mix isospin representations with different exchange symmetry properties.

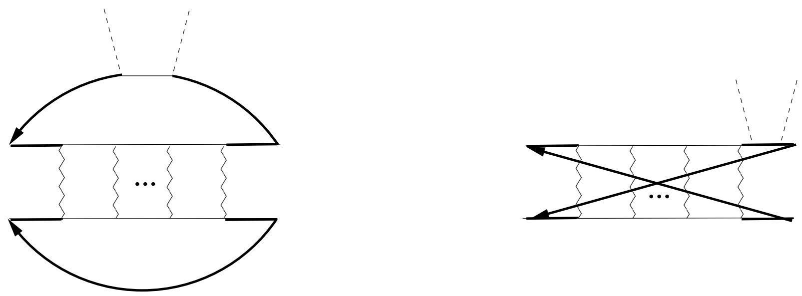

Figure 20: Contribution to the pion self-energy by dressing the nucleon propagators in fig.3 due to the in-medium nucleon-nucleon interactions. Every diagram actually represents two diagrams by the exchange of the initial and final pion lines. This is called Σ_8.

$$\begin{aligned}\Sigma_8 = & \frac{-ig_A^2}{2f^2}\frac{\mathbf{q}^2}{q^0}\varepsilon_{ij3}\sum_{\sigma_1}\int\frac{d^3k_1}{(2\pi)^3}\left(\frac{\partial\Sigma_{p,NN}}{\partial k_1^0}\theta_p^- - \frac{\partial\Sigma_{n,NN}}{\partial k_1^0}\theta_n^-\right)_{k_1^0=E(\mathbf{k}_1)} \\ & - \frac{g_A^2}{2f^2}\frac{\mathbf{q}^2}{q^{0\,2}}\delta_{ij}\sum_{\sigma_1}\int\frac{d^3k_1}{(2\pi)^3}\left(\Sigma_{p,NN}\theta_p^- + \Sigma^-_{n,NN}\theta_n^-\right) \; . \end{aligned} \tag{10.1}$$

The expression for $\Sigma_{\beta,NN}$ is given in eq.(9.4). When introduced in eq.(10.1) and performing the sums over spins and third components of orbital angular momentum, as already done for Σ_7 in section 9, see eq.(9.12), one is left with

$$\begin{aligned}\Sigma_8 = & \frac{-ig_A^2}{2f^2}\frac{\mathbf{q}^2}{q^0}\varepsilon_{ij3}\int\frac{d^3k_1}{(2\pi)^3}\int\frac{d^3k_2}{(2\pi)^3}\sum_{J,\ell,S}\chi(S\ell 1)^2(2J+1)\Bigg\{\theta(\xi_p-|\mathbf{k}_1|)\theta(\xi_p-|\mathbf{k}_2|)\frac{\partial T_{J1}^{+1}(\ell,\ell,S)}{\partial k_1^0} \\ & - \theta(\xi_n-|\mathbf{k}_1|)\theta(\xi_n-|\mathbf{k}_2|)\frac{\partial T_{J1}^{-1}(\ell,\ell,S)}{\partial k_1^0}\Bigg\}_{k_1^0=E(\mathbf{k}_1)} \\ & - \frac{g_A^2}{2f^2}\frac{\mathbf{q}^2}{q^{0\,2}}\delta_{ij}\int\frac{d^3k_1}{(2\pi)^3}\int\frac{d^3k_2}{(2\pi)^3}\sum_{J,\ell,S}(2J+1)\big(\theta(\xi_p-|\mathbf{k}_1|)\theta(\xi_p-|\mathbf{k}_2|)\chi(S\ell 1)^2T_{J1}^{+1}(\ell,\ell,S) \\ & + \theta(\xi_n-|\mathbf{k}_1|)\theta(\xi_n-|\mathbf{k}_2|)\chi(S\ell 1)^2T_{J1}^{-1}(\ell,\ell,S) + \theta(\xi_p-|\mathbf{k}_1|)\theta(\xi_n-|\mathbf{k}_2|) \\ & \times\left[\chi(S\ell 0)^2T_{J0}^{0}(\ell,\ell,S)+\chi(S\ell 1)^2T_{J1}^{0}(\ell,\ell,S)\right]\big) \; . \end{aligned} \tag{10.2}$$

Σ_8 is a P-wave self-energy contribution that comprises both a part of isovector character, the piece proportional to ε_{ij3} that we denote by Σ_8^{iv}, and another of isoscalar type, proportional to δ_{ij}, and denoted by Σ_8^{is}.

Eqs. (9.13) and (10.2) involve the knowledge of the derivative of the nucleon-nucleon partial wave amplitude with respect to the energy k_1^0. Instead of the variable k_1^0 we use the variable A, eq.(E.4), which is also the argument of L_{10} and use the relation

$$\frac{\partial}{\partial k_1^0} = \frac{\partial}{\partial k_2^0} = m\frac{\partial}{\partial A} \tag{10.3}$$

Let us rewrite eq. (8.8) as

$$T_{JI} = N_{JI} - N_{JI}\cdot L_{10}\cdot T_{JI} \; . \tag{10.4}$$

Taking the derivative to both sides of the previous equation

$$\frac{\partial T_{JI}}{\partial A} = \frac{\partial N_{JI}}{\partial A} - \frac{\partial N_{JI}}{\partial A} \cdot L_{10} \cdot T_{JI} - N_{JI} \cdot \frac{\partial L_{10}}{\partial A} \cdot T_{JI} - N_{JI} \cdot L_{10} \cdot \frac{\partial T_{JI}}{\partial A} \ . \tag{10.5}$$

It follows then

$$\frac{\partial T_{JI}}{\partial A} = D_{JI}^{-1} \cdot \frac{\partial N_{JI}}{\partial A} - D_{JI}^{-1} \cdot \frac{\partial N_{JI}}{\partial A} \cdot L_{10} \cdot D_{JI}^{-1} \cdot N_{JI} - D_{JI}^{-1} \cdot N_{JI} \cdot \frac{\partial L_{10}}{\partial A} \cdot D_{JI}^{-1} \cdot N_{JI} \ , \tag{10.6}$$

with

$$D_{JI}^{i_3}(\bar{\ell}, \ell, S) = I + N_{JI}^{i_3}(\bar{\ell}, \ell, S) \cdot L_{10}^{i_3} \ . \tag{10.7}$$

Here we have shown the explicit dependence of D_{JI} on all the discrete indices. Eq.(10.6) can be simplified by taking into account that D_{JI} and N_{JI} commute and then writing $L_{10}^{i_3} \cdot N_{JI} \left[I + N_{JI} \cdot L_{10}^{i_3} \right]^{-1} = I - D_{JI}^{-1}$. Then,

$$\frac{\partial T_{JI}}{\partial A} = D_{JI}^{-1} \cdot \left[\frac{\partial N_{JI}}{\partial A} - N_{JI}^2 \frac{\partial L_{10}}{\partial A} \right] \cdot D_{JI}^{-1} \ . \tag{10.8}$$

At LO the previous expression is

$$\left. \frac{\partial T_{JI}}{\partial A} \right|_{LO} = D_{JI}^{(0)\,-1} \cdot \left[-(N_{JI}^{(0)})^2 \frac{\partial L_{10}}{\partial A} \right] \cdot D_{JI}^{(0)\,-1} \ , \tag{10.9}$$

with

$$D_{JI}^{(0)} = I + N_{JI}^{(0)} \cdot L_{10} \ . \tag{10.10}$$

We have taken into account that

$$\frac{\partial N_{JI}^{(0)}}{\partial A} = 0 \ , \tag{10.11}$$

as it is clear from eqs.(8.2) and (8.4). At NLO eq.(10.8) reads

$$\left. \frac{\partial T_{JI}}{\partial A} \right|_{NLO} = D_{JI}^{(1)\,-1} \cdot \left[\frac{\partial (N_{JI}^{(0)} + N_{JI}^{(1)})}{\partial A} - (N_{JI}^{(0)} + N_{JI}^{(1)})^2 \frac{\partial L_{10}}{\partial A} \right] \cdot D_{JI}^{(1)\,-1} \ , \tag{10.12}$$

where

$$D_{JI}^{(1)} = I + (N_{JI}^{(0)} + N_{JI}^{(1)}) \cdot L_{10} = D_{JI}^{(0)} + N_{JI}^{(1)} \cdot L_{10} \ . \tag{10.13}$$

Expanding eq.(10.12) and neglecting terms of $\mathcal{O}(p^2)$

$$\left. \frac{\partial T_{JI}}{\partial A} \right|_{NLO} = D_{JI}^{(1)\,-1} \cdot \left[\frac{\partial L_{JI}^{(1)}}{\partial A} - \left\{ N_{JI}^{(1)}, N_{JI}^{(0)} \right\} \frac{\partial L_{10}}{\partial A} \right] \cdot D_{JI}^{(1)\,-1} \ . \tag{10.14}$$

Where we have used the standard notation $\{B, C\} = B \cdot C + C \cdot B$.

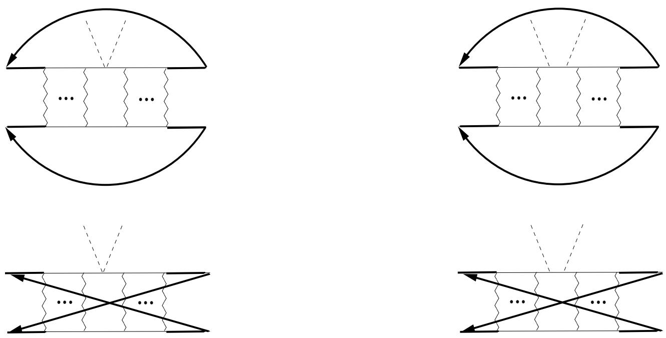

Figure 21: Diagrams Σ_9 and Σ_{10} for the pion self-energy in the medium involving the pion production in one intermediate loop. Then, this elementary process is dressed by initial and final state interactions as indicated by the iterative nucleon-nucleon interactions.

11 Non nucleon self-energy contributions with $V_\rho = 2$ (Σ_9 and Σ_{10}) and cancellation of the isovector terms

In this section we start the calculation of those contributions that originate from the diagrams in fig.21 where a pion scatters inside a two-nucleon reducible loop. This loop has to be corrected by initial (ISI) and final (FSI) state interactions as denoted in the figure by the ellipsis which represent iterated nucleon-nucleon interactions. This iteration is the same as occurs for the nucleon-nucleon scattering in the nuclear medium, see fig.11. The "elementary" nucleon-nucleon interaction N_{JI} is dressed by the iterative process which gives rise to eq.(8.8), with N_{JI} multiplied by the inverse of the matrix D_{JI}. In this way, if we denote by $\xi_{JI}(\bar{\ell},\ell,S)$ the elementary partial wave for a generic "production" process, $F_{JI}(\bar{\ell},\ell,S)$, then FSI dress it so that

$$F_{JI}(\bar{\ell},\ell,S) = D_{JI}^{-1}(\bar{\ell},\ell',S)\cdot\xi_{JI}(\ell',\ell,S) \ . \tag{11.1}$$

The matrix D_{JI} is already known by the study of the nucleon-nucleon interactions up to some order. On the other hand, ξ_{JI} can be fixed following an analogous process to that used before for determining N_{JI} in section 8.1. In this way, $\xi_{JI}^{(n)}$ is determined by expanding eq.(11.1) in powers of L_{10} up to L_{10}^n and then compare with a full CHPT calculation up to $\mathcal{O}(p^{\mu+n})$ with at most $n+1$ two-nucleon reducible diagrams. Note that we have written $\mu+n$ and $n+1$ because for our present purposes the basic process, made up by a two nucleon-reducible loop with the two pions attached to one nucleon propagator, starts at $\mathcal{O}(p^{-1})$, so that $\mu=-1$, and it implies already one two-nucleon reducible loop. Were the basic process a tree-level one then instead of $n+1$ one would have n. In addition, fig.21 also implies not only FSI but also ISI. Then, instead of eq.(11.1) we have

$$H_{JI}(\bar{\ell},\ell,S) = D_{JI}^{-1}(\bar{\ell},\ell'',S)\cdot\xi_{JI}(\ell'',\ell',S)\cdot D_{JI}^{-1}(\ell',\ell,S) \ . \tag{11.2}$$

The disposition of the indices in this equation can be easily deduced as follows. The chain of processes in fig.21, already projected in partial waves, can be schematically written as

$$\sum_{i,j} S_{\bar{\ell}i}\Pi_{ij}S_{j\ell} \ , \tag{11.3}$$

with $\Pi_{i,j}$ the production process and S the evolution operator. Now,

$$S = I + i\frac{mp}{2\pi}T = I + i\frac{mp}{2\pi}[I+N_{JI}\cdot L_{10}^{i_3}]^{-1}\cdot N_{JI} = I + i\frac{mp}{2\pi}N_{JI}\cdot[I+L_{10}^{i_3}\cdot N_{JI}]^{-1} \ , \tag{11.4}$$

where the last step follows because $T^{i_3}_{JI}(\bar{\ell},\ell,S)$ is symmetric under the exchange $\ell \leftrightarrow \bar{\ell}$, the same as $N_{JI}(\bar{\ell},\ell,S)$ and $L^{i_3}_{10}$. Equivalent forms of the terms on the right hand sides of the last two equalities in eq.(11.4) are, respectively,

$$S = \left[I + N_{JI}\cdot L^{i_3}_{10}\right]^{-1}(I+N_{JI}\cdot L^{i_3}_{10}+i\frac{mp}{2\pi}N_{JI}) = (I+L^{i_3}_{10}\cdot N_{JI}+i\frac{mp}{2\pi}N_{JI})\cdot\left[I+L^{i_3}_{10}\cdot N_{JI}\right]^{-1} \ . \quad (11.5)$$

Incorporating these last forms for S into eq.(11.3) one has

$$\begin{aligned}&\sum_{ij}\left[I+N_{JI}\cdot L^{\alpha_3}_{10}\right]^{-1}|_{\bar{\ell}i}\,\widetilde{\Pi}_{ij}\,\left[I+L^{\alpha_3}_{10}\cdot N_{JI}\right]^{-1}|_{j\ell}\\&=\sum_{i,j}D^{\alpha_3\,-1}_{JI}(\bar{\ell},i,S)\,\widetilde{\Pi}_{ij}\,D^{\alpha_3\,-1^T}_{JI}(j,\ell,S)\\&=\sum_{i,j}D^{\alpha_3\,-1}_{JI}(\bar{\ell},i,S)\,\widetilde{\Pi}_{ij}\,D^{\alpha_3\,-1}_{JI}(j,\ell,S)\ ,\end{aligned}\quad (11.6)$$

as in eq.(11.2).

The LO result requires to employ $D^{(0)}_{JI}$ and to calculate the two-nucleon reducible loop to which the two pions are attached by factorizing on-shell the nucleon-nucleon scattering amplitudes. We use the notation $D^{(n);i_3}_{JI} = I+N^{(n);i_3}_{JI}\cdot L^{i_3}_{10}$ with n the chiral order. Recall that UCHPT treats as $\mathcal{O}(p)$ the difference between a fully calculated two-nucleon reducible loop and the result obtained by the factorization on-shell the nucleon-nucleon vertices, as discussed in section 8.1 after eq.(8.19).

$$\begin{aligned}\xi^{(0)}_{JI} &= -(N^{(0)}_{JI})^2\cdot DL_{10}\ ,\\ H_{JI}|_{LO} &= D^{(0)^{-1}}_{JI}\cdot\xi^{(0)}_{JI}\cdot D^{(0)^{-1}}_{JI}\ .\end{aligned}\quad (11.7)$$

We give below explicit expressions for DL_{10} in eqs.(11.17) and (11.21).

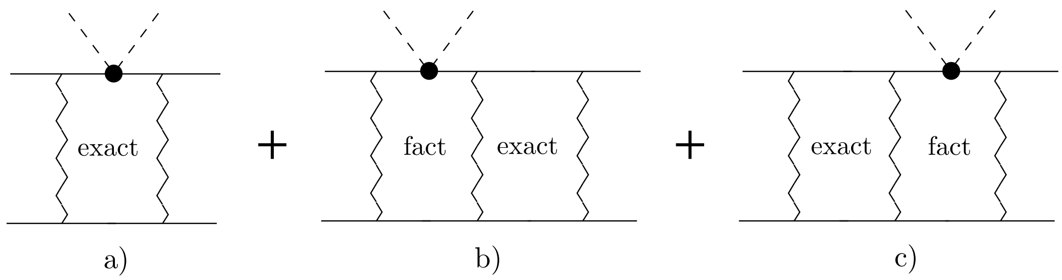

Figure 22: Diagrams that contribute for the calculation of $\xi^{(1)}_{JI}$. Those two-nucleon reducible loops that contain the remark "exact" must be calculated exactly in the EFT, while those with the work "fact" must be calculated with the factorization on-shell of the pertinent vertices. The filled circle in the figure indicates that the pion-nucleon scattering process contains both the local and Born terms, as explicitly indicated in fig.21.

At NLO one has an extra two-nucleon reducible loop. Expanding the D^{-1}_{JI} matrices in eq.(11.2) up to one L_{10} and ξ_{JI} up to $\mathcal{O}(p)$ it results

$$\xi^{(0)}_{JI}+\xi^{(1)}_{JI}-2N^{(0)}_{JI}\cdot L_{10}\cdot\xi^{(0)}_{JI}\ . \quad (11.8)$$

We now match the previous equation with the result of fig.22. In this figure we have included inside each loop the remarks "exact" or "fact" according to whether the loop is calculated exactly or factorizing on-shell the nucleon-nucleon vertices. The filled circle refers to the pion-nucleon scattering process that contains both the local and Born terms, fig.23. We denote by $L^{(1)}_{JI}$ the two-nucleon reducible loop without pions calculated exactly in CHPT and that occurs in figs.22b and 22c. There is also the new contribution of fig.22a whose exact calculation is denoted by $DL^{(1)}_{JI}$. The result is

$$DL^{(1)}_{JI} - N^{(0)}_{JI}\cdot DL_{10}\cdot L^{(1)}_{JI} - L^{(1)}_{JI}\cdot DL_{10}\cdot N^{(0)}_{JI}~. \tag{11.9}$$

The equality of eqs.(11.8) and (11.9), taking into account eq.(11.7) for $\xi^{(0)}_{JI}$, implies that

$$\begin{aligned}\xi^{(0)}_{JI}+\xi^{(1)}_{JI}-2N^{(0)}_{JI}\cdot L_{10}\cdot\xi^{(0)}_{JI} &= -(N^{(0)}_{JI})^2\cdot DL_{10}+2N^{(0)}_{JI}\cdot L_{10}\cdot\xi^{(0)}_{JI}+DL^{(1)}_{JI}\\ &\quad - L^{(1)}_{JI}\cdot DL_{10}\cdot N^{(0)}_{JI}-N^{(0)}_{JI}\cdot DL_{10}\cdot L^{(1)}_{JI}~.\end{aligned} \tag{11.10}$$

The last two terms in the previous expression correspond to the diagrams fig.22b and c, in order. Substituting eq.(11.7) in $2N^{(0)}_{JI}\cdot L_{10}\cdot\xi^{(0)}_{JI}$ one has

$$\xi^{(0)}_{JI}+\xi^{(1)}_{JI} = DL^{(1)}_{JI} - \left\{L^{(1)}_{JI}+(N^{(0)}_{JI})^2\cdot L_{10}, N^{(0)}_{JI}\right\}\cdot DL_{10}~. \tag{11.11}$$

In the last term we have the combination $L^{(1)}_{JI}+(N^{(0)}_{JI})^2\cdot L_{10}$ which is $\mathcal{O}(p)$ in our counting because it corresponds to the difference between an exact calculation of a two-nucleon reducible loop and that obtained by factorizing the vertices on-shell. The other contribution to $\xi^{(1)}_{JI}$ is given by $DL^{(1)}_{JI}-\xi^{(0)}_{JI}$, as follows from eq.(11.11), that is also $\mathcal{O}(p)$ by the same token. Finally, let us note that in the previous expression the two pions are attached to the loops $DL^{(1)}_{JI}$ and DL_{10}, while the rest of terms originate because of nucleon-nucleon scattering.

Since we are concerned with the calculation of a pion self-energy this implies that the initial and final pion is the same. As a result, the nucleon propagator before and after the filled circles in fig. 22 is also the same and then it appears squared in the corresponding two-nucleon reducible loop. Instead of a nucleon propagator squared we write,

$$\begin{aligned}&\left[\frac{\theta(\xi_o-|\mathbf{p}_1-\mathbf{k}|)}{p_1^0-k_1^0-E(\mathbf{p}_1-\mathbf{k})-i\epsilon}+\frac{\theta(|\mathbf{p}_1-\mathbf{k}|-\xi_o)}{p_1^0-k_1^0-E(\mathbf{p}_1-\mathbf{k})+i\epsilon}\right]^2\\ &=-\frac{\partial}{\partial z}\left[\frac{\theta(\xi_o-|\mathbf{p}_1-\mathbf{k}|)}{p_1^0+z-k_1^0-E(\mathbf{p}_1-\mathbf{k})-i\epsilon}+\frac{\theta(|\mathbf{p}_1-\mathbf{k}|-\xi_o)}{p_1^0+z-k_1^0-E(\mathbf{p}_1-\mathbf{k})+i\epsilon}\right]_{z=0}\\ &=-\frac{\partial}{\partial z}\left[\frac{1}{p_1^0+z-k_1^0-E(\mathbf{p}_1-\mathbf{k})+i\epsilon}+2\pi i\,\delta\big(p_1^0+z-k_1^0-E(\mathbf{p}_1-\mathbf{k})\big)\right]_{z=0}~.\end{aligned} \tag{11.12}$$

Figure 23: Nucleon pole terms in pion-nucleon scattering. The lowest order pion-nucleon vertex is given in eq.(A.1).

The filled circles in fig.22 consists of a WT pion-nucleon vertex, eq.(A.3), and of the pion-nucleon scattering Born terms shown in fig.23. Its sum is

$$-\frac{iq^0}{2f^2}\varepsilon_{ijk}\tau^k-\left(\frac{g_A}{2f}\right)^2\left\{\frac{\tau^j\tau^i(\vec{\sigma}\mathbf{q})(\vec{\sigma}\mathbf{q})}{q^0+p_1^0-k_1^0-E(\mathbf{p}_1+\mathbf{q}-\mathbf{k})+i\epsilon}+\frac{\tau^i\tau^j(\vec{\sigma}\mathbf{q})(\vec{\sigma}\mathbf{q})}{-q^0+p_1^0-k_1^0-E(\mathbf{p}_1-\mathbf{q}-\mathbf{k})+i\epsilon}\right\}~. \tag{11.13}$$

We do not include the in-medium part of the nucleon propagator because for $q^0 = \mathcal{O}(m_\pi)$ the argument of the Dirac delta function in eq.(3.2) is never satisfied as $m_\pi >> \mathcal{O}(\text{nucleon kinetic energy})$. For the same reason, when performing the k^0 integration in the loop the poles at $k^0 = p_1^0 \mp q^0$, resulting from eq.(11.13), are not considered because the nucleon propagators will not be any longer of $\mathcal{O}(p^{-2})$ but just of $\mathcal{O}(p^{-1})$ (standard counting). A contribution two orders higher would then result. Once the k^0 integration is done the latter acquires from eq.(11.12) the value $z + p_1^0 - E(\mathbf{p}_1 - \mathbf{k}) + i\varepsilon$ from the free part of the propagator by applying the Cauchy integration theorem and closing the upper k^0 complex half-plane with a infinite semicircle centered at the origin. On the other hand, the term proportional to the Dirac delta function fixes k^0 to $z + p_1^0 - E(\mathbf{p}_1 - \mathbf{k})$. The integration on k^0 for the evaluation of the two-nucleon reducible loop is analogous to the one performed in Appendix E to calculate the L_{10} function with $z = 0$. The point is that L_{10} only depends on the energy of the external legs through the variable $A = m(p_1^0 + p_2^0) - \alpha^2$, defined in eq.(E.4), that in turn only depends on the total energy. As a result, when the derivative with respect to z acts on a baryon propagator not entering in eq.(11.13) we can take instead the derivative

$$\frac{\partial L^{ab\ldots}_{k\ell,r}}{\partial z} = \frac{\partial L^{ab\ldots}_{k\ell,r}}{\partial p_1^0} = m\frac{\partial L^{ab\ldots}_{k\ell,r}}{\partial A} = m\frac{\partial L^{ab\ldots}_{k\ell,r}}{\partial p_2^0} \ . \tag{11.14}$$

Taking into account the chiral expansion of the nucleon propagator involved in eq.(11.13) the pole terms in this equations give rise to

$$\left(\frac{g_A}{2f}\right)^2 \frac{\mathbf{q}^2}{q^0}\left(\tau^i\tau^j - \tau^j\tau^i\right) = i\frac{g_A^2}{2f^2}\frac{\mathbf{q}^2}{q^0}\varepsilon_{ijk}\tau^k \ , \tag{11.15}$$

that has the same structure as the WT term. Their sum is

$$-\frac{iq^0}{2f^2}\,\kappa\,\varepsilon_{ijk}\tau^k \quad \text{with} \qquad \kappa \doteq 1 - g_A^2\frac{\mathbf{q}^2}{{q_0}^2} \ . \tag{11.16}$$

Thus, employing the latter vertex allows to discuss simultaneously all the diagrams in fig.21 for the case when the derivative with respect to z, that stems from eq.(11.12), does not act on the baryon propagators in the Born terms of eq.(11.13). Regarding the antisymmetric tensor in eq.(11.16) for the π^0 vanishes because here $i = j = 3$. For the $\pi^\pm$ i and j can be 1 or 2 and the surviving contribution is proportional to τ^3. This isospin matrix gives +2 for $i_3 = +1$, -2 for $i_3 = -1$ and vanishes for $i_3 = 0$. Notice that the pions are attached not only to the baryon 1 in the loop, as represented in eq.(11.12), but also to the other baryon and both contributions sum symmetrically. As a result we can rewrite eq.(11.7) for this case as

$$\xi^{(0)}_{JI;iv} = -i\frac{mq^0}{f^2}\left(1 - g_A^2\frac{\mathbf{q}^2}{q_0^2}\right) i_3\varepsilon_{ij3}\left[-(N^{(0)}_{JI})^2\frac{\partial L^{i_3}_{10}}{\partial A}\right] \ , \tag{11.17}$$

so that

$$DL_{10;iv} = -i\frac{mq^0}{f^2}\left(1 - g_A^2\frac{\mathbf{q}^2}{q_0^2}\right) i_3\varepsilon_{ij3}\frac{\partial L^{i_3}_{10}}{\partial A} \ . \tag{11.18}$$

Notice that in eq.(11.17) the term between brackets is the same as the one in eq.(10.9). In the previous equations we have included the subscript isv given their isovector character. In the same way for $DL^{(1)}_{JI}$ one has

$$DL^{(1)}_{JI;iv} = -i\frac{mq^0}{f^2}\left(1 - g_A^2\frac{\mathbf{q}^2}{q_0^2}\right) i_3\varepsilon_{ij3}\frac{\partial L^{(1);i_3}_{JI}}{\partial A} \ , \tag{11.19}$$

which corresponds to eq.(11.17) but substituting the term between brackets, with the nucleon-nucleon vertices on-shell, by its exact calculation. By applying eq.(11.11) we can fix $\xi^{(1);i_3}_{JI;iv}$ in terms of eqs.(11.17) and (11.19).

We now consider the case where the derivative with respect to z from eq.(11.12) acts on the baryon propagator involved in the Born terms of eq.(11.13). For that one has to take into account that $k_1^0 = p_1^0 + z - E(\mathbf{p}_1 - \mathbf{k})$. The term $E(\mathbf{p}_1-\mathbf{k}) - E(\mathbf{p}_1 - \mathbf{k} \pm \mathbf{q})$ can be neglected when summed with q^0 so that the derivative with respect to z of eq.(11.12) gives rise to the isoscalar contribution

$$-\frac{g_A^2}{2f^2}\frac{\mathbf{q}^2}{q_0^2}\delta_{ij} \ . \tag{11.20}$$

For any i_3 the spin operator just gives rise to $+2$ instead of $2i_3$ as in the isovector case. In this way, we can use eqs.(11.17) and (11.19) substituting the vertex of eq.(11.16) by eq.(11.20) and removing the action of the derivative $m\partial/\partial A$. Thus,

$$\begin{aligned}
\xi^{(0)}_{JI;is} &= -\frac{g_A^2\mathbf{q}^2}{f^2 q_0^2}\delta_{ij}\left[-(N^{(0)}_{JI})^2 L^{i_3}_{10}\right] \ , \\
DL_{10;is} &= -\frac{g_A^2\mathbf{q}^2}{f^2 q_0^2}\delta_{ij} L^{i_3}_{10} \ , \\
DL^{(1)}_{JI;is} &= -\frac{g_A^2\mathbf{q}^2}{f^2 q_0^2}\delta_{ij} L^{(1);i_3}_{JI} \ ,
\end{aligned} \tag{11.21}$$

Here we have included the subscript is given their isoscalar character. It also allows us to distinguish this case to the previous one of isovector type.

Now, we proceed to obtain the expressions for the pion self-energy corresponding to fig.21 as a sum over partial waves, similarly as for Σ_7, eq.(9.13), and Σ_8, eq.(10.2). For our present diagrams we have to correct by ISI and FSI employing eq.(11.2). Next, we have to sum over the two Fermi seas and we take also into account the simplification analogous to eq.(9.12) after summing over the quantum numbers of the Fermi seas. Additionally one has to include a symmetric factor 1/2 given the symmetry under the exchange of the two external lines when they are finally closed. The isovector and isovector contributions from the diagrams of fig.21 are denoted by $\Sigma_9 + \Sigma_{10}^{iv}$ and Σ_{10}^{is}, in that order.

The contributions at lowest order for these self-energy contributions employ $\xi^{(0)}_{JI}$, eqs.(11.17) and (11.21). The following expressions result

$$\begin{aligned}
\left(\Sigma_9 + \Sigma_{10}^{iv}\right)_{LO} &= i\varepsilon_{ij3}\frac{mq^0}{2f^2}\left(1 - g_A^2\frac{\mathbf{q}^2}{q_0^2}\right)\sum_{J,\ell,S}\chi(S\ell 1)^2(2J+1)\int\frac{d^3k_1}{(2\pi)^3}\int\frac{d^3k_2}{(2\pi)^3}\,\theta^{--}_{+1}\frac{\partial L_{10}^{+1}}{\partial A}\cdot\Big([D^{(0);+1}_{J1}]^{-1} \\
&\quad (N^{(0)}_{J1})^2\cdot[D^{(0),+1}_{J1}]^{-1} - \theta^{--}_{-1}\frac{\partial L^{-1}_{10}}{\partial A}[D^{(0);-1}_{J1}]^{-1}\cdot(N^{(0)}_{J1})^2\cdot[D^{(0);-1}_{J1}]^{-1}\Big) \ ,
\end{aligned} \tag{11.22}$$

where

$$\theta^{--}_{i_3} = \begin{cases} i_3 = +1 & \theta(\xi_p - |\mathbf{k}_1|)\theta(\xi_p - |\mathbf{k}_2|) \\ i_3 = 0 & \theta(\xi_1 - |\mathbf{k}_1|)\theta(\xi_2 - |\mathbf{k}_2|) \\ i_3 = -1 & \theta(\xi_n - |\mathbf{k}_1|)\theta(\xi_n - |\mathbf{k}_2|) \end{cases} \tag{11.23}$$

with $\xi_1 = min(\xi_p, \xi_n)$ and $\xi_2 = max(\xi_p, \xi_n)$ as in Appendix F.

$$\Sigma^{is}_{10}\big|_{LO} = \delta_{ij}\frac{g_A^2\mathbf{q}^2}{2f^2q_0^2}\sum_{J,\ell,S}\sum_{I,i_3}\chi(S\ell I)^2(2J+1)\int\frac{d^3k_1}{(2\pi)^3}\int\frac{d^3k_2}{(2\pi)^3}\theta^{--}_{i_3}\,[D^{(0);i_3}_{JI}]^{-1}\cdot\left[(N^{(0)}_{JI})^2L^{i_3}_{10}\right]\cdot[D^{(0);i_3}_{JI}]^{-1} \tag{11.24}$$

where we have omitted the arguments (ℓ,ℓ,S) in both $D_{JI}^{(0);i_3}$ and $N_{JI}^{(0)}$. As usual the chiral order is shown as a superscript in the corresponding symbols. From the previous expression it is clear that Σ_{10}^{is} is at least $\mathcal{O}(p^6)$, one order higher than $\Sigma_9+\Sigma_{10}^{iv}$.

Including those contributions to $\Sigma_9+\Sigma_{10}^{iv}$ and Σ_{10}^{is} up to one more order, $D_{JI}^{(1)}$ and $\xi_{JI}=\xi_{JI}^{(0)}+\xi_{JI}^{(1)}$, eq.(11.11), must be employed. The input functions $DL_{JI}^{(1)}$ is given in eqs.(11.19) and (11.21) for the isovector and isoscalar cases, respectively. In the same order the also needed function DL_{10} can be found in eqs.(11.18) and (11.21).

$$
\begin{aligned}
\left(\Sigma_9+\Sigma_{10}^{iv}\right)_{NLO} &= -i\varepsilon_{ij3}\frac{mq^0}{2f^2}\left(1-g_A^2\frac{\mathbf{q}^2}{q_0^2}\right)\sum_{J,\ell,S}\chi(S\ell 1)^2(2J+1)\int\frac{d^3p_1}{(2\pi)^3}\int\frac{d^3p_2}{(2\pi)^3} \\
&\cdot\Bigg[\theta_{+1}^{--}[D_{J1}^{(1);+1}]^{-1}\cdot\left(\frac{\partial L_{JI}^{(1);+1}}{\partial A}-\left\{L_{JI}^{(1);+1}+(N_{JI}^{(0)})^2\cdot L_{10}^{+1},N_{JI}^{(0)}\right\}\cdot\frac{\partial L_{10}^{+1}}{\partial A}\right)\cdot[D_{J1}^{(1);+1}]^{-1} \\
&-\theta_{-1}^{--}[D_{J1}^{(1);-1}]^{-1}\cdot\left(\frac{\partial L_{JI}^{(1);-1}}{\partial A}-\left\{L_{JI}^{(1),-1}+(N_{JI}^{(0)})^2\cdot L_{10}^{-1},N_{JI}^{(0)}\right\}\cdot\frac{\partial L_{10}^{-1}}{\partial A}\right)\cdot[D_{J1}^{(1);-1}]^{-1}\Bigg]\,,
\end{aligned}
\tag{11.25}
$$

$$
\begin{aligned}
\Sigma_{10}^{is}\big|_{NLO} &= -\delta_{ij}\frac{g_A^2\mathbf{q}^2}{2f^2q_0^2}\sum_{J,\ell,S}\sum_{I,i_3}\chi(S\ell 1)^2(2J+1)\int\frac{d^3p_1}{(2\pi)^3}\int\frac{d^3p_2}{(2\pi)^3}\,\theta_{i_3}^{--}[D_{JI}^{(1);i_3}]^{-1} \\
&\cdot\left(L_{JI}^{(1);i_3}-\left\{L_{JI}^{(1);i_3}+(N_{JI}^{(0)})^2\cdot L_{10}^{i_3},N_{JI}^{(0)}\right\}\cdot L_{10}^{i_3}\right)\cdot[D_{J1}^{(1);i_3}]^{-1}\,.
\end{aligned}
\tag{11.26}
$$

In ref.[1] we established that at $\mathcal{O}(p^5)$ all the contributions to the pion self-energy involving nucleon-nucleon interaction ($V_\rho=2$) vanish. This implies that the contributions from figs.18, 20 and 21 must vanish at this order. The argument followed in ref.[1] was a general one without any mention to a specific process for resumming and evaluating nucleon-nucleon interactions. We now show that UCHPT fulfills this requirement. For that one has to substitute in the expressions for Σ_7, eq.(9.13), and Σ_8^{iv}, eq.(10.2), $\partial T_{JI}/\partial A$ and check that their sum cancels with $\Sigma_9+\Sigma_{10}^{iv}$. At leading order the previous derivative is given in eq.(10.9) and then

$$
\begin{aligned}
\left(\Sigma_7+\Sigma_8^{iv}\right)_{LO} &= i\varepsilon_{ij3}\frac{mq^0}{2f^2}\left(1-g_A^2\frac{\mathbf{q}^2}{q_0^2}\right)\sum_{J,\ell,S}\chi(S\ell 1)^2(2J+1)\int\frac{d^3k_1}{(2\pi)^3}\int\frac{d^3k_2}{(2\pi)^3}\Big[\theta_{+1}^{--}[D_{J1}^{(0);+1}]^{-1} \\
&\cdot\left[-(N_{JI}^{(0)})^2\cdot\frac{\partial L_{10}^{+1}}{\partial A}\right]\cdot[D_{J1}^{(0);+1}]^{-1}-\theta_{-1}^{--}[D_{J1}^{(0);-1}]^{-1}\cdot\left[-(N_{JI}^{(0)})^2\cdot\frac{\partial L_{10}^{-1}}{\partial A}\right]\cdot[D_{J1}^{(0);-1}]^{-1}\Big]\,.
\end{aligned}
\tag{11.27}
$$

This equation is the same as eq.(11.22) but with opposite sign so that the cancellation takes place.

We also show this cancellation between the isovector contributions with $V_\rho=2$ for one order higher, $\mathcal{O}(p^6)$. In this case we substitute in eqs.(9.13) and (10.2) the derivative $\partial T_{JI}/\partial A$ at NLO given in eq.(10.14) with the result

$$\left(\Sigma_7+\Sigma_8^{iv}\right)_{NLO}=i\varepsilon_{ij3}\frac{mq^0}{2f^2}\left(1-g_A^2\frac{\mathbf{q}^2}{q_0^2}\right)\sum_{J,\ell,S}\chi(S\ell 1)^2(2J+1)\int\frac{d^3k_1}{(2\pi)^3}\int\frac{d^3k_2}{(2\pi)^3}$$
$$\cdot\left[\theta_{+1}^{--}[D_{J1}^{(1);+1}]^{-1}\cdot\left(\frac{\partial L_{JI}^{(1);+1}}{\partial A}-\left\{L_{JI}^{(1);+1}+(N_{JI}^{(0)})^2\cdot L_{10}^{+1},N_{JI}^{(0)}\right\}\frac{\partial L_{10}^{+1}}{\partial A}\right)\cdot[D_{JI}^{(1);+1}]^{-1}\right.$$
$$\left.-\theta_{-1}^{--}[D_{J1}^{(1);-1}]^{-1}\cdot\left(\frac{\partial L_{JI}^{(1);-1}}{\partial A}-\left\{L_{JI}^{(1);-1}+(N_{JI}^{(0)})^2\cdot L_{10}^{-1},N_{JI}^{(0)}\right\}\frac{\partial L_{10}^{-1}}{\partial A}\right)\cdot[D_{JI}^{(1);-1}]^{-1}\right]\ , \tag{11.28}$$

that exactly cancels with $\Sigma_9+\Sigma_{10}^{iv}$ calculated up to $\mathcal{O}(p^6)$, eq.(11.25). In the previous expression we have replaced $N_{JI}^{(1)}$ by its explicit expression in terms of $L_{JI}^{(1)}$. Notice that this cancellation is less trivial than at $\mathcal{O}(p^5)$ because it involves a precise balance between loops calculated exactly or factorizing vertices as required in fig.22. For comparison we give here the isoscalar expressions Σ_8^{is}

$$\Sigma_8^{is}\big|_{LO}=-\delta_{ij}\frac{g_A^2\mathbf{q}^2}{2f^2q_0^2}\sum_{J,\ell,S}\sum_{I,i_3}\chi(S\ell I)^2(2J+1)\int\frac{d^3k_1}{(2\pi)^3}\int\frac{d^3k_2}{(2\pi)^3}\,\theta_{i_3}^{--}[D_{JI}^{(0);i_3}]^{-1}\cdot N_{JI}^{(0)} \tag{11.29}$$
$$\Sigma_8^{is}\big|_{NLO}=-\delta_{ij}\frac{g_A^2\mathbf{q}^2}{2f^2q_0^2}\sum_{J,\ell,S}\sum_{I,i_3}\chi(S\ell I)^2(2J+1)\int\frac{d^3k_1}{(2\pi)^3}\int\frac{d^3k_2}{(2\pi)^3}\,\theta_{i_3}^{--}[D_{JI}^{(1);i_3}]^{-1}\cdot\left(N_{JI}^{(0)}+N_{JI}^{(1)}\right)\ . \tag{11.30}$$

For the calculation of the diagrams in fig.22 one has to recall that the exchange of a wiggly line corresponds to local plus one-pion exchange terms, as indicated in fig.10. When the nucleon loop to which the two pion lines are attached includes only nucleon-nucleon local vertices, we then have T_{10} and T_{11} for the isovector and isoscalar cases, respectively. When one of the nucleon-nucleon vertices in this loop is due to one-pion exchange then T_{12} and T_{13} result. Finally, when both vertices are due to one-pion exchange one has T_{14} and T_{15}. We denote by $\mathcal{T}_{1x;JI}(\bar{\ell},\ell,S)$ the resulting partial waves for T_{1x} at that order. Summing over the previous partial waves it follows that

$$DL_{JI;is}^{(1)}=\mathcal{T}_{11}+\mathcal{T}_{13}+\mathcal{T}_{15}\ . \tag{11.31}$$

Comparing with eq.(11.21) it is straightforward to determine $L_{JI}^{(1)}$. The explicit calculations of the box diagrams T_{10}–T_{15} and their partial waves $\mathcal{T}_{10}$–$\mathcal{T}_{15}$ will be the topic of the following three sections.

12 Explicit calculation of T_{10} and T_{11}

In this section we evaluate explicitly T_{10} and T_{11} up to and including $\mathcal{O}(p^6)$. The basic needed input is $L_{JI}^{(1)}$ since it enters for fixing $N_{JI}^{(1)}$ and $\xi_{JI}^{(1)}$. In order to illustrate with explicit calculations some steps introduced in the derivations of the previous section we evaluate explicitly the required two-nucleon reducible loop in the nuclear medium with the pions attached to it corresponding to fig.22.

We first consider the two-nucleon reducible loop with only local vertices, fig.24. For the isovector case the derivative with respect to z from eq.(11.12) acts on a nucleon propagator not entering in eq.(11.13).

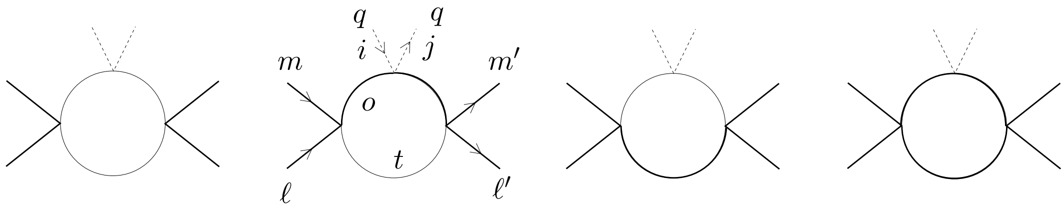

Figure 24: Two-nucleon reducible loop with only local vertices. The free part of the in-medium nucleon propagator in eq.(3.2) is indicated by a thin line while the in-medium part, proportional to the Dirac delta function, is denoted by a thick line.

With the four nucleon local vertex of eq.(8.2) we then have for T_{10},

$$
\begin{aligned}
T_{10} = & -\frac{\kappa q^0 \varepsilon_{ijk}}{2f^2}\left\{C_S\left(\delta_{\alpha_{m'}\alpha}\delta_{\alpha_{\ell'}\beta}\delta_{m'o}\delta_{\ell' t} - \delta_{\alpha_{m'}\beta}\delta_{\alpha_{\ell'}\alpha}\delta_{m't}\delta_{\ell' o}\right) + C_T\left(\vec{\sigma}_{\alpha_{m'}\alpha}\vec{\sigma}_{\alpha_{\ell'}\beta}\delta_{m'o}\delta_{\ell' t} - \vec{\sigma}_{\alpha_{m'}\beta}\vec{\sigma}_{\alpha_{\ell'}\alpha}\delta_{m't}\delta_{\ell' o}\right)\right\} \\
& \times \tau^k_{oo}\frac{\partial}{\partial p_1^0}\int\frac{d^4k}{(2\pi)^4}S(o,p_1-k)S(t,p_2+k)\times\left\{C_S\left(\delta_{\alpha_m\alpha}\delta_{\alpha_\ell\beta}\delta_{mo}\delta_{\ell t} - \delta_{\alpha_m\beta}\delta_{\alpha_\ell\alpha}\delta_{mt}\delta_{\ell o}\right)\right. \\
& \left. + C_T\left(\vec{\sigma}_{\alpha\alpha_m}\vec{\sigma}_{\beta\alpha_\ell}\delta_{mo}\delta_{\ell t} - \vec{\sigma}_{\beta\alpha_m}\vec{\sigma}_{\alpha\alpha_\ell}\delta_{mt}\delta_{\ell o}\right)\right\}~. \qquad (12.1)
\end{aligned}
$$

With $S(o,p_1-k)$ the nucleon propagator with flavor index o and four-momentum p_1-k, and similarly for $S(t,p_2+k)$. The spin indices are indicated with Greek letters. The momentum integration in the previous equation is the same as for the function L_{10}, evaluated in Appendix E. This function only depends on the energy of the external legs through the variable A, defined in eq.(E.4), so that eq.(11.14) holds. In this way, after some straightforward algebra one can write

$$
\begin{aligned}
T_{10} = & \frac{i\kappa q^0 \epsilon_{ijk}}{2f^2} m \frac{\partial L_{10}^{i_3}}{\partial A}\left\{\left[C_S^2\delta_{\alpha_{m'}\alpha_m}\delta_{\alpha_{\ell'}\alpha_\ell} + C_T^2(\vec{\sigma}_{\alpha_{m'}\alpha}\vec{\sigma}_{\alpha_{\ell'}\beta})(\vec{\sigma}_{\alpha\alpha_m}\vec{\sigma}_{\beta\alpha_\ell}) + 2C_SC_T\vec{\sigma}_{\alpha_{m'}\alpha_m}\vec{\sigma}_{\alpha_{\ell'}\alpha_\ell}\right]\right. \\
& \times\left(\delta_{\ell'\ell}\tau^k_{m'm} + \delta_{m'm}\tau^k_{\ell'\ell}\right) - \left[C_S^2\delta_{\alpha_{m'}\alpha_\ell}\delta_{\alpha_{\ell'}\alpha_m} + C_T^2(\vec{\sigma}_{\alpha_{m'}\alpha}\vec{\sigma}_{\alpha_{\ell'}\beta})(\vec{\sigma}_{\alpha\alpha_\ell}\vec{\sigma}_{\beta\alpha_m}) + 2C_SC_T\vec{\sigma}_{\alpha_{m'}\alpha_\ell}\vec{\sigma}_{\alpha_{\ell'}\alpha_m}\right] \\
& \left.\times\left(\delta_{m'\ell}\tau^k_{\ell'm} + \delta_{\ell'm}\tau^k_{m'\ell}\right)\right\} \qquad (12.2)
\end{aligned}
$$

after taking the derivative one has to fix $A\to\mathbf{p}^2$. This equation contains both the direct and exchange terms, the latter corresponding to the last contribution between squared brackets preceded by a minus sign. However, as explained in eq.(B.31) the direct term is the only one needed to evaluate the different partial waves. Notice that since $L_{10}^{i_3}$ is pure S-wave, because it only depends on A and α^2, eq.(12.2) only contributes to the partial waves $^1S_0$ and $^3S_1$. We work out now the spin and isospin projections of the direct term in eq.(12.2) with the result

$$
\begin{array}{rcc}
 & S=0 & S=1 \\
\delta_{\alpha_{m'}\alpha_m}\delta_{\alpha_{\ell'}\alpha_\ell} & 1 & 1 \\
\vec{\sigma}_{\alpha_{m'}\alpha_m}\vec{\sigma}_{\alpha_{\ell'}\alpha_\ell} & -3 & 1 \\
\left(\vec{\sigma}_{\alpha_{m'}\alpha}\vec{\sigma}_{\alpha_{\ell'}\beta}\right)\left(\vec{\sigma}_{\alpha\alpha_m}\vec{\sigma}_{\beta\alpha_\ell}\right) & 9 & 1
\end{array}~. \qquad (12.3)
$$

As it should, we have the combinations $(C_S-3C_T)^2$ and $(C_S+C_T)^2$ for $S=0$ and 1, respectively.

The isospin projection corresponding to the operator $(\delta_{\ell'\ell}\tau^3_{m'm}+\delta_{m'm}\tau^3_{\ell'\ell})$, which is the relevant combination for eq.(12.2), is 2 for $i_3=+1$, -2 for $i_3=-1$ and 0 for $i_3=0$, which excludes $I=0$ altogether. Keeping only the direct term in eq.(12.2) we then have

$$
T_{10,d}^{i_3} = i_3\frac{i\kappa m q^0 \epsilon_{ij3}}{f^2}(C_S-3C_T)^2\frac{\partial L_{10}^{i_3}}{\partial A}~. \qquad (12.4)
$$

In this equation the subscript d indicates that we are keeping only the direct contribution.

Next, we consider the isoscalar contribution $T_{11,d}$ making use of the vertex in eq.(11.20). Instead of eq.(12.2) one has now

$$T_{11}^{is} = \frac{g_A^2 \delta_{ij} |\mathbf{q}|^2}{2f^2 q_0^2} L_{10}^{i_3} \left\{ \left[C_S^2 \delta_{\alpha_{m'}\alpha_m} \delta_{\alpha_{\ell'}\alpha_\ell} + C_T^2 (\vec{\sigma}_{\alpha_{m'}\alpha} \vec{\sigma}_{\alpha_{\ell'}\beta})(\vec{\sigma}_{\alpha\alpha_m} \vec{\sigma}_{\beta\alpha_\ell}) + 2C_S C_T \vec{\sigma}_{\alpha_{m'}\alpha_m} \vec{\sigma}_{\alpha_{\ell'}\alpha_\ell} \right] \right.$$
$$\left. \times\, 2\delta_{\ell'\ell}\delta_{m'm} - \left[C_S^2 \delta_{\alpha_{m'}\alpha_\ell} \delta_{\alpha_{\ell'}\alpha_m} + C_T^2 (\vec{\sigma}_{\alpha_{m'}\alpha} \vec{\sigma}_{\alpha_{\ell'}\beta})(\vec{\sigma}_{\alpha\alpha_\ell} \vec{\sigma}_{\beta\alpha_m}) + 2C_S C_T \vec{\sigma}_{\alpha_{m'}\alpha_\ell} \vec{\sigma}_{\alpha_{\ell'}\alpha_m} \right] 2\delta_{m'\ell}\delta_{\ell'm} \right\} \tag{12.5}$$

The spin projection is the same as before, eq.(12.3). However, the isospin operator now is different and for the direct term just corresponds to twice the identity operator. Then, we always have $+2$, for $i_3 = \pm 1$, 0. As a result

$$T_{11,d}^{i_3} = \frac{g_A^2 \mathbf{q}^2}{f^2 q_0^2} (C_S + (4S-3)C_T)^2 L_{10}^{i_3} \ . \tag{12.6}$$

13 Explicit calculation of T_{12} and T_{13}

We now consider T_{12} and T_{13} when one-pion exchange vertex and one local vertex happen in the two-nucleon reducible loop at which the two pions are attached. Similarly as in the previous section, we start by considering the isovector case. The pion can be exchanged between the final two nucleons, fig.25, or between the initial ones, fig.26. They are denoted by T_{12}^f and T_{12}^i, respectively.

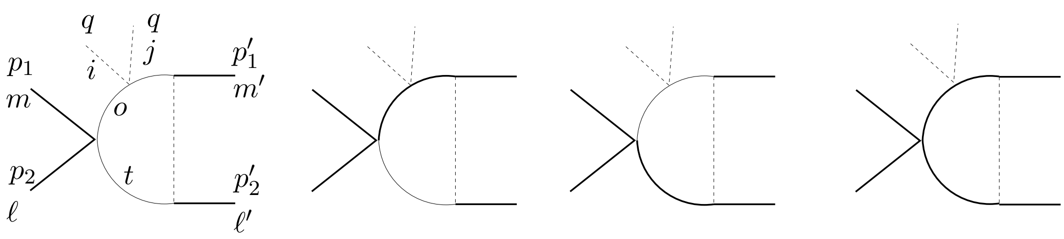

Figure 25: Two-nucleon reducible loop with one local and one-pion exchange vertices between the final nucleons. The free part of the in-medium nucleon propagator in eq.(3.2) is indicated by a thin line while the in-medium part, proportional to the Dirac delta function, is denoted by a thick line.

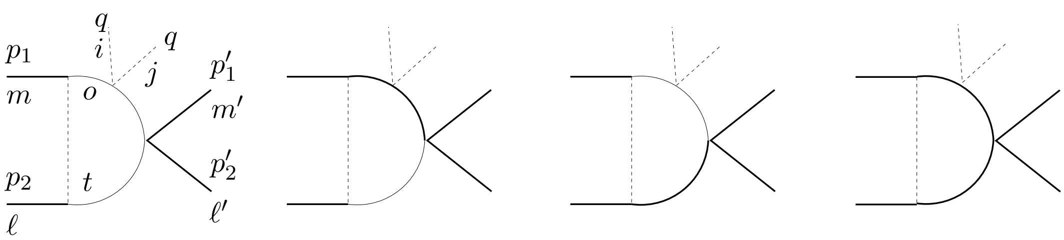

Figure 26: Two-nucleon reducible loop with one local and one-pion exchange vertices between the initial nucleons. The free part of the in-medium nucleon propagator in eq.(3.2) is indicated by a thin line while the in-medium part is denoted by a thick line.

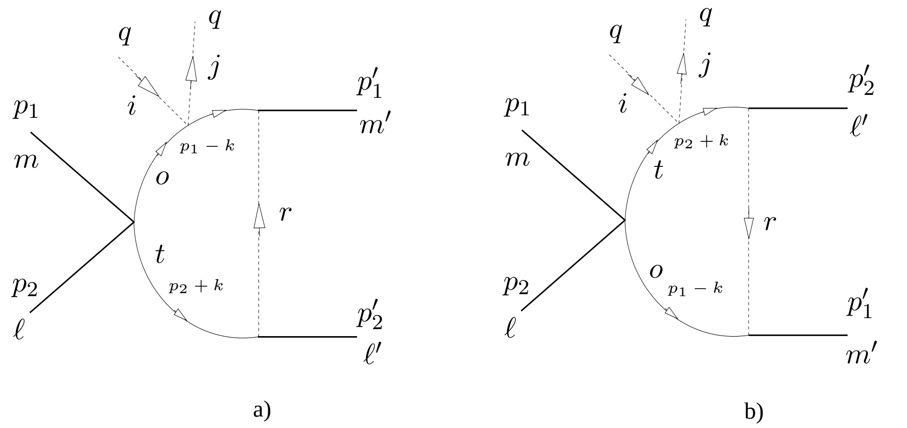

Figure 27: The internal four-momenta and discrete indices are indicated on the two figures. The figure on the left corresponds to $T_{12}^{f,a}$ and the one on the right to $T_{12}^{f,b}$. Note that the final pions are exchanged in $T_{12}^{f,b}$ with respect to $T_{12}^{f,a}$.

Taking into account the labelling of four-momenta in the internal lines, given in fig.27a, and the vertices of eq.(8.2) and (A.1), one has

$$T_{12}^{f,a} = \frac{\kappa q^0 \varepsilon_{ij3}}{2f^2}\left(\frac{g_A}{2f}\right)^2 \int \frac{d^4k}{(2\pi)^4} \left\{\vec{\sigma}_{\alpha_{m'}\alpha} \cdot \mathbf{r}\, \tau^3_{oo} \tau^a_{m'o} \tau^a_{\ell' t}\, \vec{\sigma}_{\alpha_{\ell'}\beta} \cdot \mathbf{r}\right\} \left\{C_S(\delta_{\alpha\alpha_m}\delta_{\beta\alpha_\ell}\delta_{mo}\delta_{\ell t} - \delta_{\alpha\alpha_\ell}\delta_{\beta\alpha_m}\delta_{mt}\delta_{\ell o}) \right.$$
$$\left. + C_T(\delta_{mo}\delta_{\ell t}\vec{\sigma}_{\alpha\alpha_m}\cdot\vec{\sigma}_{\beta\alpha_\ell} - \delta_{mt}\delta_{\ell o}\vec{\sigma}_{\beta\alpha_m}\cdot\vec{\sigma}_{\alpha\alpha_\ell})\right\} S(o,p_1-k)^2 S(t,p_2+k)\frac{1}{r^2 - m_\pi^2 + i\epsilon} \;, \tag{13.1}$$

where repeated indices are summed and $r = p_1' - p_1 + k$ from fig.27a. In the previous equation, $S(o, p_1 - k)$ and $S(t, p_2 + k)$ are the in-medium nucleon propagators, (3.2), with isospin indices o and t and four-momenta $p_1 - k$ and $p_2 + k$, respectively. Instead of keeping $S(o)^2$ we take $-\left.\partial S(o)/\partial z\right|_{z=0}$, as done in eq.(11.12). On the other hand, for the pion propagator we neglect its dependence on r_0^2, since it is $\mathcal{O}(p^4)$, while $\mathbf{r}^2$ is $\mathcal{O}(p^2)$. Then, the energy dependence enters in T_{12}^f similarly as in L_{10} and the derivative is taken with respect to the variable A, analogously to eq.(11.14). Then, it results

$$T_{12}^{f,a} = \frac{\kappa q^0 \varepsilon_{ij3}}{2f^2}\left(\frac{g_A}{2f}\right)^2 \frac{m\partial}{\partial A}\int \frac{d^4k}{(2\pi)^4}\left\{\left[C_S(\vec{\sigma}_{\alpha_{m'}\alpha_m}\cdot\mathbf{r})(\vec{\sigma}_{\alpha_{\ell'}\alpha_\ell}\cdot\mathbf{r}) + C_T(\vec{\sigma}_{\alpha_{m'}\alpha}\cdot\mathbf{r})(\vec{\sigma}_{\alpha_{\ell'}\beta}\cdot\mathbf{r})(\vec{\sigma}_{\alpha\alpha_m}\cdot\vec{\sigma}_{\beta\alpha_\ell})\right]\right.$$
$$\times\, \tau^3_{mm}\vec{\tau}_{m'm}\vec{\tau}_{\ell'\ell} - \left[C_S(\vec{\sigma}_{\alpha_{m'}\alpha_\ell}\cdot\mathbf{r})(\vec{\sigma}_{\alpha_{\ell'}\alpha_m}\cdot\mathbf{r}) + C_T(\vec{\sigma}_{\alpha_{m'}\alpha}\cdot\mathbf{r})(\vec{\sigma}_{\alpha_{\ell'}\beta}\cdot\mathbf{r})(\vec{\sigma}_{\alpha\alpha_\ell}\cdot\vec{\sigma}_{\beta\alpha_m})\right]\tau^3_{\ell\ell}\vec{\tau}_{m'\ell}\cdot\vec{\tau}_{\ell' m}\Big\}$$
$$\times\, S(m,p_1-k)S(\ell,p_2+k)\frac{1}{\mathbf{r}^2 + m_\pi^2} \tag{13.2}$$

with A fixed to $\mathbf{p}^2$ after the derivative is taken. One still has a similar diagram to that of fig.27a but with the two final nucleons exchanged, this is fig.27b, denoted by $T_{12}^{f,b}$. Note also the different disposition of the internal four-momentum lines as compared with fig.27a. Its expression is

$$T_{12}^{f,b} = \frac{\kappa q^0 \varepsilon_{ij3}}{2f^2}\left(\frac{g_A}{2f}\right)^2 \frac{m\partial}{\partial A}\int \frac{d^4k}{(2\pi)^4}\left\{-\left[C_S(\vec{\sigma}_{\alpha_{\ell'}\alpha_m}\cdot\mathbf{r})(\vec{\sigma}_{\alpha_{m'}\alpha_\ell}\cdot\mathbf{r}) + C_T(\vec{\sigma}_{\alpha_{\ell'}\alpha}\cdot\mathbf{r})(\vec{\sigma}_{\alpha_{m'}\beta}\cdot\mathbf{r})(\vec{\sigma}_{\alpha\alpha_m}\cdot\vec{\sigma}_{\beta\alpha_\ell})\right]\right.$$
$$\times \tau^3_{mm}\vec{\tau}_{\ell' m}\cdot\vec{\tau}_{m'\ell} + \left[C_S(\vec{\sigma}_{\alpha_{\ell'}\alpha_\ell}\cdot\mathbf{r})(\vec{\sigma}_{\alpha_{m'}\alpha_m}\cdot\mathbf{r}) + C_T(\vec{\sigma}_{\alpha_{\ell'}\alpha}\cdot\mathbf{r})(\vec{\sigma}_{\alpha_{m'}\beta}\cdot\mathbf{r})(\vec{\sigma}_{\alpha\alpha_\ell}\cdot\vec{\sigma}_{\beta\alpha_m})\right]$$
$$\times \tau^3_{\ell\ell}\vec{\tau}_{\ell'\ell}\vec{\tau}_{m'm}\Big\}\, S(m,p_1-k)S(\ell,p_2+k)\frac{1}{\mathbf{r}^2 + m_\pi^2} \;. \tag{13.3}$$

Summing eqs.(13.2) and (13.3),

$$T_{12}^f = \frac{\kappa q^0 \varepsilon_{ij3}}{2f^2}\left(\frac{g_A}{2f}\right)^2 \frac{m\partial}{\partial A}\int \frac{d^4k}{(2\pi)^4}\left\{\left[C_S(\vec{\sigma}_{\alpha_{m'}\alpha_m}\cdot\mathbf{r})(\vec{\sigma}_{\alpha_{\ell'}\alpha_\ell}\cdot\mathbf{r}) + C_T(\vec{\sigma}_{\alpha_{m'}\alpha}\cdot\mathbf{r})(\vec{\sigma}_{\alpha_{\ell'}\beta}\cdot\mathbf{r})(\vec{\sigma}_{\alpha\alpha_m}\cdot\vec{\sigma}_{\beta\alpha_\ell})\right]\right.$$
$$\times\, \tau^a_{m'm}\tau^a_{\ell'\ell} - \left.\left[C_S(\vec{\sigma}_{\alpha_{m'}\alpha_\ell}\cdot\mathbf{r})(\vec{\sigma}_{\alpha_{\ell'}\alpha_m}\cdot\mathbf{r}) + C_T(\vec{\sigma}_{\alpha_{m'}\alpha}\cdot\mathbf{r})(\vec{\sigma}_{\alpha_{\ell'}\beta}\cdot\mathbf{r})(\vec{\sigma}_{\alpha\alpha_\ell}\cdot\vec{\sigma}_{\beta\alpha_m})\right]\tau^a_{m'\ell}\tau^a_{\ell'm}\right\}$$
$$\times\left(\tau^3_{mm}+\tau^3_{\ell\ell}\right) S(m,p_1-k)S(\ell,p_2+k)\frac{1}{\mathbf{r}^2+m_\pi^2}\ . \tag{13.4}$$

Now we proceed to the isospin and spin projections for the direct term, given by the first square bracket in the previous equation, which is the only one needed for the calculation of the partial waves. One has the isospin operator

$$\tau^a_{m'm}\tau^a_{\ell'\ell}(\tau^3_{mm}+\tau^3_{\ell\ell})\ , \tag{13.5}$$

whose projection between states of well defined isospin is $2i_3$. This implies that for $i_3 = 0$ there is no contribution.

For the case of spin we first perform the following manipulation of the term multiplied by C_T,

$$(\vec{\sigma}_{\alpha_{m'}\alpha}\cdot\mathbf{r})(\vec{\sigma}_{\alpha_{\ell'}\beta}\cdot\mathbf{r})(\vec{\sigma}_{\alpha\alpha_m}\cdot\vec{\sigma}_{\beta\alpha_\ell}) = \mathbf{r}^2\left[\delta_{\alpha_{m'}\alpha_m}\delta_{\alpha_{\ell'}\alpha_\ell} - \vec{\sigma}_{\alpha_{m'}\alpha_m}\cdot\vec{\sigma}_{\alpha_{\ell'}\alpha_\ell}\right] + (\vec{\sigma}_{\alpha_{m'}\alpha_m}\cdot\mathbf{r})(\vec{\sigma}_{\alpha_{\ell'}\alpha_\ell}\cdot\mathbf{r})\ . \tag{13.6}$$

The last structure is common with that multiplied by C_S in eq.(13.4) for the direct term. Here, we have made use of the well known identity,

$$\sigma^a\cdot\sigma^b = \delta^{ab}I + i\varepsilon^{abc}\sigma^c\ . \tag{13.7}$$

Then, we have for the first square bracket in eq.(13.4),

$$(C_S+C_T)(\vec{\sigma}_{\alpha_{m'}\alpha_m}\cdot\mathbf{r})(\vec{\sigma}_{\alpha_{\ell'}\alpha_\ell}\cdot\mathbf{r}) + C_T\,\mathbf{r}^2\left(\delta_{\alpha_{m'}\alpha_m}\delta_{\alpha_{\ell'}\alpha_\ell} - \vec{\sigma}_{\alpha_{m'}\alpha_m}\cdot\vec{\sigma}_{\alpha_{\ell'}\alpha_\ell}\right)\ . \tag{13.8}$$

The structures $\delta_{\alpha_{m'}\alpha_m}\delta_{\alpha_{\ell'}\alpha_\ell}$ and $\vec{\sigma}_{\alpha_{m'}\alpha_m}\cdot\vec{\sigma}_{\alpha_{\ell'}\alpha_\ell}$ are already projected in eq.(12.3) for the different spin states. There is, however, the new structure $(\vec{\sigma}_{\alpha_{m'}\alpha_m}\cdot\mathbf{r})(\vec{\sigma}_{\alpha_{\ell'}\alpha_\ell}\cdot\mathbf{r})$. Its projection for different spin states is

$$S=0\ ,\quad -\mathbf{r}^2\ ,$$
$$S=1$$
$$||B_{s_3's_3}|| = \left(\begin{array}{c|ccc} & -1 & 0 & +1 \\ \hline -1 & r_3^2 & -\sqrt{2}r_3(r_1+ir_2) & (r_1+ir_2)^2 \\ 0 & -\sqrt{2}r_3(r_1-ir_2) & \mathbf{r}^2-2r_3^2 & \sqrt{2}r_3(r_1+ir_2) \\ +1 & (r_1-ir_2)^2 & \sqrt{2}r_3(r_1-ir_2) & r_3^2 \end{array}\right)\ . \tag{13.9}$$

In this matrix the rows correspond to the final third component of the total spin, s_3', and the columns to the initial one, s_3. Notice that for $S=1$, because of eq.(12.3), there is no contribution proportional to $C_T\mathbf{r}^2$ from eq.(13.8). For $S=0$ the net result is $-(C_S-3C_T)\mathbf{r}^2$. Then we can write for the direct term

$$T_{12,d}^{f,S=0} = -i_3\frac{\kappa q^0\varepsilon_{ij3}}{f^2}\left(\frac{g_A}{2f}\right)^2 (C_S-3C_T)\frac{m\partial}{\partial A}\int\frac{d^4k}{(2\pi)^4}S(m,p_1-k)S(\ell,p_2+k)\frac{\mathbf{r}^2}{\mathbf{r}^2+m_\pi^2}\ . \tag{13.10}$$

Here, we have included the superscript $S=0$ to indicate that it corresponds to $S=0$ and the subscript d referring that only the direct term is shown. Only partial waves with $\ell =$ even contribute because of the rule $S+I+\ell$ =odd and the fact that only the $I=1$ contribution does not vanish.

For $S=1$ and the transition from a state with total third component of spin s_3 to another of s_3', one has from eqs.(13.4) and (13.9),

$$T_{12,d}^{f,S=1}(s_3',s_3)=i_3\frac{\kappa q^0\varepsilon_{ij3}}{f^2}\left(\frac{g_A}{2f}\right)^2(C_S+C_T)\frac{m\partial}{\partial A}\int\frac{d^4k}{(2\pi)^4}S(m,p_1-k)S(\ell,p_2+k)\frac{1}{\mathbf{r}^2+m_\pi^2}B_{s_3' s_3}\ . \tag{13.11}$$

In this case, the rule $S+\ell+I=$odd requires ℓ to be odd.

For the integrals in eqs.(13.10) and (13.11) it is convenient to perform the shift of integration variable

$$k\to\frac{p_1-p_2}{2}+k=p+k\ , \tag{13.12}$$

so that

$$\begin{aligned}p_1-k&\to\frac{Q}{2}-k\ ,\ Q=\frac{p_1+p_2}{2}\ ,\\ p_2+k&\to\frac{Q}{2}+k\ ,\\ r=p_1'-p_1+k&\to p'+k\ ,\ p'=\frac{p_1'-p_2'}{2}\ ,\end{aligned} \tag{13.13}$$

In this way,

$$\begin{aligned}S(m,p_1-k)S(\ell,p_2+k)=&\left[\frac{\theta(\xi_m-|\vec{\alpha}-\mathbf{k}|)}{Q^0/2-k^0-w(\vec{\alpha}-\mathbf{k})-i\epsilon}+\frac{\theta(|\vec{\alpha}-\mathbf{k}|-\xi_m)}{Q^0/2-k^0-w(\vec{\alpha}-\mathbf{k})+i\epsilon}\right]\\ &\times\left[\frac{\theta(\xi_\ell-|\vec{\alpha}+\mathbf{k}|)}{Q^0/2+k^0-w(\vec{\alpha}+\mathbf{k})-i\epsilon}+\frac{\theta(|\vec{\alpha}+\mathbf{k}|-\xi_\ell)}{Q^0/2+k^0-w(\vec{\alpha}+\mathbf{k})+i\epsilon}\right]\ ,\end{aligned} \tag{13.14}$$

with the same four-momenta in the propagators as in eq.(E.1) for L_{10}. The vector $\vec{\alpha}=(\mathbf{p}_1+\mathbf{p}_2)/2$, defined in eq.(E.2). Because of the factor i_3 in front of $T_{12}^{f,S}$ there is contribution only when $i_3=\pm1$ and then $m=\ell=\pm1$.

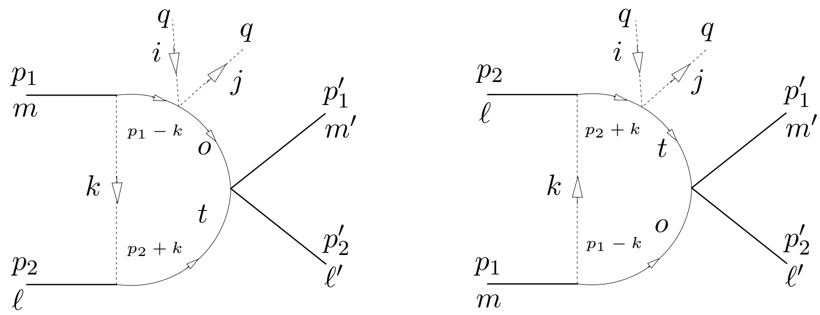

Figure 28: The internal four-momenta and discrete indices are shown on this figure contributing to T_{12}^i.

We now consider the one-pion exchange between the initial nucleons, fig.26. The disposition of four-momenta, spin and isospin indices is indicated in fig.28. Proceeding in the same way as before when calculating T_{12}^{f}, eq.(13.4), we have for the sum of the diagrams in the previous figure,

$$T_{12}^{i} = \frac{\kappa q^0 \varepsilon_{ij3}}{2f^2}\left(\frac{g_A}{2f}\right)^2 \frac{m\partial}{\partial A}\int\frac{d^4k}{(2\pi)^4}\left\{\left[C_S(\vec{\sigma}_{\alpha_{m'}\alpha_m}\cdot\mathbf{k})(\vec{\sigma}_{\alpha_{\ell'}\alpha_\ell}\cdot\mathbf{k}) + C_T(\vec{\sigma}_{\alpha_{m'}\alpha}\cdot\vec{\sigma}_{\alpha_{\ell'}\beta})(\vec{\sigma}_{\alpha\alpha_m}\cdot\mathbf{k})(\vec{\sigma}_{\beta\alpha_\ell}\cdot\mathbf{k})\right]\right.$$
$$\times\,\vec{\tau}_{m'm}\vec{\tau}_{\ell'\ell} - \left.\left[C_S(\vec{\sigma}_{\alpha_{m'}\alpha_\ell}\cdot\mathbf{k})(\vec{\sigma}_{\alpha_{\ell'}\alpha_m}\cdot\mathbf{k}) + C_T(\vec{\sigma}_{\alpha_{\ell'}\alpha}\cdot\vec{\sigma}_{\alpha_{m'}\beta})(\vec{\sigma}_{\alpha\alpha_m}\cdot\mathbf{k})(\vec{\sigma}_{\beta\alpha_\ell}\cdot\mathbf{k})\right]\vec{\tau}_{\ell'm}\vec{\tau}_{m'\ell}\right\}$$
$$\times\left(\tau^3_{m'm'}+\tau^3_{\ell'\ell'}\right)S(m,p_1-k)S(\ell,p_2+k)\frac{1}{\mathbf{k}^2+m_\pi^2}\ . \tag{13.15}$$

The shift in the integration variables,

$$k \to \frac{p_1-p_2}{2}+k = p+k\ , \tag{13.16}$$

is taken as in eq.(13.12). The same expressions for $T_{12}^{f,S=0}$ and $T_{12}^{f,S=1}$, after incorporating in eqs.(13.10) and (13.11) the change of variables of eq.(13.13), can be used for $T_{12}^{i,S=0}$ and $T_{12}^{i,S=1}$ with the exchange $\mathbf{p'}\to\mathbf{p}$. T_{12}^{i} only contributes to S-wave because it depends on $\mathbf{p}$ but not on $\mathbf{p'}$.

We now introduce the functions L_{11}, L_{11}^{a} and L_{11}^{ab} given by

$$L_{11}(\mathbf{r}) = i\int\frac{d^4k}{(2\pi)^4}\frac{1}{(\mathbf{k}+\mathbf{r})^2+m_\pi^2}\left[\frac{\theta(\xi_m-|\vec{\alpha}-\mathbf{k}|)}{Q^0/2-k^0-w(\vec{\alpha}-\mathbf{k})-i\epsilon}+\frac{\theta(|\vec{\alpha}-\mathbf{k}|-\xi_m)}{Q^0/2-k^0-w(\vec{\alpha}-\mathbf{k})+i\epsilon}\right]$$
$$\times\left[\frac{\theta(\xi_\ell-|\vec{\alpha}+\mathbf{k}|)}{Q^0/2+k^0-w(\vec{\alpha}+\mathbf{k})-i\epsilon}+\frac{\theta(|\vec{\alpha}+\mathbf{k}|-\xi_\ell)}{Q^0/2+k^0-w(\vec{\alpha}+\mathbf{k})+i\epsilon}\right]\ ,$$
$$L_{11}^{a}(\mathbf{r}) = i\int\frac{d^4k}{(2\pi)^4}\frac{k^a}{(\mathbf{k}+\mathbf{r})^2+m_\pi^2}\left[\frac{\theta(\xi_m-|\vec{\alpha}-\mathbf{k}|)}{Q^0/2-k^0-w(\vec{\alpha}-\mathbf{k})-i\epsilon}+\frac{\theta(|\vec{\alpha}-\mathbf{k}|-\xi_m)}{Q^0/2-k^0-w(\vec{\alpha}-\mathbf{k})+i\epsilon}\right]$$
$$\times\left[\frac{\theta(\xi_\ell-|\vec{\alpha}+\mathbf{k}|)}{Q^0/2+k^0-w(\vec{\alpha}+\mathbf{k})-i\epsilon}+\frac{\theta(|\vec{\alpha}+\mathbf{k}|-\xi_\ell)}{Q^0/2+k^0-w(\vec{\alpha}+\mathbf{k})+i\epsilon}\right]\ ,$$
$$L_{11}^{ab}(\mathbf{r}) = i\int\frac{d^4k}{(2\pi)^4}\frac{k^ak^b}{(\mathbf{k}+\mathbf{r})^2+m_\pi^2}\left[\frac{\theta(\xi_m-|\vec{\alpha}-\mathbf{k}|)}{Q^0/2-k^0-w(\vec{\alpha}-\mathbf{k})-i\epsilon}+\frac{\theta(|\vec{\alpha}-\mathbf{k}|-\xi_m)}{Q^0/2-k^0-w(\vec{\alpha}-\mathbf{k})+i\epsilon}\right]$$
$$\times\left[\frac{\theta(\xi_\ell-|\vec{\alpha}+\mathbf{k}|)}{Q^0/2+k^0-w(\vec{\alpha}+\mathbf{k})-i\epsilon}+\frac{\theta(|\vec{\alpha}+\mathbf{k}|-\xi_\ell)}{Q^0/2+k^0-w(\vec{\alpha}+\mathbf{k})+i\epsilon}\right]\ . \tag{13.17}$$

These integrals will be evaluated in Appendix G. For our present purposes it is enough to write them taking explicitly into account their tensor structure in terms of a few scalar integrals:

$$L_{11}^{a}(\mathbf{r}) = L_{11}^{\alpha}\alpha^a + L_{11}^{p}r^a\ ,$$
$$L_{11}^{ab}(\mathbf{r}) = L_{11}^{Tg}\delta^{ab} + L_{11}^{T\alpha}\alpha^a\alpha^b + L_{11}^{Tp}r^ar^b + L_{11}^{T\alpha p}(\alpha^ar^b+\alpha^br^a)\ , \tag{13.18}$$

where we have taken into account that L_{11}^{ab} is symmetric in the superscripts as follows from its definition, eq.(13.17). In terms of these integrals one can write the different matrix elements of $T_{12}^{f,S}$ (similar expressions hold for $T_{12}^{i,S}$ with the exchange of $\mathbf{p'}\to\mathbf{p}$, as discussed above).

$$T_{12,d}^{f,S=0} = i_3\frac{i\kappa q^0\varepsilon_{ij3}}{f^2}\left(\frac{g_A}{2f}\right)^2(C_S-3C_T)\frac{m\partial}{\partial A}(L_{10}-m_\pi^2L_{11})\ , \tag{13.19}$$
$$T_{12,d}^{f,S=1}(1,1) = -i_3\frac{i\kappa q^0\varepsilon_{ij3}}{f^2}\left(\frac{g_A}{2f}\right)^2(C_S+C_T)\frac{m\partial}{\partial A}\left[{p'_3}^2\left(L_{11}+2L_{11}^{p}+L_{11}^{Tp}\right)+\alpha_3^2L_{11}^{T\alpha}\right.$$
$$\left.+\,2\alpha_3p'_3\left(L_{11}^{\alpha}+L_{11}^{T\alpha p}\right)+L_{11}^{Tg}\right]\ ,$$

$$
\begin{aligned}
T^{f,S=1}_{12,d}(1,0) &= -i_3 \frac{i\kappa q^0 \varepsilon_{ij3}}{f^2}\left(\frac{g_A}{2f}\right)^2 \sqrt{2}(C_S+C_T)\frac{m\partial}{\partial A}\Big[(\alpha_1 - i\alpha_2)\Big\{L^{T\alpha}_{11}\alpha_3 + (L^{T\alpha p}_{11} + L^{\alpha}_{11})p'_3\Big\} \\
&+ (p'_1 - ip'_2)\Big\{(L^{T\alpha p}_{11} + L^{\alpha}_{11})\alpha_3 + (L^{Tp}_{11} + L_{11} + 2L^{p}_{11})p'_3\Big\}\Big] \ , \\
T^{f,S=1}_{12,d}(1,-1) &= -i_3 \frac{i\kappa q^0 \varepsilon_{ij3}}{f^2}\left(\frac{g_A}{2f}\right)^2 (C_S+C_T)\frac{m\partial}{\partial A}\Big[(\alpha_1 - i\alpha_2)^2 L^{T\alpha}_{11} + (p'_1 - ip'_2)^2(L^{Tp}_{11} + 2L^{p}_{11} + L_{11}) \\
&+ 2(p'_1 - ip'_2)(\alpha_1 - i\alpha_2)(L^{T\alpha p}_{11} + L^{\alpha}_{11})\Big] \ , \\
T^{f,S=1}_{12,d}(0,1) &= -i_3 \frac{i\kappa q^0 \varepsilon_{ij3}}{f^2}\left(\frac{g_A}{2f}\right)^2 \sqrt{2}(C_S+C_T)\frac{m\partial}{\partial A}\Big[(\alpha_1 + i\alpha_2)\Big\{L^{T\alpha}_{11}\alpha_3 + (L^{T\alpha p}_{11} + L^{\alpha}_{11})p'_3\Big\} \\
&+ (p'_1 + ip'_2)\Big\{(L^{T\alpha p}_{11} + L^{\alpha}_{11})\alpha_3 + (L^{Tp}_{11} + L_{11} + 2L^{p}_{11})p'_3\Big\}\Big] \ , \\
T^{f,S=1}_{12,d}(0,0) &= -i_3 \frac{i\kappa q^0 \varepsilon_{ij3}}{f^2}\left(\frac{g_A}{2f}\right)^2 (C_S+C_T)\frac{m\partial}{\partial A}\Big[L_{10} - m_\pi^2 L_{11} - 2L^{Tg}_{11} - 2{\alpha_3}^2 L^{T\alpha}_{11} \\
&- 2{p'_3}^2(L_{11} + L^{Tp}_{11} + 2L^{p}_{11}) - 4\alpha_3 p'_3(L^{T\alpha p}_{11} + L^{\alpha}_{11})\Big] \ , \\
T^{f,S=1}_{12,d}(-1,1) &= -i_3 \frac{i\kappa q^0 \varepsilon_{ij3}}{f^2}\left(\frac{g_A}{2f}\right)^2 (C_S+C_T)\frac{m\partial}{\partial A}\Big[(\alpha_1 + i\alpha_2)^2 L^{T\alpha}_{11} + (p'_1 + ip'_2)^2(L^{Tp}_{11} + 2L^{p}_{11} + L_{11}) \\
&+ 2(p'_1 + ip'_2)(\alpha_1 + i\alpha_2)(L^{T\alpha p}_{11} + L^{\alpha}_{11})\Big] \ , \\
T^{f,S=1}_{12,d}(0,-1) &= -T^{f,S=1}_{12,d}(1,0) \ , \\
T^{f,S=1}_{12,d}(-1,0) &= -T^{f,S=1}_{12,d}(0,1) \ , \\
T^{f,S=1}_{12,d}(-1,-1) &= T^{f,S=1}_{12,d}(1,1) \ .
\end{aligned} \tag{13.20}
$$

The other contribution originates by taking the derivative of the intermediate nucleon propagator in fig.23. We have the same expression as for T^f_{12}, eq.(13.4), but removing the operator $m\partial/\partial A$ and with the replacement

$$
i\frac{q^0\varepsilon_{ij3}}{f^2}\tau^3_{nn'} \rightarrow \frac{g_A^2}{f^2}\frac{|\mathbf{q}|^2}{q_0^2}\delta_{nn'} \ . \tag{13.21}
$$

As a result, the isospin operator changes and for the direct term it is given now by

$$
2\vec{\tau}_{m'm}\vec{\tau}_{\ell'\ell}S(m,Q/2-k)S(\ell,Q/2+k) \ . \tag{13.22}
$$

The projection for a state with $i_3 = \pm 1$ is $2(4I-3)S(\pm 1/2, Q/2-k)S(\pm 1/2, Q/2+k)$, instead of $2i_3 S(\pm 1/2, Q/2-k)S(\pm 1/2, Q/2+k)$ previously introduced for T_{12}. For $\alpha_3 = 0$ there is now a contribution from eq.(13.22) and it is given by

$$
2(4I-3)\frac{1}{2}\left[S(+1/2,Q/2-k)S(-1/2,Q/2+k) + S(-1/2,Q/2-k)S(+1/2,Q/2+k)\right] \ . \tag{13.23}
$$

Hence, instead of eqs.(13.10) and (13.11) one has now, respectively,

$$
\begin{aligned}
T_{13,d}^{f,S=0} &= i(4I-3)\frac{g_A^2}{f^2}\frac{\mathbf{q}^2}{q_0^2}\delta_{ij}\left(\frac{g_A}{2f}\right)^2(C_S-3C_T)\int\frac{d^4k}{(2\pi)^4}\frac{\mathbf{r}^2}{\mathbf{r}^2+m_\pi^2} \\
&\times\frac{1}{2}\left[S(m,Q/2-k)S(\ell,Q/2+k)+S(\ell,Q/2-k)S(m,Q/2+k)\right] \\
T_{13,d}^{f,S=1}(s_3',s_3) &= -i(4I-3)\frac{g_A^2}{f^2}\frac{\mathbf{q}^2}{q_0^2}\delta_{ij}\left(\frac{g_A}{2f}\right)^2(C_S+C_T)\int\frac{d^4k}{(2\pi)^4}\frac{1}{\mathbf{r}^2+m_\pi^2}B_{s_3',s_3} \\
&\times\frac{1}{2}\left[S(m,Q/2-k)S(\ell,Q/2+k)+S(\ell,Q/2-k)S(m,Q/2+k)\right]~, \qquad (13.24)
\end{aligned}
$$

such that $m=\ell=\pm 1/2$ for $i_3=\pm 1$ and $m=+1/2$, $\ell=-1/2$ for $i_3=0$. Due to their isoscalar character we have included the subscript iss. We now have instead of eqs.(13.19) and (13.20)

$$
\begin{aligned}
T_{13,d}^{f,S=0} &= (4I-3)\frac{g_A^2}{f^2}\frac{\mathbf{q}^2}{q_0^2}\left(\frac{g_A}{2f}\right)^2(C_S-3C_T)(\widetilde{L}_{10}-m_\pi^2\widetilde{L}_{11})~, \\
T_{13,d}^{f,S=1}(1,1) &= -(4I-3)\frac{g_A^2}{f^2}\frac{\mathbf{q}^2}{q_0^2}\left(\frac{g_A}{2f}\right)^2(C_S+C_T)\left[{p_3'}^2\left(\widetilde{L}_{11}+2\widetilde{L}_{11}^{p}+\widetilde{L}_{11}^{Tp}\right)+\alpha_3^2\widetilde{L}_{11}^{T\alpha}\right. \\
&\left.+\,2\alpha_3 p_3'\left(\widetilde{L}_{11}^{\alpha}+\widetilde{L}_{11}^{T\alpha p}\right)+\widetilde{L}_{11}^{Tg}\right]~, \\
T_{13,d}^{f,S=1}(1,0) &= -(4I-3)\frac{g_A^2}{f^2}\frac{\mathbf{q}^2}{q_0^2}\left(\frac{g_A}{2f}\right)^2\sqrt{2}(C_S+C_T)\left[(\alpha_1-i\alpha_2)\left\{\widetilde{L}_{11}^{T\alpha}\alpha_3+(\widetilde{L}_{11}^{T\alpha p}+\widetilde{L}_{11}^{\alpha})p_3'\right\}\right. \\
&\left.+\,(p_1'-ip_2')\left\{(\widetilde{L}_{11}^{T\alpha p}+\widetilde{L}_{11}^{\alpha})\alpha_3+(\widetilde{L}_{11}^{Tp}+\widetilde{L}_{11}+2\widetilde{L}_{11}^{p})p_3'\right\}\right]~, \\
T_{13,d}^{f,S=1}(1,-1) &= -(4I-3)\frac{g_A^2}{f^2}\frac{\mathbf{q}^2}{q_0^2}\left(\frac{g_A}{2f}\right)^2(C_S+C_T)\left[(\alpha_1-i\alpha_2)^2\widetilde{L}_{11}^{T\alpha}+(p_1'-ip_2')^2(\widetilde{L}_{11}^{Tp}+2\widetilde{L}_{11}^{p}+\widetilde{L}_{11})\right. \\
&\left.+\,2(p_1'-ip_2')(\alpha_1-i\alpha_2)(\widetilde{L}_{11}^{T\alpha p}+\widetilde{L}_{11}^{\alpha})\right]~, \\
T_{13,d}^{f,S=1}(0,1) &= -(4I-3)\frac{g_A^2}{f^2}\frac{\mathbf{q}^2}{q_0^2}\left(\frac{g_A}{2f}\right)^2\sqrt{2}(C_S+C_T)\left[(\alpha_1+i\alpha_2)\left\{\widetilde{L}_{11}^{T\alpha}\alpha_3+(\widetilde{L}_{11}^{T\alpha p}+\widetilde{L}_{11}^{\alpha})p_3'\right\}\right. \\
&\left.+\,(p_1'+ip_2')\left\{(\widetilde{L}_{11}^{T\alpha p}+\widetilde{L}_{11}^{\alpha})\alpha_3+(\widetilde{L}_{11}^{Tp}+\widetilde{L}_{11}+2\widetilde{L}_{11}^{p})p_3'\right\}\right]~, \\
T_{13,d}^{f,S=1}(0,0) &= -(4I-3)\frac{g_A^2}{f^2}\frac{\mathbf{q}^2}{q_0^2}\left(\frac{g_A}{2f}\right)^2(C_S+C_T)\left[\widetilde{L}_{10}-m_\pi^2\widetilde{L}_{11}-2\widetilde{L}_{11}^{Tg}-2{\alpha_3}^2\widetilde{L}_{11}^{T\alpha}\right. \\
&\left.-\,2{p_3'}^2(\widetilde{L}_{11}+\widetilde{L}_{11}^{Tp}+2\widetilde{L}_{11}^{p})-4\alpha_3 p_3'(\widetilde{L}_{11}^{Tp}+\widetilde{L}_{11}^{\alpha})\right]~, \\
T_{13,d}^{f,S=1}(-1,1) &= -(4I-3)\frac{g_A^2}{f^2}\frac{\mathbf{q}^2}{q_0^2}\left(\frac{g_A}{2f}\right)^2(C_S+C_T)\left[(\alpha_1+i\alpha_2)^2\widetilde{L}_{11}^{T\alpha}+(p_1'+ip_2')^2(\widetilde{L}_{11}^{Tp}+2\widetilde{L}_{11}^{p}+\widetilde{L}_{11})\right. \\
&\left.+\,2(p_1'+ip_2')(\alpha_1+i\alpha_2)(\widetilde{L}_{11}^{T\alpha p}+\widetilde{L}_{11}^{\alpha})\right]~, \\
T_{13,d}^{f,S=1}(0,-1) &= -T_{13,d}^{f,S=1}(1,0)~, \\
T_{13,d}^{f,S=1}(-1,0) &= -T_{13,d}^{f,S=1}(0,1)~, \\
T_{13,d}^{f,S=1}(-1,-1) &= T_{13,d}^{f,S=1}(1,1)~. \qquad (13.25)
\end{aligned}
$$

Where the tilde indicates the symmetric form

$$\widetilde{L}_{ij}^{ab\ldots} = \frac{1}{2}\left(L_{ij}^{ab\ldots}(m,\ell) + L_{ij}^{ab\ldots}(\ell,m)\right) \ , \tag{13.26}$$

with m and ℓ given in terms of α_3 as explained above.

We now proceed to the partial wave projection of $T_{12}^S = T_{12}^{f,S} + T_{12}^{i,S}$ and $T_{13}^S = T_{13}^{f,S} + T_{13}^{i,S}$. As discussed in the clarifying remark at the end of Appendix B, we can still use eq. (B.31), valid in the vacuum, for T_{12}^f and T_{13}^f though we are evaluating transition amplitudes in the nuclear medium. For T_{12}^i and T_{13}^i it was also established that the same partial waves as for T_{12}^f and T_{13}^f result with the exchange $\ell \leftrightarrow \ell'$. We denote the corresponding partial waves by $\mathcal{T}_{12;JI}^f(\ell,\bar{\ell},S)$ and $\mathcal{T}_{13;JI}^f(\ell,\bar{\ell},S)$. To simplify the notation we omit the subscript 12 and 13. Then,

$$\begin{aligned}
\mathcal{T}_{\ell 1}^f(\ell,\ell,0) &= \frac{Y_\ell^0(\hat{\mathbf{z}})}{2\ell+1}\int d\hat{\mathbf{p}}' T_d^{S=0} Y_\ell^0(\hat{\mathbf{p}}')^* \ , \\
\mathcal{T}_{J1}^f(\ell,\bar{\ell},1) &= \frac{Y_{\bar{\ell}}^0(\hat{\mathbf{z}})}{2J+1}\left\{\int d\hat{\mathbf{p}}' Y_\ell^0(\hat{\mathbf{p}}')\left[T_d^{S=1}(0,0)(000|\bar{\ell}1J)(000|\ell 1J)\right.\right. \\
&\quad \left. + \left(T_d^{S=1}(+1,+1) + T_d^{S=1}(-1,-1)\right)(011|\bar{\ell}1J)(011|\ell 1J)\right] \\
&\quad - \int d\hat{\mathbf{p}}'\left(Y_\ell^{-1}(\hat{\mathbf{p}}')T_d^{S=1}(-1,0) + Y_\ell^1(\hat{\mathbf{p}}')T_d^{S=1}(+1,0)\right)(000|\bar{\ell}1J)(1-10|\ell 1J) \\
&\quad - \int d\hat{\mathbf{p}}'\left(Y_\ell^{-1}(\hat{\mathbf{p}}')T_d^{S=1}(0,+1) + Y_\ell^1(\hat{\mathbf{p}}')T_d^{S=1}(0,-1)\right)(011|\bar{\ell}1J)(101|\ell 1J) \\
&\quad \left. + \int d\hat{\mathbf{p}}'\left(Y_\ell^{-2}(\hat{\mathbf{p}}')T_d^{S=1}(-1,+1) + Y_\ell^2(\hat{\mathbf{p}}')T_d^{S=1}(+1,-1)\right)(011|\bar{\ell}1J)(2-11|\ell 1J)\right\} \ .
\end{aligned} \tag{13.27}$$

Let us recall that in order to apply eq.(13.27) the vector $\mathbf{p}$ must be taken along the z-axis. Namely, for the partial wave projections, we take the reference frame with axes

$$\begin{aligned}
\hat{\mathbf{z}} &= \hat{\mathbf{p}} \ , \\
\hat{\mathbf{x}} &= \frac{\hat{\mathbf{p}} \times \hat{\alpha}}{\sin\beta} \ , \\
\hat{\mathbf{y}} &= \frac{\hat{\mathbf{p}} \times (\hat{\mathbf{p}} \times \hat{\alpha})}{\sin\beta} = \hat{\mathbf{p}}\,\mathrm{ctg}\beta - \hat{\alpha}\,\mathrm{csec}\beta \ .
\end{aligned} \tag{13.28}$$

As follows from the discussion at the end of Appendix B for T_{12}, that requires $I=1$, only the partial wave $\mathcal{T}_{12;01}(0,0,0) = 2\mathcal{T}_{12;01}^f(0,0,0)$ is not zero. On the other hand, for T_{13} both isospin combinations occur. Then one has the partial waves $\mathcal{T}_{13;01}(0,0,0) = 2\mathcal{T}_{13;01}^f(0,0,0)$, $\mathcal{T}_{13;10}(0,0,1) = 2\mathcal{T}_{13;10}^f(0,0,1)$, $\mathcal{T}_{13;10}(2,0,1) = \mathcal{T}_{13;10}^f(2,0,1)$ and $\mathcal{T}_{13;10}(0,2,1) = \mathcal{T}_{13;10}^f(2,0,1)$.

14 Explicit calculation of T_{14} and T_{15}

Let us move to now to the evaluation of T_{14} and T_{15} where both vertices in the two-nucleon reducible loop at which the two pions are attached correspond to one-pion exchange. In fig.29 a specific arrange of pion four-momenta and labels is shown for this case. As in the previous cases we start by calculating the isovector contribution. We restrict from the beginning to the direct term, corresponding to the diagrams in fig.30, the one that is finally used when evaluating the associated partial wave amplitudes.

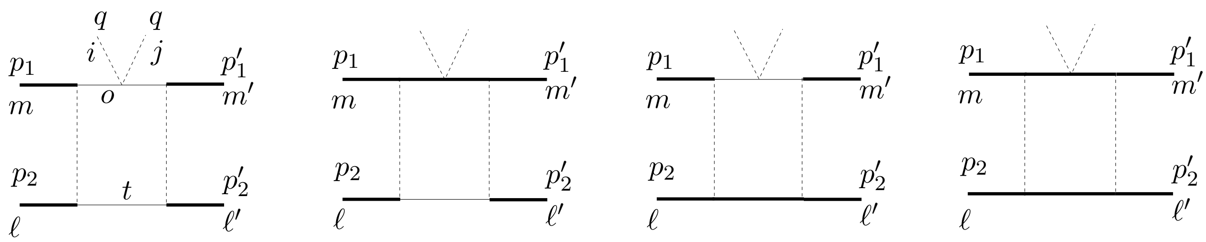

Figure 29: Two-nucleon reducible loop with two one-pion exchange vertices between the initial and final nucleons. The free part of the in-medium nucleon propagator in eq.(3.2) is indicated by a thin line while the in-medium part, proportional to the Dirac delta function, is denoted by a thick line.

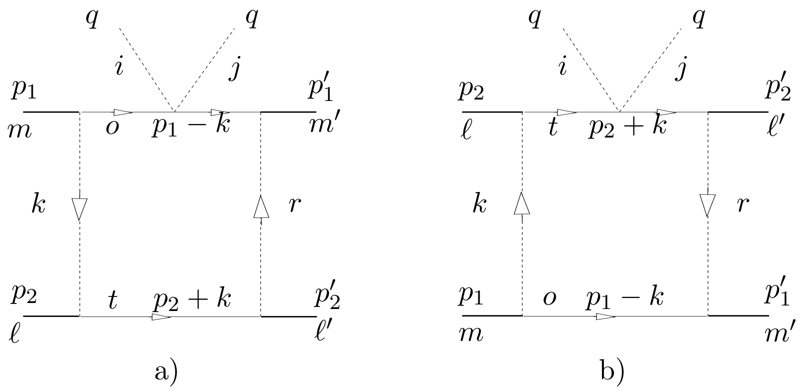

Figure 30: The internal four-momenta and discrete indices are indicated on the two figures whose sum determines $T_{14,d}$. Note that the pion labels and four-momenta are exchanged for the initial and final states separately between the two figures. The one on the left is $T^a_{14,d}$ and that on the right is $T^b_{14,d}$.

For the diagram of fig.30a, denoted by $T^a_{14,d}$, one has

$$T^a_{14,d} = -\left(\frac{g_A}{2f}\right)^4 \frac{\kappa q^0 \varepsilon_{ij3}}{2f^2} \tau^a_{m'o}\tau^c_{om}\tau^3_{oo}\tau^a_{\ell' t}\tau^c_{t\ell} \frac{\partial}{\partial p_1^0} \int \frac{d^4k}{(2\pi)^4} (\vec{\sigma}\cdot\mathbf{r})_{\alpha_{\ell'}\beta}(\vec{\sigma}\cdot\mathbf{r})_{\alpha_{m'}\alpha}(\vec{\sigma}\cdot\mathbf{k})_{\alpha\alpha_m}(\vec{\sigma}\cdot\mathbf{k})_{\beta\alpha_\ell}$$
$$\times \frac{1}{\mathbf{r}^2+m_\pi^2}\frac{1}{\mathbf{k}^2+m_\pi^2} S(o,p_1-k)S(t,p_2+k)~. \tag{14.1}$$

Where $r = p_1' - p_1 + k$ and we have also neglected energy dependences in the pion propagators since they are of two chiral orders higher. $T^b_{14,d}$, corresponding to fig.30b, reads

$$T^b_{14,d} = -\left(\frac{g_A}{2f}\right)^4 \frac{\kappa q^0 \varepsilon_{ij3}}{2f^2} \tau^a_{\ell' t}\tau^c_{t\ell}\tau^3_{tt}\tau^a_{\ell' o}\tau^c_{o\ell} \frac{\partial}{\partial p_2^0} \int \frac{d^4k}{(2\pi)^4} (\vec{\sigma}\cdot\mathbf{r})_{\alpha_{\ell'}\beta}(\vec{\sigma}\cdot\mathbf{r})_{\alpha_{m'}\alpha}(\vec{\sigma}\cdot\mathbf{k})_{\alpha\alpha_m}(\vec{\sigma}\cdot\mathbf{k})_{\beta\alpha_\ell}$$
$$\times \frac{1}{\mathbf{r}^2+m_\pi^2}\frac{1}{\mathbf{k}^2+m_\pi^2} S(o,p_1-k)S(t,p_2+k)~. \tag{14.2}$$

The energy dependence on p_1^0 and p_2^0 enters into these integrals only through the variable A, similarly as for L_{10}, and we apply eq.(11.14). In this way, $T^b_{14,d}$ is given by the same expression as $T^a_{14,d}$ with the exchanges $\ell \leftrightarrow m$ and $\ell' \leftrightarrow m'$. Their sum is

$$
\begin{aligned}
T_{14,d} &= T^a_{14,d} + T^b_{14,d} \\
&= -\left(\frac{g_A}{2f}\right)^4 \frac{\kappa q^0 \varepsilon_{ij3}}{2f^2} \left(\tau^a_{m'o}\tau^c_{om}\tau^3_{oo}\tau^a_{\ell' t}\tau^c_{t\ell} + \tau^a_{\ell' t}\tau^c_{t\ell}\tau^3_{tt}\tau^a_{m'o}\tau^c_{om}\right) \frac{m\partial}{\partial A} \int \frac{d^4k}{(2\pi)^4} (\vec{\sigma}_{\alpha_{\ell'}\beta}\cdot \mathbf{r})(\vec{\sigma}_{\alpha_{m'}\alpha}\cdot \mathbf{r}) \\
&\times (\vec{\sigma}_{\alpha\alpha_m}\cdot \mathbf{k})(\vec{\sigma}_{\beta\alpha_\ell}\cdot \mathbf{k}) \frac{1}{\mathbf{r}^2+m_\pi^2}\frac{1}{\mathbf{k}^2+m_\pi^2} S(o,p_1-k)S(t,p_2+k) \ . \qquad (14.3)
\end{aligned}
$$

We now evaluate the diagonal matrix elements of the isospin operator

$$
\left(\tau^a_{m'o}\tau^c_{om}\tau^3_{oo}\tau^a_{\ell' t}\tau^c_{t\ell} + \tau^a_{\ell' t}\tau^c_{t\ell}\tau^3_{tt}\tau^a_{m'o}\tau^c_{om}\right) S(o,p_1-k)S(t,p_2+k) \ , \qquad (14.4)
$$

present in eq.(14.3), between states with well defined isospin. For $i_3=\pm 1$ then $\ell=\ell'=m=m'=o=t=\pm 1/2$ and the result is

$$
i_3=\pm 1 \ , \ 2i_3\, S(\pm\frac{1}{2},p_1-k)S(\pm\frac{1}{2},p_2+k) \ . \qquad (14.5)
$$

For $i_3=0$ it is required by charge conservation that $o+t=0$, so that o and t have different sign. Working out explicitly the matrix elements for $I=1$, and 0 one has,

$$
\begin{aligned}
&\sum_{a,c}\left\{\tau^a_{1o}\tau^3_{oo}\tau^c_{o1}\tau^a_{-1t}\tau^c_{t-1} \pm \tau^a_{1o}\tau^3_{oo}\tau^c_{o-1}\tau^a_{-1t}\tau^c_{t1} \pm \tau^a_{-1o}\tau^3_{oo}\tau^c_{o1}\tau^a_{1t}\tau^c_{t-1} + \tau^a_{-1o}\tau^3_{oo}\tau^c_{o-1}\tau^a_{1t}\tau^c_{t1} + (t\leftrightarrow o)\right\} \\
&\times \frac{1}{2} S(o,p_1-k)S(t,p_2+k) \ . \qquad (14.6)
\end{aligned}
$$

Let us stress that here o and t are fixed. It is straightforward to show that the previous equation is zero for $o=\pm 1/2=-t$. Thus, for $i_3=0$ eq.(14.3) is zero. Then we can write $2i_3$ for the needed expectation values of the isospin operator in eq.(14.4).

Regarding to spin we can rewrite,

$$
\begin{aligned}
&(\vec{\sigma}\cdot\mathbf{r})_{\alpha_{\ell'}\beta}(\vec{\sigma}\cdot\mathbf{r})_{\alpha_{m'}\alpha}(\vec{\sigma}\cdot\mathbf{k})_{\alpha\alpha_m}(\vec{\sigma}\cdot\mathbf{k})_{\beta\alpha_\ell} = (\mathbf{r}\cdot\mathbf{k})^2\delta_{\alpha_{\ell'}\alpha_\ell}\delta_{\alpha_{m'}\alpha_m} - \left[(\mathbf{r}\times\mathbf{k})\cdot\vec{\sigma}_{\alpha_{\ell'}\alpha_\ell}\right]\left[(\mathbf{r}\times\mathbf{k})\cdot\vec{\sigma}_{\alpha_{m'}\alpha_m}\right] \\
&+ i(\mathbf{r}\times\mathbf{k})\cdot\vec{\sigma}_{\alpha_{\ell'}\alpha_\ell}(\mathbf{r}\cdot\mathbf{k})\delta_{\alpha_{m'}\alpha_m} + i(\mathbf{r}\times\mathbf{k})\cdot\vec{\sigma}_{\alpha_{m'}\alpha_m}(\mathbf{r}\cdot\mathbf{k})\delta_{\alpha_{\ell'}\alpha_\ell} \ . \qquad (14.7)
\end{aligned}
$$

The matrix elements of the spin operators $\delta_{\alpha_{m'}\alpha_m}\delta_{\alpha_{\ell'}\alpha_\ell}$ and $(\vec{\sigma}_{\alpha_{m'}\alpha_m}\cdot\mathbf{v})(\vec{\sigma}_{\alpha_{\ell'}\alpha_\ell}\cdot\mathbf{v})$ between states with well defined total spin were already worked in eqs.(12.3) and (13.9), respectively. We have now in addition the operator

$$
\left(\delta_{\alpha_{m'}\alpha_m}\vec{\sigma}_{\alpha_{\ell'}\alpha_\ell} + \delta_{\alpha_{\ell'}\alpha_\ell}\vec{\sigma}_{\alpha_{m'}\alpha_m}\right)\cdot\mathbf{v} \ , \qquad (14.8)
$$

which in matrix notation is

$$
(\vec{\sigma}_1\otimes I_2 + I_1\otimes\vec{\sigma}_2)\cdot\mathbf{v} \ . \qquad (14.9)
$$

Its matrix elements are

$$
\left(\begin{array}{r|rrr}
 & -1 & 0 & +1 \\
\hline
-1 & -2v_3 & \sqrt{2}(v_1+iv_2) & 0 \\
0 & \sqrt{2}(v_1-iv_2) & 0 & \sqrt{2}(v_1+iv_2) \\
+1 & 0 & \sqrt{2}(v_1-iv_2) & 2v_3
\end{array}\right) \ , \qquad (14.10)
$$

and 0 for $S=0$. We follow in the previous matrix the same notation as in eq.(13.9). Taking into account these matrix elements and those from eqs.(12.3) and (13.9) one has for the operator of eq.(14.7):

$$
\begin{aligned}
S=0\rightarrow S=0 &: \ \mathbf{r}^2\,\mathbf{k}^2 ~,\\
S=1\rightarrow S=1 &: \\
+1\rightarrow +1 &: \ \left[\mathbf{r}\cdot\mathbf{k}+i\,(\mathbf{r}\times\mathbf{k})_3\right]^2 ~,\\
-1\rightarrow -1 &: \ \left[\mathbf{r}\cdot\mathbf{k}-i\,(\mathbf{r}\times\mathbf{k})_3\right]^2 ~,\\
0\rightarrow 0 &: \ 2(\mathbf{r}\cdot\mathbf{k})^2-\mathbf{r}^2\mathbf{k}^2+2[(\mathbf{r}\times\mathbf{k})_3]^2 ~,\\
+1\rightarrow 0 &: \ -\sqrt{2}\,[(\mathbf{r}\times\mathbf{k})_1+i\,(\mathbf{r}\times\mathbf{k})_2]\,[(\mathbf{r}\times\mathbf{k})_3-i\,\mathbf{r}\cdot\mathbf{k}] ~,\\
0\rightarrow +1 &: \ -\sqrt{2}\,[(\mathbf{r}\times\mathbf{k})_1-i\,(\mathbf{r}\times\mathbf{k})_2]\,[(\mathbf{r}\times\mathbf{k})_3-i\,\mathbf{r}\cdot\mathbf{k}] ~,\\
+1\rightarrow -1 &: \ -\left[(\mathbf{r}\times\mathbf{k})_1+i\,(\mathbf{r}\times\mathbf{k})_2\right]^2 ~,\\
-1\rightarrow +1 &: \ -\left[(\mathbf{r}\times\mathbf{k})_1-i\,(\mathbf{r}\times\mathbf{k})_2\right]^2 ~,\\
0\rightarrow -1 &: \ \sqrt{2}\,[(\mathbf{r}\times\mathbf{k})_1+i\,(\mathbf{r}\times\mathbf{k})_2]\,[(\mathbf{r}\times\mathbf{k})_3+i\,\mathbf{r}\cdot\mathbf{k}] ~,\\
-1\rightarrow 0 &: \ \sqrt{2}\,[(\mathbf{r}\times\mathbf{k})_1-i\,(\mathbf{r}\times\mathbf{k})_2]\,[(\mathbf{r}\times\mathbf{k})_3+i\,\mathbf{r}\cdot\mathbf{k}] ~,
\end{aligned}
\tag{14.11}
$$

where the Cartesian coordinates of $\mathbf{r}\times\mathbf{k}$ are indicated with subscripts. Inserting into eq.(14.3) these matrix elements for spin and $2i_3S(\pm1/2)S(\pm1/2)$ for the isospin operator of eq.(14.4), the amplitudes $T_{14,d}^{S=0}$ and $T_{14,d}^{S=1}(s_3',s_3)$ are determined. These amplitudes are then implemented in eq.(13.27), instead of using $T_{12,d}^{S=0,1}$, and the partial waves $\mathcal{T}_{14;JI}(\bar{\ell},\ell,S)$ are evaluated.

The other contribution stems by taking the derivative of the intermediate nucleon propagator in fig.23 with respect to z, as discussed in eq.(11.12). We have the same expression as for $T_{14,d}$, eq.(14.3), but removing the derivative $m\partial/\partial A$ and with the replacement of eq.(13.21). As a consequence the isospin operator in eq.(14.3) changes and now it is given by

$$
2\tau^a_{m'o}\tau^c_{om}\tau^a_{\ell' t}\tau^c_{t\ell}S(o,p_1-k)S(t,p_2+k) ~, \tag{14.12}
$$

One can work out straightforwardly its diagonal matrix elements between states with definite isospin with the result,

$$
2(9-8I)\frac{1}{2}\left\{S(o,p_1-k)S(t,p_2+k)+S(t,p_1-k)S(o,p_2+k)\right\} ~. \tag{14.13}
$$

with $o+t=i_3$. Then, instead of eq.(14.3) one has now,

$$
\begin{aligned}
T^{is}_{15,d} &= i\left(\frac{g_A}{2f}\right)^4\frac{g_A^2}{f^2}\frac{\mathbf{q}^2}{q_0^2}(9-8I)\\
&\times\int\frac{d^4k}{(2\pi)^4}\frac{1}{2}\left\{S(o,p_1-k)S(t,p_2+k)+S(t,p_1-k)S(o,p_2+k)\right\}\\
&\times(\vec{\sigma}_{\alpha_{\ell'}\beta}\cdot\mathbf{r})(\vec{\sigma}_{\alpha_{m'}\alpha}\cdot\mathbf{r})(\vec{\sigma}_{\alpha\alpha_m}\cdot\mathbf{k})(\vec{\sigma}_{\beta\alpha_\ell}\cdot\mathbf{k})\frac{1}{\mathbf{r}^2+m_\pi^2}\frac{1}{\mathbf{k}^2+m_\pi^2} ~,
\end{aligned}
\tag{14.14}
$$

with $o+t=i_3$ as before. Of course, the spin operator is the same as for T_{14} and the results of eq.(14.11) are used again. The partial wave amplitudes $\mathcal{T}^{is}_{15;JI}$ are then determined by employing eq.(13.27) in terms of $T^{is}_{15,d}$.

The tensor integrals required by eq.(14.11) and involving one intermediate two-nucleon state with two one-pion exchanges are calculated in Appendix H.

15 Numerical results

We now discuss here the numerical results that we obtain for the π^- self-energy in the nuclear medium according to our approach. We distinguish between the different contributions. For the case of the π^+ the isovector contributions will have opposite sign to those for the π^- self-energy. For the π^0 they are absent. The isoscalar contributions are the same for all the three charge species of pions.

15.1 Isoscalar S-wave contribution

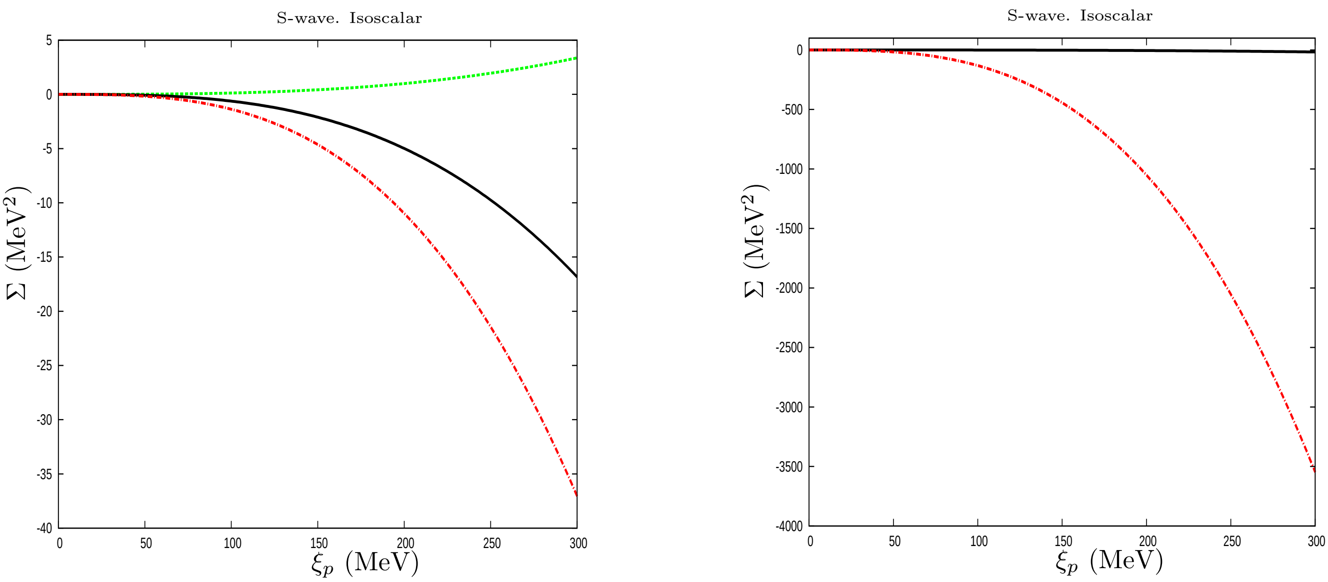

Figure 31: The left panel is the isoscalar contribution Σ_4 for $w = m_\pi$ and $\mathbf{q} = 0$. The solid, dashed and dot-dashed lines correspond to the central value of a^+_{0+}, plus and minus one σ, in that order, according to eq.(15.3). The right panel is the comparison between Σ_4 calculated with the central value of a^+_{0+}, $\mathbf{q} = 0$ and $w = m_\pi$ (solid line) and $w = 1.1 m_\pi$ (dot-dashed line).

The only isoscalar S-wave contribution that we have stems from Σ_4, eq.(6.1), without the term proportional to $c_3\mathbf{q}^2$ which is P-wave. Recall that Σ_4 is already a NLO contribution. It depends on the $\mathcal{O}(p^4)$ CHPT counterterms c_1, c_2 and c_3. It is more accurate to rewrite it in terms of measured quantities. For that, taking into account ref.[53], one has

$$a^+_{0+} = \frac{2m_\pi^2}{f^2}\left(-2c_1 + c_2 + c_3 - \frac{g_A^2}{8m}\right) + \mathcal{O}(p^3) \ . \tag{15.1}$$

Then eq.(6.1) can be expressed as

$$\Sigma_4 = \left[a^+_{0+} + (w^2 - m_\pi^2)\left(\frac{a^+_{0+}}{m_\pi^2} + \frac{4c_1}{f^2}\right) - \frac{2c_3|\mathbf{q}|^2}{f^2}\right](\rho_p + \rho_n)\delta_{ij} \ . \tag{15.2}$$

For the isoscalar S-wave πN scattering length a^+_{0+} we use its recent determination [54]

$$a^+_{0+} = (-0.0010 \pm 0.0012)m_\pi^{-1} \ , \tag{15.3}$$

and we take the value

$$c_1 = (-0.81 \pm 0.12)\ \text{GeV}^{-1}\ , \tag{15.4}$$

from ref.[55]. This value is based on a chiral expansion of the πN scattering amplitude inside the Mandelstam variable where it is better behaved. We show in fig.31 the S-wave part of Σ_4 as a function of ξ_p with $\omega = m_\pi$ and $|\mathbf{q}| = 0$, at the pion threshold. In the following we fix

$$\xi_n = 1.157\xi_p\ , \tag{15.5}$$

with the proportionality factor corresponding to neutron rich nuclei like ${}^{208}_{82}$Pb (that has $\xi_p \simeq 241$ MeV and $\xi_n \simeq 279$ MeV).

The solid, dashed and dot-dashed lines on the left panel of fig.31 correspond to the central value of a^+_{0+}, plus and minus one σ, in order, as given in eq.(15.3). One sees that this contribution is very small and compatible with zero, as it is a^+_{0+}, because for $\omega = m_\pi$ and $|\mathbf{q}| = 0$ the two quantities, Σ_4 and a^+_{0+}, are proportional. At the order of our calculation we cannot address the interesting problem of the missing repulsion having to do with a repulsive contribution in the S-wave isoscalar piece of the pion optical potential [28, 35, 56], needed to fit data from pionic atoms. New contributions to the S-wave isoscalar pion self-energy will emerge in a calculation just one order higher to the one performed here. E.g. the well known Ericson-Ericson Pauli corrected S-wave rescattering term [31] would appear at $\mathcal{O}(p^6)$ because it involves an extra nucleon propagator that is not enhanced. Of course, since a^0_{0+} is so tiny higher order corrections will have impact. Nevertheless, as soon as w^2 departs slightly from m_π^2 in eq. (15.2), the modulus of Σ_4 becomes much larger as shown by the dot-dashed line on the right-hand panel of fig. 31. The solid line is the same as in the left panel. This is an important source for the missing pion repulsion as first noticed in ref.[23]. It will be discussed below in more detail in section 15.5.

15.2 Isovector S-wave contribution

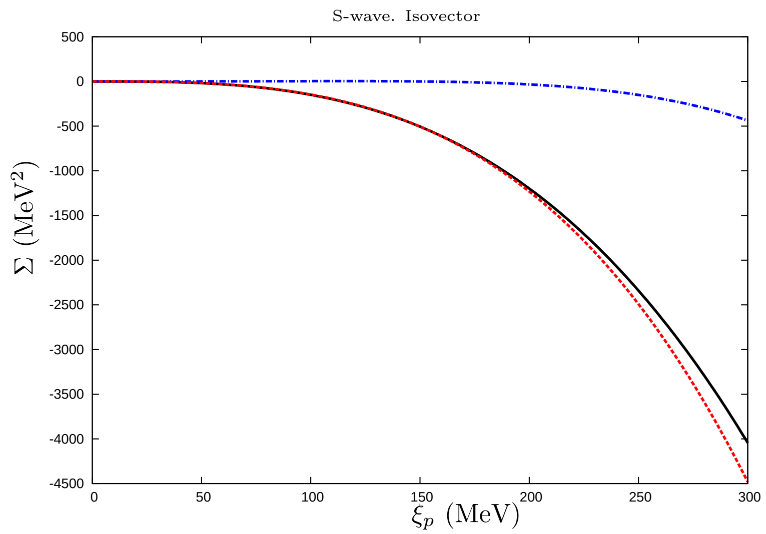

Figure 32: The non-vanishing S-wave isovector self-energy contributions calculated here. The solid line is Σ_1 and the dot-dashed one is Σ_5. The sum of both is given by the dashed line.

The calculated isovector S-wave contribution are Σ_1, eq. (3.5), Σ_5, eq. (7.11), Σ_7, eq.(9.13), and $\Sigma_9 + \Sigma_{10}^{iv}$, eq. (11.22). Of them, the LO contribution is Σ_1, Σ_7 cancels with $\Sigma_9 + \Sigma_{10}^{iv}$ and $\Sigma_5 = \mathcal{O}(p^6)$ or

N^{2}LO. The contributions Σ_1 and and Σ_5 are purely real because they only involve a one nucleon process summed all over the Fermi seas.

Σ_1 and Σ_5 are shown in fig.32 for the π^- self-energy and $\mathbf{q} = 0$. For the π^+ one has the same results but with opposite sign and for the π^0 they are zero. On the figure, Σ_1 is the solid line and Σ_5 is the dot-dashed line. The dashed line is the sum of both contributions and runs very close to Σ_1 because Σ_5 is much smaller than the latter.

15.3 Isoscalar P-wave contribution

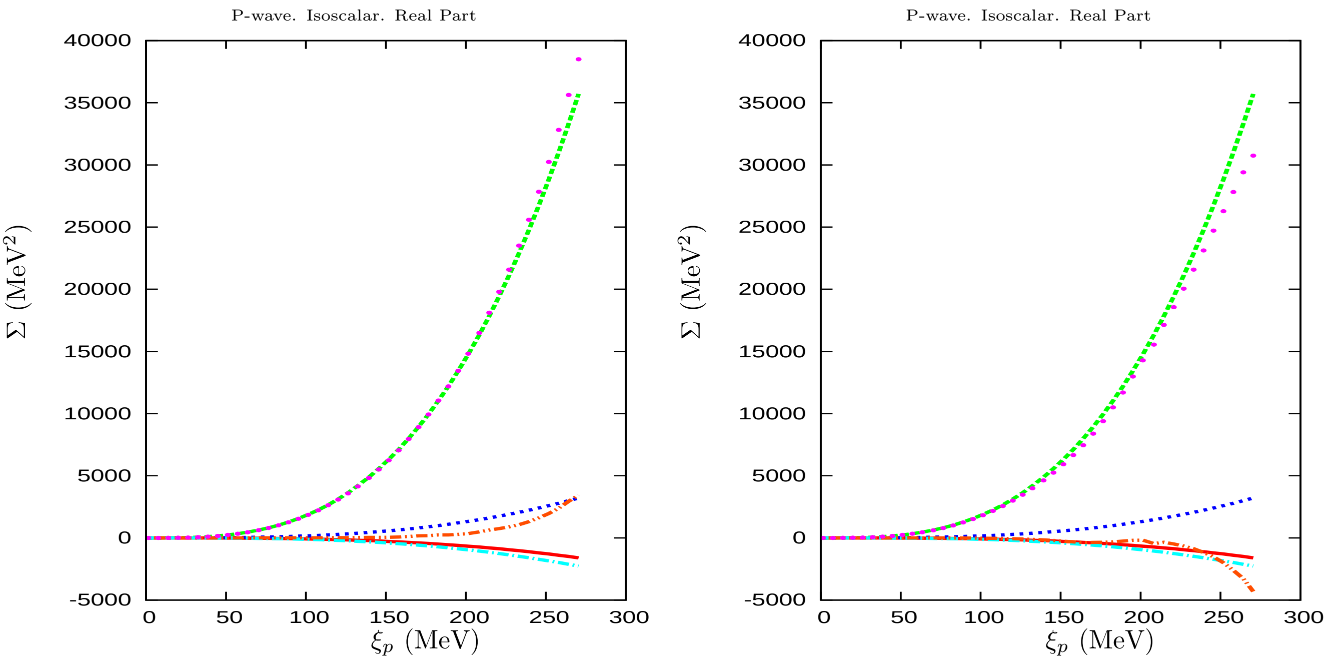

Figure 33: The two panels correspond to the real part of the the different contributions to the P-wave isoscalar pion self-energy for $q^0 = |\mathbf{q}| = m_\pi$. The meaning of the lines is the following: Σ_2^{is} is the solid line, Σ_3 is the downwards dashed line, the P-wave contribution of Σ_4 is given by the upper dashed line, Σ_6^{is} is the dot-dashed line and $\Sigma_8^{is} + \Sigma_{10}^{is}$ correspond to the dashed double-dotted line. The solid circles correspond to the sum of all these contributions. On the panel to the left $\Sigma_8^{is} + \Sigma_{10}^{is}$ is calculated at LO while on the panel to the right it is calculated at NLO.

The different contributions for the real part of the isoscalar P-wave pion self-energy are shown in fig.33: Σ_2^{is} is the solid line, Σ_3 is the bottom dashed line while the upper dashed line is the P-wave contribution of Σ_4, Σ_6^{is} is the dot-dashed line and $\Sigma_8^{is} + \Sigma_{10}^{is}$ correspond to the dashed double-dotted line. On the left panel the leading contribution to $\Sigma_8^{is} + \Sigma_{10}^{is}$ is given while on the panel to the right the latter is calculated one order higher. The sum of all the contributions is indicated by the filled circles and $q^0 = |\mathbf{q}| = m_\pi$ in all the curves. It is clear that the real part of the isoscalar P-wave pion self-energy is overwhelmingly dominated by the P-wave part of Σ_4, that is proportional to c_3, eq. (6.1). In turn, this low-energy constant is dominated by the Δ resonance [60], which stresses the important role played by this resonance in the P-wave pion self-energy [59]. The results presented in this figure have been calculated with the central value of $c_3 = (-4.66 \pm 0.36)$ GeV^{-1} [55]. The imaginary part of the isoscalar P-wave pion self-energy is shown in fig. 34. The only source of an imaginary part is from $\Sigma_8^{is} + \Sigma_{10}^{is}$. The solid line corresponds to its leading order calculation, while the dot-dashed line shows its NLO evaluation. Comparing the $V_\rho = 2$

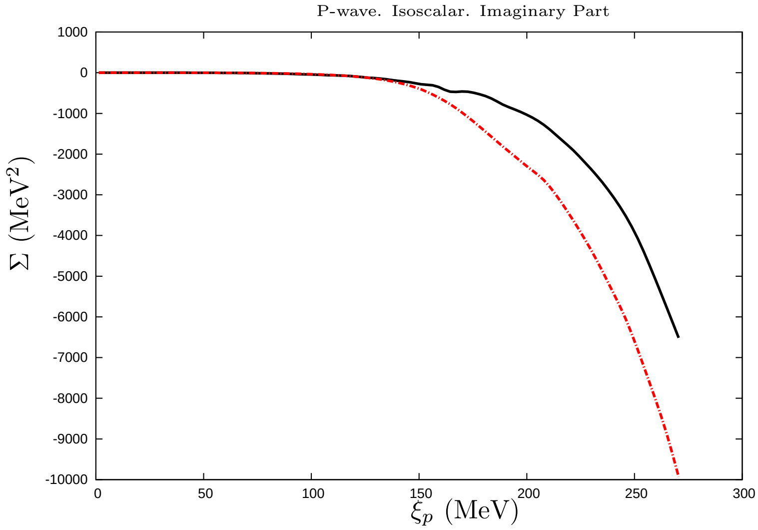

Figure 34: Imaginary part of the P-wave isoscalar pion self-energy for $q^0 = |\mathbf{q}| = m_\pi$. The only source of an imaginary part is $\Sigma_8^{is} + \Sigma_{10}^{is}$. The solid line corresponds to its leading contribution while the dot-dashed line stems from its calculation at NLO.

contributions calculated at LO and at NLO in figures 33 and 34, one observes small corrections up to around $\xi_p \simeq 200$ MeV.

15.4 Isovector P-wave contribution

The isovector P-wave contributions stem only from Σ_2^{iv}, eq. (4.5), and Σ_6^{iv}, eq. (7.14), both being real. Let us recall that the sum of Σ_7, Σ_8^{iv}, Σ_9 and Σ_{10}^{iv} cancel each other [1], as shown in section 11 within UCHPT. We show in fig. 35 by the solid and dot-dashed lines the contributions Σ_2^{iv} and Σ_6^{iv}, respectively. Their sum is given by the dashed line. The curves are calculated for $w = |\mathbf{q}| = m_\pi$. We see that, similar to the case for the isovector S-wave, fig. 32, Σ_6^{iv} is much smaller in modulus than Σ_2^{iv}.

15.5 The π^- mass in nuclear matter

We now discuss the pion mass in the nuclear medium. The latter, denoted by $m_\pi^{\rm eff}$, is defined by

$$(m_\pi^{\rm eff})^2 = m_\pi^2 - {\rm Re}\Sigma(\omega = m_\pi^{eff}, \mathbf{q} = 0) \ . \tag{15.6}$$

It corresponds to the energy at rest of the nuclear pionic modes. When applying this equation in an asymmetric nuclear medium one has to distinguish between the $\pi^\pm$ and π^0 masses. We first discuss the π^- mass, since this is the one involved in pionic atoms. On the other hand, given the relation between the pion self-energy and the pion optical potential (U), with $\Pi = 2wU$, eq. (15.6) can be used to compare our values for the effective pion masses with those stemming from potentials fitted to pionic atom data.

Ref. [57] fitted some terms in the pion-nucleus optical potential to accommodate the values of the energy and width of the deeply bound pionic atoms there discovered. The other terms were taken from fits to the bulk of pionic atom data. In ref. [57], eq. (15.6) was not solved self-consistently but perturbatively so that the equation actually employed is

$$m_\pi^{\rm eff} = m_\pi - \frac{1}{2m_\pi}{\rm Re}\Sigma(\omega = m_\pi, \mathbf{q} = 0) \ , \tag{15.7}$$

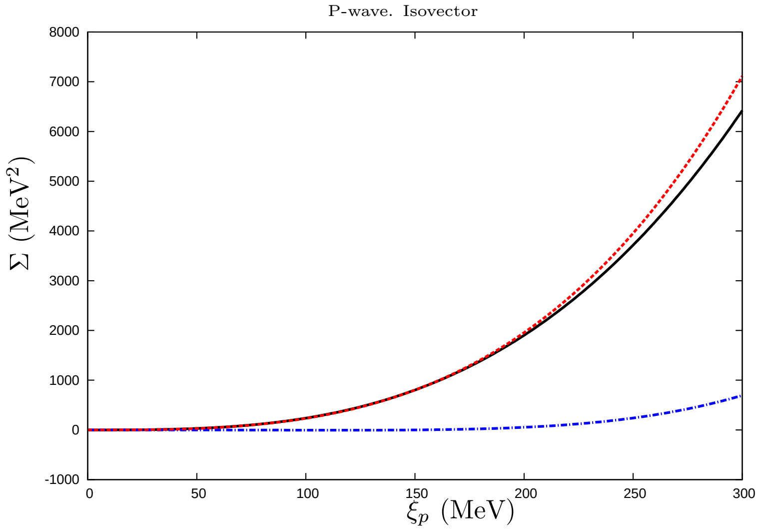

Figure 35: The non-vanishing P-wave isovector self-energy contributions. The solid line is Σ_2^{iv} and the dot-dashed one is Σ_6^{iv}. The sum of both is given by the dashed line.

with the threshold energy of the pion fixed to its vacuum mass. This is the so-called energy-independent approximation for the pion optical potential. Ref. [57] reports

$$\Delta m_\pi \equiv m_\pi^{\rm eff} - m_\pi = 23 - 27 \text{ MeV} . \tag{15.8}$$

We now give a similar discussion on the in-medium pion mass as in ref. [1]. However, we now include in addition Σ_5 and present slightly different results. We first consider eq. (15.7). We have

$$\Delta m_\pi = -\frac{1}{2m_\pi}\text{Re}\left(\Sigma_1 + \Sigma_4 + \Sigma_5\right) , \tag{15.9}$$

with the argument ω in the different Σ_i fixed to m_π. We obtain the result

$$\Delta m_\pi = 8 \text{ MeV} . \tag{15.10}$$

This number is very similar to that of ref. [1] because Σ_5 is much smaller than Σ_1, as shown on the right panel of fig. 32. Furthermore, the contribution from Σ_4 is negligible given the smallness of a_{0+}^+.

It is worth stressing that the energy dependence of the pion optical potential cannot be neglected when studying pionic atoms since this is an important source of repulsion as shown in refs. [58, 35]. When solving the Klein-Gordon equation a proper treatment of the Coulomb potential, $V_c(r)$, requires the argument of the pion self-energy to be $\Sigma(\omega - V_c(r))$ [35], instead of $\Sigma(\omega = m_\pi)$. As a result one should expect a mismatch between a first principle calculation of the pionic potential potential and its parameterization from purely phenomenological studies [57], where the energy is fixed to m_π. In order to take care of this we use eq. (15.7) but evaluated at $\omega = m_\pi + 10$ MeV and $\omega = m_\pi + 20$ MeV, similarly as in ref. [28]. Note that $-V_c(r)$ is ~ 16 MeV at around the nuclear surface and ~ 25 MeV at the center, see e.g. fig.10 of ref. [57]. The resulting values obtained for Δm_π are

$$\begin{aligned} w &= m_\pi + 10 \text{ MeV} , \ \Delta m_\pi = (13.6 \pm 0.7) \text{ MeV} , \\ w &= m_\pi + 20 \text{ MeV} , \ \Delta m_\pi = (19.3 \pm 1.5) \text{ MeV} . \end{aligned} \tag{15.11}$$

As announced, the increase of $\omega > m_\pi$ gives rise to an extra repulsion and then an enhanced π^- mass as compared to eq. (15.10). The leading Weinberg-Tomozawa linear density term experiences an increase of $+(8-9)$ MeV. The main source of the ω-dependence in Δm_π as ω slightly increases in eq. (15.11) is the quadratic term in ω present in Σ_4, eq. (6.1). This term is zero for $\omega = m_\pi$ but $+10$ MeV for $\omega = m_\pi + 20$ MeV, see also the right panel of fig. 31. Here we have used for c_1 the value in eq. (15.4). The errors quoted in eq. (15.11) are indeed given by the error in c_1 since the contribution from the uncertainty in a^+_{0+} is negligible.

In ref.[34] the Ericson-Ericson rescattering term was estimated to contribute an extra +6 MeV to Δ_π. If this N^{2}LO piece is added to the result in the second line of eq. (15.11) from our NLO calculation, then one would obtain $\Delta m_\pi \sim 25$ MeV, in agreement with the result of ref. [57]. However, the full rescattering model of ref. [28], where the in-medium isovector amplitude is used in the rescattering, obtains a significant reduction of the Ericson-Ericson repulsion or even an attraction. From our side a full N^{2}LO calculation is mandatory. The self-consistent solution of eq. (15.6) gives rise to results very similar to those with $\omega = m_\pi + 20$ MeV in eq. (15.11). In order to distinguish to the previous case we denote this quantity by $\Delta m_\pi^\star$. Self-consistency leads to an algebraic equation in ω (note that Σ_5 is linear in ω)

$$(\omega^2 - m_\pi^2)\left[1 + \left(\frac{4c_1}{f^2} + \frac{a^+_{0+}}{m_\pi^2}\right)(\rho_p+\rho_n)\right] + \omega\left[\frac{\rho_p-\rho_n}{2f^2} + \frac{\Sigma_5}{\omega}\right] + a^+_{0+}(\rho_p+\rho_n) = 0 \;. \tag{15.12}$$

Solving it one has

$$\Delta m_\pi^\star = (16 \pm 2)\ \text{MeV} \;, \tag{15.13}$$

with the error bar due to that of c_1. Eq (15.12) can be solved to a good approximation in terms of $\delta\omega^2 = \omega^2 - m_\pi^2$, as it was done in ref. [23]. In this way it is clear why the previous result is around a factor two larger than the perturbative solution in eq. (15.10). With $\omega = \sqrt{\omega^2} \simeq m_\pi + \delta\omega^2/2m_\pi$, neglecting terms of order $\delta\omega^2/m_\pi^2 \ll 1$, the solution is approximately given by

$$\begin{aligned}\delta\omega^2 &= \frac{-a^+_{0+}(\rho_p+\rho_n) + m_\pi\left[(\rho_n-\rho_p)/2f^2 - \Sigma_5/\omega\right]}{1 + \left(4c_1/f^2 + a^+_{0+}/m_\pi^2\right)(\rho_p+\rho_n) + (\rho_p-\rho_n)/4f^2 m_\pi + \Sigma_5/\omega 2m_\pi} \;, \\ \frac{\delta\omega^2}{2m_\pi} &= (17\pm 2)\ \text{MeV} \;,\end{aligned} \tag{15.14}$$

notice that Σ_5/ω, eq. (7.11), is a constant independent of ω. The denominator in eq. (15.14) equation is around 0.5 instead of 1, where the correction is dominated by the term proportional to c_1 with $4c_1(\rho_p + \rho_n)/f^2 = -0.46$. This denominator corresponds to the square of the wave function renormalization of pions in the medium [23], and it is also the major source for dressing the pion decay constant in a nuclear environment [23]. Its importance for the study of the pion mass in the medium was first shown in ref. [23]. For the π^+ we have the shift $\Delta m_\pi^\star = (-14.4 \pm 1.7)$ MeV. The shift in the π^0 mass is negligible at this order.

16 Conclusions

We have derived a promising scheme for an EFT in the nuclear medium based on a chiral power counting that combines both short-range and pion-mediated internucleon interactions. The power counting is bound from below and at a given order it requires to calculate a finite number of contributions, which typically implies the resummation of an infinite string of two-nucleon reducible diagrams with the leading multi-nucleon CHPT amplitudes. These resummations arise because this power counting takes into account from the onset the presence of enhanced nucleon propagators and it can also be applied to multi-nucleon forces. We have developed the required non-perturbative techniques that perform these resummations both in scattering as well as in production processes. This non-perturbative method is based on Unitary CHPT, which is adapted now to the nuclear medium by implementing the new power counting. Using these non-perturbative techniques we have first calculated the LO and NLO vacuum nucleon-nucleon interactions. For the in-medium case the LO nucleon-nucleon scattering is given as well. Then, the pion self-energy in nuclear matter up to $\mathcal{O}(p^5)$ was determined together with some other contributions at N^{2}LO. The latter are calculated to further illustrate the application of these non-perturbative techniques to non-trivial calculations and for first studies on the issue of the size of higher orders corrections. In particular, the resulting NLO nucleon-nucleon contribution to the isoscalar P-wave in-medium pion self-energy is of similar size to other NLO contributions obtained by closing meson-baryon diagrams, and larger than the N^{2}LO nucleon-nucleon one for Fermi momenta up to around 200 MeV. The cancellation between all leading corrections to the linear density approximation for the pion self-energy is explicitly shown here for the amplitudes calculated with the non-perturbative methods developed. This cancellation also affects some other N^{2}LO contributions and this is a good check for the consistency of the full approach. A complete $\mathcal{O}(p^6)$ (or N^{2}LO) calculation of the pion self-energy is a very interesting task and is underway employing the present techniques. It will merge important meson-baryon mechanisms like e.g. the Ericson-Ericson-Pauli rescattering effect [31], with novel multi-nucleon contributions that can be worked out systematically within our EFT. More calculations and applications of the present theory to other interesting physical problems should be pursued.

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Appendices

A Vertices

Figure 36: One and two pion-nucleon vertices.

1. Two-nucleon and one-pion vertex, fig.36a.

From $\mathcal{L}_{\pi N}^{(1)}$:

$$\frac{i g_A}{2f} \vec{\sigma} \cdot \mathbf{q} \, \vec{\tau} \cdot \vec{\pi} \ . \tag{A.1}$$

From $\mathcal{L}_{\pi N}^{(2)}$:

$$\frac{-i g_A}{4Mf} q^0 \, \vec{\sigma} \cdot (\mathbf{p} + \mathbf{p}') \, \vec{\tau} \cdot \vec{\pi} \ . \tag{A.2}$$

2. Two-nucleon and two-pion vertex, fig.36b.

From $\mathcal{L}_{\pi N}^{(1)}$:

$$\frac{-i}{4f^2} \varepsilon_{\ell m k} \tau^k (q_1^0 + q_2^0) \ . \tag{A.3}$$

From $\mathcal{L}_{\pi N}^{(2)}$:

$$\frac{i \varepsilon_{\ell m k} \tau^k}{2 m f^2} \mathbf{p} \cdot \mathbf{q}_2 - \frac{4 c_1 m_\pi^2}{f^2} \delta_{\ell m} + 2(c_2 - \frac{g_A^2}{8M}) \frac{q_2^{0\,2}}{f^2} \delta_{\ell m} + \frac{2 c_3}{f^2} q_2^2 \delta_{\ell m} \ . \tag{A.4}$$

We have not shown the contribution from c_5 because it violates isospin symmetry.

B Partial wave decomposition of nucleon-nucleon amplitudes

We now derive the partial wave decomposition of the nucleon-nucleon scattering amplitudes in the CM frame. Our states are normalized as,

$$\begin{aligned}
&1 - \text{particle state: } \langle \mathbf{p}', j | \mathbf{p}, i \rangle = \delta_{ij} (2\pi)^3 \delta(\mathbf{p}' - \mathbf{p}) \\
&2 - \text{particle state: } \langle \mathbf{p}', j_1 j_2 | \mathbf{p}, i_1 i_2 \rangle = \delta_{j_1 i_1} \delta_{j_2 i_2} (2\pi)^4 \delta(\mathcal{P}_f - \mathcal{P}_i) \frac{4\pi^2 W}{p E_1 E_2} \delta(\Omega - \Omega') \ ,
\end{aligned} \tag{B.1}$$

Here, $\mathcal{P}_f$ corresponds to the total four-momentum of the final state and $\mathcal{P}_i$ to that of the initial one, with $W = \mathcal{P}_i^0 = \mathcal{P}_f^0$, the total CM energy. In addition, E_1 and E_2 are the energies of the particles 1 and 2, in order. The indices i and j refer to any discrete quantum number used to characterize the states. The solid angle in the CM frame is denoted by Ω. Finally, $p = |\mathbf{p}|$ is the modulus of the three-momentum in the CM frame. The two-particle states with well defined orbital angular momentum are defined as,

$$|\ell m, i_1, i_2\rangle = \frac{1}{\sqrt{4\pi}} \int d\hat{\mathbf{p}} \, Y_\ell^m(\hat{\mathbf{p}})^* |\mathbf{p}, i_1 i_2\rangle \ . \tag{B.2}$$

Taking into account eq.(B.1) it follows then

$$\langle \ell', m', j_1 j_2 | \ell, m, i_1 i_2 \rangle = \frac{\pi W}{p E_1 E_2} \delta_{\ell' \ell} \delta_{m' m} \delta_{j_1 i_1} \delta_{j_2 i_2} \ . \tag{B.3}$$

The decomposition in states with well defined total angular momentum J, third component μ, orbital angular momentum ℓ and total spin S is given by,

$$|\mathbf{p}, \sigma_1 \sigma_2\rangle = \sqrt{4\pi} \sum_{J,S,\ell,m} (\sigma_1 \sigma_2 s_3 | s_1 s_2 S)(m_3 s_3 \mu | \ell S J) Y_\ell^m(\hat{\mathbf{p}})^* | J \mu \ell S s_1 s_2 \rangle \ , \tag{B.4}$$

Where the indices σ_1 and σ_2 refer to the third components of spin, and s_1 and s_2 to their maximum values and m to the third component of the orbital angular momentum ℓ. Now, we introduce the isospin indices

α_1, α_2, for the third components, τ_1, τ_2, for their maximum values, and decompose the free state in terms of states that have well defined total isospin I and third component α_3. In addition, the antisymmetric nature of a two fermion state is introduced, as corresponds to a nucleon-nucleon state.

$$\frac{1}{\sqrt{2}}\left(|\mathbf{p},\sigma_1\alpha_1\sigma_2\alpha_2\rangle - |-\mathbf{p},\sigma_2\alpha_2\sigma_1\alpha_1\rangle\right) = \sqrt{4\pi}\sum\{(\sigma_1\sigma_2 s_3|s_1 s_2 S)(m s_3 \mu|\ell S J)(\alpha_1\alpha_2\alpha_3|\tau_1\tau_2 I)$$
$$\times\, Y_\ell^m(\hat{\mathbf{p}})^*|J\mu\ell S s_3 I\alpha_3\rangle - (\sigma_2\sigma_1 s_3|s_2 s_1 S)(m s_3\mu|\ell S J)(\alpha_2\alpha_1\alpha_3|\tau_2\tau_1 I)Y_\ell^m(-\hat{\mathbf{p}})^*|J\mu\ell S s_3 I\alpha_3\rangle\}\ , \tag{B.5}$$

with the repeated indices to be summed. This convention is used along this section. To simplify the notation we use in the following

$$|\mathbf{p},\sigma_1\alpha_1\sigma_2\alpha_2\rangle_A = \frac{1}{\sqrt{2}}\left(|\mathbf{p},\sigma_1\alpha_1\sigma_2\alpha_2\rangle - |-\mathbf{p},\sigma_2\alpha_2\sigma_1\alpha_1\rangle\right)\ . \tag{B.6}$$

with the subscript A indicating that the state is antisymmetrized. Applying the symmetry relations,

$$\begin{aligned} Y_\ell^m(-\hat{\mathbf{p}}) &= (-1)^\ell Y_\ell^m(\hat{\mathbf{p}})\ ,\\ (\sigma_2\sigma_1 s_3|s_2 s_1 S) &= (-1)^{S-s_1-s_2}(\sigma_1\sigma_2 s_3|s_1 s_2 S)\ ,\\ (\alpha_2\alpha_1\alpha_3|\tau_1\tau_3 I) &= (-1)^{I-\tau_1-\tau_2}(\alpha_1\alpha_2\alpha_3|\tau_1\tau_2 I)\ , \end{aligned} \tag{B.7}$$

eq.(B.5) for the nucleon-nucleon case ($s_1 = s_2 = \tau_1 = \tau_2 = 1/2$) simplifies to

$$|\mathbf{p},\sigma_1\alpha_1\sigma_2\alpha_2\rangle_A = \sqrt{4\pi}\sum_{J,S,\ell,m,I,\alpha_3}(\sigma_1\sigma_2 s_3|s_1 s_2 S)(m s_3\mu|\ell S J)Y_\ell^m(\hat{\mathbf{p}})^*\chi(\ell S I)|J\mu\ell S s_3 I\alpha_3\rangle\ , \tag{B.8}$$

with

$$\chi(S\ell I) = \frac{1-(-1)^{\ell+S+I}}{\sqrt{2}} = \begin{cases}\sqrt{2} & \ell+S+I = \text{ odd}\\ 0 & \ell+S+I = \text{ even}\end{cases} \tag{B.9}$$

In this way, $\chi(S\ell I)$ ensures the well known rule that a partial wave contributes to nucleon-nucleon scattering only if $S+\ell+I$ is odd. Using the decomposition eq.(B.8) we have for the scattering amplitude,

$$_A\langle\mathbf{p}',\sigma_1'\alpha_1'\sigma_2'\alpha_2'|T(\vec{\alpha})|\mathbf{p},\sigma_1\alpha_1\sigma_2\alpha_2\rangle_A = 4\pi\sum(\sigma_1'\sigma_2's_3'|s_1 s_2 S')(m's_3'\mu'|\ell'S'J')(\sigma_1\sigma_2 s_3|s_1 s_2 S)(m s_3\mu|\ell S J)$$
$$\times\,(\alpha_1'\alpha_2' i_3|\tau_1\tau_2 I)(\alpha_1\alpha_2 i_3|\tau_1\tau_2 I)Y_{\ell'}^{m'}(\hat{\mathbf{p}}')Y_\ell^m(\hat{\mathbf{p}})^*\chi(S'\ell' I)\chi(S\ell I)T_{J'JI}(\ell'S';\ell S)\ . \tag{B.10}$$

Here, $T_{J'JI}(\ell'S';\ell S)$ is the partial wave with final total angular momentum J', initial one J, final total spin S', initial one S, isospin I and final and initial orbital angular momenta ℓ' and ℓ, respectively. Notice that in the previous equation we have distinguished between the final and initial total angular momenta J' and J, and similarly for the total spins S' and S. For free two nucleon scattering we have of course $J' = J$ because of angular momentum conservation. This conservation law, the conservation of parity and the rule $S+\ell+I =$odd imply that $S' = S$. However, the resulting matrix elements in the nuclear medium depend additionally on the total three-momentum of the two nucleons because the medium rest-frame does not coincide in general with their center-of-mass. This is why we have included $\vec{\alpha}$ as an argument in the scattering operator, with the former defined in (F.4). Employing the orthogonality properties of the Clebsch-Gordan coefficients and spherical harmonics, one can invert eq. (B.10) with the result,

$$4\pi\chi(S'\ell'I)\chi(S\ell I)T_{J'JI}(\ell'S';\ell S) = \sum\int d\hat{\mathbf{p}}'\int d\hat{\mathbf{p}}\ _A\langle\mathbf{p}',\sigma_1'\alpha_1'\sigma_2'\alpha_2'|T(\vec{\alpha})|\mathbf{p},\sigma_1\alpha_1\sigma_2\alpha_2\rangle_A(\sigma_1'\sigma_2's_3'|s_1 s_2 S')$$
$$\times\,(m's_3'\mu'|\ell'S'J')(\sigma_1\sigma_2 s_3|s_1 s_2 S)(m s_3\mu|\ell S J)(\alpha_1'\alpha_2' i_3|\tau_1\tau_2 I)(\alpha_1\alpha_2 i_3|\tau_1\tau_2 I)Y_{\ell'}^{m'}(\hat{\mathbf{p}}')^*Y_\ell^m(\hat{\mathbf{p}})\ . \tag{B.11}$$

This expression can be further reduced by making use of properties under rotational invariance so that the initial relative three-momentum $\mathbf{p}$ can be taken parallel to the z-axis. In deriving this simplification we omit the isospin indices that do not play any role in the following considerations, and introduce the symbol

$$T^{\sigma_1\alpha_1\sigma_2\alpha_2}_{\sigma'_1\alpha'_1\sigma'_2\alpha'_2}(\mathbf{p}',\mathbf{p},\vec{\alpha}) = {}_A\langle \mathbf{p}',\sigma'_1\alpha'_1\sigma'_2\alpha'_2|T(\vec{\alpha})|\mathbf{p},\sigma_1\alpha_1\sigma_2\alpha_2\rangle_A \ . \tag{B.12}$$

Eq.(B.11) can be further reduced so that the angular integration over the initial relative three-momentum $\mathbf{p}$ can be removed making use of properties under rotational invariance. In deriving this simplification we omit the isospin indices that do not play any role in the following considerations. Let $R(\hat{p})$ such that $R(\hat{p})\hat{\mathbf{z}} = \hat{\mathbf{p}}$ and consisting first of a rotation around the y-axis of angle θ and then a rotation around the z-axis of angle ϕ, with θ and ϕ the polar and azimuthal angles of $\hat{\mathbf{p}}$. We could also have taken first an arbitrary rotation of angle γ around the z-axis. Then,

$$\begin{aligned}
R(\hat{\mathbf{p}})^\dagger|\mathbf{p},\sigma_1\sigma_2\rangle &= \sum_{\bar{s}_1,\bar{s}_2} D^{(1/2)}_{\bar{s}_1\sigma_1}(R^\dagger)D^{(1/2)}_{\bar{s}_2\sigma_2}(R^\dagger)|p\hat{\mathbf{z}},\bar{s}_1\bar{s}_2\rangle \ ,\\
R(\hat{\mathbf{p}})^\dagger|\mathbf{p}',\sigma'_1\sigma'_2\rangle &= \sum_{\bar{\sigma}_1,\bar{\sigma}_2} D^{(1/2)}_{\bar{\sigma}_1\sigma'_1}(R^\dagger)D^{(1/2)}_{\bar{\sigma}_2\sigma'_2}(R^\dagger)|\mathbf{p}'',\bar{\sigma}_1\bar{\sigma}_2\rangle \ ,
\end{aligned} \tag{B.13}$$

with $\mathbf{p}'' = R(\hat{\mathbf{p}})^{-1}\mathbf{p}'$ and $\vec{\alpha}'' = R(\hat{\mathbf{p}})^{-1}\vec{\alpha}$. The dependence on the total three-momentum has been made explicit in the state vectors to emphasize that the total three-momentum also is rotated. Inserting eq. (B.13) into eq. (B.11) we have,

$$\begin{aligned}
&4\pi\chi(S'\ell'I)\chi(S\ell I)T_{J'JI}(\ell'S';\ell S) = \sum \int d\hat{\mathbf{p}}' \int d\hat{\mathbf{p}}\, T^{\bar{s}_1\bar{s}_2}_{\bar{s}'_1\bar{s}'_2}(\mathbf{p}'',p\hat{\mathbf{z}},\vec{\alpha}'')D^{(1/2)}_{\bar{s}'_1\sigma'_1}(R^\dagger)^*D^{(1/2)}_{\bar{s}'_2\sigma'_2}(R^\dagger)^*D^{(1/2)}_{\bar{s}_1\sigma_1}(R^\dagger)D^{(1/2)}_{\bar{s}_2\sigma_2}(R^\dagger)\\
&\times Y^{m'}_{\ell'}(\hat{\mathbf{p}}')^*Y^m_\ell(\hat{\mathbf{p}})(\sigma'_1\sigma'_2s'_3|s_1s_2S')(m's'_3\mu'|\ell'S'J')(\sigma_1\sigma_2s_3|s_1s_2S)(ms_3\mu|\ell SJ) \ .
\end{aligned} \tag{B.14}$$

As usual in this section repeated indices are summed. The spherical harmonics satisfy the following transformation properties under rotations,

$$\begin{aligned}
Y^{m'}_{\ell'}(\hat{\mathbf{p}}') &= \sum_{\bar{m}'} D^{(\ell')}_{\bar{m}'m'}(R^\dagger)Y^{\bar{m}'}_{\ell'}(\hat{\mathbf{p}}'') \ ,\\
Y^m_\ell(\hat{\mathbf{p}}) &= \sum_{\bar{m}} D^{(\ell)}_{\bar{m}m}(R^\dagger)Y^{\bar{m}}_\ell(\hat{\mathbf{z}}) \ .
\end{aligned} \tag{B.15}$$

Inserting these equalities into eq. (B.14) we are then left with the following product of rotation matrices,

$$D^{(1/2)}_{\bar{s}'_1\sigma'_1}(R^\dagger)^*D^{(1/2)}_{\bar{s}'_2\sigma'_2}(R^\dagger)^*D^{(\ell')}_{\bar{m}'m'}(R^\dagger)^*D^{(1/2)}_{\bar{s}_1\sigma_1}(R^\dagger)D^{(1/2)}_{\bar{s}_2\sigma_2}(R^\dagger)D^{(\ell)}_{\bar{m}m}(R^\dagger) \ . \tag{B.16}$$

We now take into account the Clebsch-Gordan composition of the rotation matrices [40],

$$\sum_{M'} D^{(L)}_{M'M}(R)(m'_1m'_2M'|\ell_1\ell_2L) = \sum_{m_1,m_2} D^{(\ell_1)}_{m'_1m_1}(R)D^{(\ell_2)}_{m'_2m_2}(R)(m_1m_2M|\ell_1\ell_2L) \ . \tag{B.17}$$

Since eq.(B.16) appears in eq.(B.14) times Clebsch-Gordan coefficients we can make use of the previous composition repeatedly. First,

$$\begin{aligned}
&\sum D^{(1/2)}_{\bar{s}'_1\sigma'_1}(R^\dagger)D^{(1/2)}_{\bar{s}'_2\sigma'_2}(R^\dagger)(\sigma'_1\sigma'_2s'_3|s_1s_2S') = \sum D^{(S')}_{\bar{\sigma}'_3s'_3}(R^\dagger)(\bar{s}'_1\bar{s}'_2\bar{\sigma}'_3|s_1s_2S') \ ,\\
&\sum D^{(1/2)}_{\bar{s}_1\sigma_1}(R^\dagger)D^{(1/2)}_{\bar{s}_2\sigma_2}(R^\dagger)(\sigma_1\sigma_2s_3|s_1s_2S) = \sum D^{(S)}_{\bar{\sigma}_3s_3}(R^\dagger)(\bar{s}_1\bar{s}_2\bar{\sigma}_3|s_1s_2S) \ .
\end{aligned} \tag{B.18}$$

The rotation matrix $D^{(S')}_{\bar{\sigma}'_3 s'_3}$, that appears on the right-hand-side of the first of the previous equalities, can be combined in eq.(B.14) such that

$$\sum D^{(S')}_{\bar{\sigma}'_3 s'_3}(R^\dagger) D^{(\ell')}_{\bar{m}' m'}(R^\dagger)(m' s'_3 \mu'|\ell' S' J') = \sum D^{(J')}_{\bar{\mu}'\mu'}(R^\dagger)(\bar{m}'\bar{\sigma}'_3\bar{\mu}'|\ell' S' J') \ . \tag{B.19}$$

Similarly

$$\sum D^{(S)}_{\bar{\sigma}_3 s_3}(R^\dagger) D^{(\ell)}_{\bar{m} m}(R^\dagger)(m s_3 \mu|\ell S J) = \sum D^{(J)}_{\bar{\mu}\mu}(R^\dagger)(\bar{m}\bar{\sigma}_3\bar{\mu}|\ell S J) \ . \tag{B.20}$$

Incorporating eqs. (B.19) and (B.20) in eq. (B.14), the latter takes the form

$$4\pi\chi(S'\ell' I)\chi(S\ell I) T_{J'JI}(\ell' S'; \ell S) = \sum \int d\hat{\mathbf{p}}' \int d\hat{\mathbf{p}}\, T^{\sigma_1\sigma_2}_{\sigma'_1\sigma'_2}(\mathbf{p}'', p\mathbf{z}, \vec{\alpha}'') Y^{\bar{m}'}_{\ell'}(\hat{\mathbf{p}}'')^* Y^{\bar{m}}_{\ell}(\hat{\mathbf{z}}) D^{(J')}_{\bar{\mu}'\mu'}(R^\dagger) D^{(J)}_{\bar{\mu}\mu}(R^\dagger)$$
$$\times (\bar{m}'\bar{\sigma}'_3\bar{\mu}'|\ell' S' J')(\bar{s}'_1\bar{s}'_2\bar{\sigma}'_3|s_1 s_2 S')(\bar{m}\bar{\sigma}_3\bar{\mu}|\ell S J)(\bar{s}_1\bar{s}_2\bar{\sigma}_3|s_1 s_2 S) \ . \tag{B.21}$$

Let us first consider the vacuum case where the scattering amplitude does not depend on $\vec{\alpha}$. In this way the integration over $\hat{\mathbf{p}}$ in the previous equation can be done explicitly taking into account the orthogonality relation between two rotation matrices [40]. For that let us recall our previous remark about the fact that an arbitrary initial rotation over the z-axis and angle γ can also be included. In this way we take

$$\frac{1}{2\pi}\int_0^{2\pi} d\gamma \int d\hat{\mathbf{p}} D^{(J')}_{\bar{\mu}'\mu'}(R^\dagger)^* D^{(J)}_{\bar{\mu}\mu}(R^\dagger) = \frac{4\pi}{2J+1}\delta_{\bar{\mu}'\bar{\mu}}\delta_{\mu'\mu}\delta_{JJ'} \ . \tag{B.22}$$

Inserting this back to eq. (B.21) one arrives at

$$\chi(S\ell' I)\chi(S\ell I) T_{JI}(\ell', \ell, S) = \frac{Y^0_\ell(\hat{\mathbf{z}})\delta_{JJ'}\delta_{\mu'\mu}}{2J+1}\sum \int d\hat{\mathbf{p}}'' T^{\bar{s}_1\bar{s}_2}_{\bar{s}'_1\bar{s}'_2}(\mathbf{p}'', p\mathbf{z}) Y^{\bar{m}'}_{\ell'}(\hat{\mathbf{p}}'')(\bar{m}'\bar{\sigma}'_3\bar{\sigma}_3|\ell' S J)(0\bar{\sigma}_3\bar{\sigma}_3|\ell S J)$$
$$\times (\bar{s}'_1\bar{s}'_2\bar{\sigma}'_3|s_1 s_2 S)(\bar{s}_1\bar{s}_2\bar{\sigma}_3|s_1 s_2 S) \ . \tag{B.23}$$

In this expression we have made use that only $\bar{m} = 0$ gives a contribution to $Y^{\bar{m}}_\ell(\hat{\mathbf{z}})$ and, as explained after eq. (B.10), $S' = S$. In addition, we have also used that $d\hat{\mathbf{p}}' = d\hat{\mathbf{p}}''$, since both vectors are related by a rotation. The subscript J' in $T_{J'JI}$ is suppressed because $J' = J$ and it is redundant. Also, we have employed the notation for the partial waves of section 8, $T_{JI}(\ell', \ell, S)$.

We now come back to the in-medium case and keep the dependence on $\vec{\alpha}$. Here also $\bar{m} = 0$ so that $\bar{\mu} = \bar{\sigma}_3$. Let us show first that a Fermi sea with all the free three-momentum states filled up to ξ has total spin zero. This is required because for a given three-momentum $\mathbf{p}_1$ one has two spin states that must be combined antisymmetrically because of the Fermi statistics so that $S = 0$ for this pair. Then, since this happens for any pair, the total spin of the Fermi sea must be zero. Regarding total angular momentum we now give a non-relativistic argument to claim that the orbital angular momentum must also be zero. This is due to the fact that the nuclear medium in the CM of the two nucleons that scatter is seen with a velocity parallel to $-\vec{\alpha}$. In this way, both the CM position vector and the total three-momentum of the nuclear medium are also parallel so that their cross product vanishes. As a result, since the intrinsic orbital angular momentum of the medium is also zero, one expects that the total angular momentum is zero for the system also in the CM frame of the two nucleons. Thus, $J' = J$ also in this case and then, because of the same reasons as in vacuum, $S' = S$. Let us recall the remark after eq. (9.12) to justify that I is conserved also in the nuclear medium. In addition the third component of total angular momentum must be conserved, $\mu = \mu'$, and summing over μ one has

$$\frac{1}{2J+1}\sum_\mu D^{(J)}_{\bar{\mu}'\mu}(R^\dagger)^* D^{(J)}_{\bar{\mu}\mu}(R) = \frac{\delta_{\bar{\mu}'\bar{\mu}}}{2J+1} \ , \tag{B.24}$$

given the unitary character of the rotation matrices. Then,

$$\chi(S\ell' I)\chi(S\ell I)T_{JI}(\ell',\ell,S) = \frac{Y_\ell^0(\hat{\mathbf{z}})}{4\pi(2J+1)}\sum\int d\hat{\alpha}''\int d\hat{\mathbf{p}}'' T^{\bar{s}_1\bar{s}_2}_{\bar{s}'_1\bar{s}'_2}(\mathbf{p}'',p\hat{z},\vec{\alpha}'')(\bar{s}'_1\bar{s}'_2\bar{\sigma}'_3|s_1s_2S')(\bar{s}_1\bar{s}\bar{\sigma}_3|s_1s_2S)$$
$$\times Y_{\ell'}^{m'}(\hat{\mathbf{p}}'')^*(\bar{m}'\bar{\sigma}'_3\bar{\sigma}_3|\ell' SJ)(0\bar{\sigma}_3\bar{\sigma}_3|\ell SJ)\ . \tag{B.25}$$

This expression reduces to the one in the vacuum, eq. (B.31), when the dependence on $\hat{\alpha}$ in the scattering operator can be neglected once

$$\sum\int d\hat{\mathbf{p}}'' T^{\bar{s}_1\bar{s}_2}_{\bar{s}'_1\bar{s}'_2}(\mathbf{p}'',p\hat{z},\vec{\alpha}'')(\bar{s}'_1\bar{s}'_2\bar{\sigma}'_3|s_1s_2S')(\bar{s}_1\bar{s}\bar{\sigma}_3|s_1s_2S)Y_{\ell'}^{m'}(\hat{\mathbf{p}}'')^*(\bar{m}'\bar{\sigma}'_3\bar{\mu}'|\ell' SJ)(0\bar{\sigma}_3\bar{\sigma}_3|\ell SJ) \tag{B.26}$$

is performed. In that case the integration over $d\hat{\alpha}''$ is simply 4π and eq. (B.23) is restored.

Eq.(B.25) can be further simplified because for evaluating a nucleon-nucleon partial wave amplitude one only needs to consider the direct term in the nucleon-nucleon scattering amplitude. This follows because

$$T^{\sigma_1\alpha_1\sigma_2\alpha_2}_{\sigma'_1\alpha'_1\sigma'_2\alpha'_2}(\mathbf{p}',\mathbf{p},\vec{\alpha}) = \frac{1}{2}\left(\langle\mathbf{p}',\sigma'_1\alpha'_1\sigma'_2\alpha'_2| - \langle-\mathbf{p}',\sigma'_2\alpha'_2\sigma'_1\alpha'_1|\right)T(\vec{\alpha})\left(|\mathbf{p},s_1\alpha_1s_2\alpha_2\rangle - |-\mathbf{p},s_2\alpha_2s_1\alpha_1\rangle\right)\ , \tag{B.27}$$

and the operator T is Bose symmetric under the simultaneous exchange of particles 1 and 2 so that,

$$T^{\sigma_1\alpha_1\sigma_2\alpha_2}_{\sigma'_1\alpha'_1\sigma'_2\alpha'_2}(\mathbf{p}',\mathbf{p},\vec{\alpha}) = \langle\mathbf{p}',\sigma'_1\alpha'_1\sigma'_2\alpha'_2|T(\vec{\alpha})|\mathbf{p},\sigma_1\alpha_1\sigma_2\alpha_2\rangle - \langle-\mathbf{p}',\sigma'_2\alpha'_2\sigma'_1\alpha'_1|T(\vec{\alpha})|\mathbf{p},\sigma_1\alpha_1\sigma_2\alpha_2\rangle\ . \tag{B.28}$$

When implementing the second or exchange term in eq. (B.25), reincluding the isospin indices as well, and using the above referred symmetry properties of the Clebsch-Gordan coefficients and spherical harmonics, one is left with the same expression as for the direct term in eq. (B.28) except for the global sign $-(-1)^{S+\ell'+I}$. Summing both expressions the factor

$$1-(-1)^{S+\ell'+I} \tag{B.29}$$

arises. Given the definition of $\chi(S\ell I)$ in eq. (B.9) and imposing the rule that $\ell+S+I=$odd and $\ell'+S+I=$ odd, the factor $\chi(S\ell I)\chi(S\ell' I)$ can be simplified on both sides of eq. (B.25). The latter then reads

$$T^{i_3}_{JI}(\ell',\ell,S) = \frac{Y_\ell^0(\hat{\mathbf{z}})}{4\pi(2J+1)}\sum(\sigma'_1\sigma'_2s'_3|s_1s_2S)(\sigma_1\sigma_2s_3|s_1s_2S)(0s_3s_3|\ell SJ)(m's'_3s_3|\ell' SJ)$$
$$\times(\alpha'_1\alpha'_2i_3|\tau_1\tau_2I)(\alpha_1\alpha_2i_3|\tau_1\tau_2I)\int d\hat{\alpha}\int d\hat{\mathbf{p}}\,\langle\mathbf{p},\sigma'_1\alpha'_1\sigma'_2\alpha'_2|T_d(\vec{\alpha})|p\hat{\mathbf{z}},\sigma_1\alpha_1\sigma_2\alpha_2\rangle Y_{\ell'}^{m'}(\mathbf{p})^*\ , \tag{B.30}$$

in terms of only the direct term, as indicated by the subscript d in the scattering operator. For the particular case of the vacuum nucleon-nucleon scattering the previous expression simplifies to

$$T_{JI}(\ell',\ell,S) = \frac{Y_\ell^0(\hat{\mathbf{z}})}{2J+1}\sum(\sigma'_1\sigma'_2s'_3|s_1s_2S)(\sigma_1\sigma_2s_3|s_1s_2S)(0s_3s_3|\ell SJ)(m's'_3s_3|\ell' SJ)$$
$$\times(\alpha'_1\alpha'_2i_3|\tau_1\tau_2I)(\alpha_1\alpha_2i_3|\tau_1\tau_2I)\int d\hat{\mathbf{p}}\,\langle\mathbf{p},\sigma'_1\alpha'_1\sigma'_2\alpha'_2|T_d|p\hat{\mathbf{z}},\sigma_1\alpha_1\sigma_2\alpha_2\rangle Y_{\ell'}^{m'}(\mathbf{p})^*\ . \tag{B.31}$$

For our practical applications after performing the operations of eq. (B.26) one can proceed in a way such that the integration over $\hat{\alpha}$ in eq.(B.30) becomes trivial and is equal to 4π. For the once iterated one-pion exchange, fig. 30, that gives rise to the amplitudes T_{14} and T_{15}, this has been checked numerically and it holds at the level of one per mil. This is similar to the numerical accuracy to which the in-medium integrations have been calculated. For the diagrams in fig. 27, with the one-pion exchange between the final nucleons, this is clearly the case because T_{12}^f and T_{13}^f only depend on $\hat{\alpha}$ through its scalar product with $\mathbf{p}''$. Thus, there is no angular dependence on $\vec{\alpha}$ once the integration over $d\hat{\mathbf{p}}''$ is performed. For the case when the pion is exchanged between the initial nucleons, fig. 28, the resulting T_{12}^i and T_{13}^i do not depend on $\hat{\mathbf{p}}''$. In this way, the integration over $d\hat{\mathbf{p}}''$ can not remove the dependence on $\hat{\alpha}$. This also implies that this diagram only can contribute to partial waves with $\ell'=0$, that is, ${}^3S_1$ and ${}^3D_1 \to {}^3S_1$. However, as remarked above after eq. (13.13), the exchange $\mathbf{p}' \leftrightarrow \mathbf{p}$ transforms T_{12}^f, T_{13}^f into T_{12}^i, T_{13}^i and vice versa. In addition, one has to notice the symmetry between $\mathbf{p}$ and $\mathbf{p}'$ in eq. (B.11) for the partial wave decomposition. It is then clear that the same partial waves result for the diagrams of figs. 27 and 28 with the exchange $\ell' \leftrightarrow \ell$. Thus, we can still use eq. (B.31) but using the diagrams with the pion exchanged between the final nucleons. The elastic partial wave ${}^3S_1$ is exactly the same for both diagrams and ${}^3D_1 \to {}^3S_1$ is equal to ${}^3S_1 \to {}^3D_1$ evaluated as discussed. When only local vertices are involved in the evaluation of the two-nucleon reducible loops, fig. 24, there is no dependence on $\hat{\alpha}$ but just on $|\vec{\alpha}|$. It follows from this discussion that for all the actual calculations performed here we can use the simpler eq. (B.31) which does not require to integrate over $d\hat{\alpha}$.

C Lorentz transformations

A nucleon state is defined by means of a Lorentz transformation of reference acting on a nucleon state at rest with third component of spin σ,

$$|\mathbf{p},\sigma\rangle = U(L_{\mathbf{p}})|\mathbf{0},\sigma\rangle \ , \tag{C.1}$$

where $U(L_{\mathbf{p}})$ is a Lorentz boost in the direction of the three-momentum $\mathbf{p}$,

$$U(L_{\mathbf{p}}) = R(\hat{\mathbf{p}})B(v\hat{\mathbf{z}})R(\hat{\mathbf{p}})^{-1} \ . \tag{C.2}$$

In this expression $R(\hat{\mathbf{p}})$ is the rotation already introduced in Appendix B such that $R(\hat{\mathbf{p}})\hat{\mathbf{z}} = \hat{\mathbf{p}}$,

$$R(\hat{\mathbf{p}}) = e^{-i\phi J_3} e^{-i\theta J_2} \ , \tag{C.3}$$

with ϕ and θ the azimuthal and polar angles of $\hat{\mathbf{p}}$. On the other hand, $B(v\hat{\mathbf{z}})$ is a Lorentz boost along the z-axis and velocity $-|\mathbf{p}|/E$, with $E=\sqrt{m^2+\mathbf{p}^2}$. Simple expressions can be worked out for a rotation and $B(v\hat{\mathbf{z}})$ acting on a particle of spin 1/2:

$$\begin{aligned} e^{-i\alpha\hat{\mathbf{n}}\vec{\sigma}/2} &= \cos\frac{\alpha}{2} - i\sin\frac{\alpha}{2}\hat{\mathbf{n}}\vec{\sigma} \ , \\ B(v\hat{\mathbf{z}}) = e^{\phi\gamma^3\gamma^0} &= \cosh\frac{\phi}{2} + \sinh\frac{\phi}{2}\gamma^3\gamma^0 \ , \end{aligned} \tag{C.4}$$

with

$$\gamma^3\gamma^0 = \begin{pmatrix} 0 & -\sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} \ . \tag{C.5}$$

On the other hand, for the boost from the rest frame to the moving one at rest with the particle one has $\sinh\phi/2 = -p/\sqrt{2m(E+m)}$ and $\cosh\phi/2 = \sqrt{(m+E)/2m}$. It is then straightforward to obtain the Dirac spinors,

$$u_r(\mathbf{p}) = B(\mathbf{p})\begin{pmatrix} \xi_r \\ 0 \end{pmatrix} = \sqrt{\frac{E+m}{2m}}\begin{pmatrix} \xi_r \\ \frac{\vec{\sigma}\cdot\mathbf{p}}{E+m}\xi_r \end{pmatrix} \ , \tag{C.6}$$

with

$$\xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \ \xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ . \tag{C.7}$$

Now, the transformation of $|\mathbf{p}, \sigma\rangle$ under a Lorentz transformation $U(\Lambda)$ can be described in terms of the so called Little Wigner rotation. For that let us consider the manipulation

$$U(\Lambda)|\mathbf{p}\sigma\rangle = U(\Lambda)U(L_{\mathbf{p}})|0\sigma\rangle = U(L_{\Lambda p})U^{-1}(L_{\Lambda p})U(\Lambda)U(L_{\mathbf{p}})|0\sigma\rangle \ . \tag{C.8}$$

The point is that the transformation

$$R = U^{-1}(L_{\Lambda p})U(\Lambda)U(L_{\mathbf{p}}) \tag{C.9}$$

is a rotation in the rest frame of the particle. Here, $L_{\Lambda p}$ is the reference Lorentz boost for the particle with four-momentum Λp. The fact that R in eq.(C.9) is a rotation follows because this transformation leaves invariant the four-vector $n = (1, \vec{0})$. Note that L_p acting on $m\,n = (m, \vec{0})$ gives rise to the four-momentum p, then Λ transforms it to Λp and, finally, the inverse of $L_{\Lambda p}$ returns it to n.

For our studies of nucleon-nucleon interactions we are particularly interested of passing to the rest frame of the two scattering nucleons,

$$\mathbf{P}' = \mathbf{p}_1' + \mathbf{p}_2' = \vec{0} = \gamma(\mathbf{P} - \mathbf{v}W) \rightarrow \mathbf{v} = \frac{\mathbf{P}}{W} \ , \tag{C.10}$$

with $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$, $W = E_1 + E_2$ and $\gamma = 1/\sqrt{1 - v^2}$. The velocity for the reference boost for p is $\mathbf{w} = -\mathbf{p}/E$, and $\vec{\rho} = -\mathbf{p}'/E'$ that for the final four-momentum Λp. We also make use of the general expression for a Lorentz boost of velocity $\mathbf{v}$,

$$\begin{aligned} \mathbf{p}' &= \mathbf{p} + [(\mathbf{p}\hat{\mathbf{v}})(\gamma - 1) - \gamma E v]\,\hat{\mathbf{v}} \ , \\ E' &= \gamma(E - \mathbf{v}\mathbf{p}) \end{aligned} \tag{C.11}$$

By definition we choose the vector $\mathbf{p}||\hat{\mathbf{x}}$ and the velocity $\mathbf{v}$ contained in the plane $\hat{xy}$. In order to obtain an expression for the angle φ of the little Wigner rotation we apply the different Lorentz transformations of eq.(C.9) to a four-vector a. This is chosen such that its transformation under L_p is

$$\begin{aligned} & a^0 - \mathbf{w}\mathbf{a} = 0 \ , \\ & \mathbf{a} + (\mathbf{a}\hat{\mathbf{w}})\hat{\mathbf{w}}(\gamma_0 - 1) - \gamma_0 a^0 \omega = N \begin{pmatrix} \hat{v}_y \\ -\hat{v}_x \\ 0 \end{pmatrix} \ , \end{aligned} \tag{C.12}$$

where $\gamma_0 = 1/\sqrt{1 - \omega^2}$ and note that the vector $(v_y, -v_x, 0)$ in this equation is orthogonal to $\mathbf{v}$. In addition, the normalization condition

$$\mathbf{a}\hat{\mathbf{w}} = 1 \tag{C.13}$$

is imposed. These conditions fix

$$\begin{aligned} \mathbf{a} &= (\omega, -1, \frac{\text{ctg}\theta}{\gamma_0}, 0) \equiv \left(w, -\hat{\mathbf{x}} + \frac{\text{ctg}\theta}{\gamma_0}\hat{\mathbf{y}}\right) \ , \\ N &= -1/\gamma_0 \sin\theta \ . \end{aligned} \tag{C.14}$$

Where θ is the polar angle of $\mathbf{v} = v(\cos\theta, \sin\theta, 0)$. We define

$$b \equiv L_p a = \left(\begin{array}{c} 0 \\ \frac{1}{\gamma_0}\left[-\hat{\mathbf{x}} + \text{ctg}\theta\hat{\mathbf{y}}\right] \end{array} \right) , \tag{C.15}$$

and $\mathbf{bv} = 0$, because of eq.(C.12). This makes that the transformation of b under Λ is trivial

$$\Lambda b = b , \tag{C.16}$$

the real reason for having imposed the conditions in eq.(C.12). The last transformation is

$$\begin{aligned} c = L_{\Lambda p}^{-1} b &= \left(\begin{array}{c} -\gamma_2 \mathbf{bu} \\ \mathbf{b} + (\mathbf{b}\hat{\mathbf{u}})(\gamma_2 - 1)\hat{\mathbf{u}} \end{array} \right) \\ \mathbf{u} = \frac{\mathbf{p}'}{E'} &= \frac{\mathbf{p} + \left[(\mathbf{p}\hat{\mathbf{v}})(\gamma - 1) - \gamma E v\right]\hat{\mathbf{v}}}{\gamma(E - \mathbf{vp})} \\ \mathbf{bu} = \frac{\mathbf{bp}}{\gamma(E - \mathbf{vp})} &= -\frac{p/\gamma_0}{\gamma(E - \mathbf{vp})} . \end{aligned} \tag{C.17}$$

It is interesting to check that $c^0 = a^0$ since R must be a rotation. For that let us take into account that $E' = \gamma_2 m = \gamma(E - \mathbf{vp})$, so that

$$\gamma_2 = \gamma(E - \mathbf{vp})/m . \tag{C.18}$$

Then, substituting this expression in eq.(C.17),

$$c^0 = -\gamma_2 \mathbf{bu} = \frac{\gamma_2 p}{\gamma_0 \gamma(E - \mathbf{vp})} = \frac{p}{E} = \omega = a^0 . \tag{C.19}$$

The expression for $\mathbf{c}$ from eq.(C.17) can be worked straightforwardly taking into account the last two lines of eq.(C.17) and γ_2 in eq.(C.18), that we use in the equivalent form $\gamma_2 = \gamma\gamma_0(1 + \mathbf{vw})$. Then,

$$\begin{aligned} \mathbf{c} = &-\frac{\hat{\mathbf{x}}}{\gamma_0}\left[1 + \frac{w^2\gamma_0^2}{1 + \gamma\gamma_0(1 + \mathbf{vw})}\left\{1 + \cos^2\theta(\gamma - 1) - \gamma\frac{v}{w}\cos\theta\right\}\right] \\ &+ \frac{\hat{\mathbf{y}}}{\gamma_0}\left[\text{ctg}\theta - \frac{w^2\gamma_0^2}{1 + \gamma\gamma_0(1 + \mathbf{vw})}\left\{\cos\theta(\gamma - 1) - \gamma\frac{v}{w}\right\}\sin\theta\right] . \end{aligned} \tag{C.20}$$

The resulting Wigner rotation is then a rotation around the z-axis whose general form is

$$\begin{aligned} x'' &= x\cos\varphi - y\sin\varphi , \\ y'' &= x\sin\varphi + y\cos\varphi . \end{aligned} \tag{C.21}$$

Comparing this general expression with eq.(C.20), and keeping in mind the original vector $\mathbf{a}$ given in eq.(C.14), one has,

$$\sin\varphi = -\frac{y'' + \text{ctg}\theta x''/\gamma_0}{1 + \text{ctg}^2\theta/\gamma_0^2} . \tag{C.22}$$

Since v and w are both $\mathcal{O}(p)$ one can check straightforwardly that φ is $\mathcal{O}(p^2)$. Then for our present calculation of the pion self-energy they are N^3LO.

D One-pion exchange nucleon-nucleon partial waves

The one-pion exchange nucleon-nucleon partial waves $\mathcal{T}_{JI}^{1\pi}(\bar{\ell},\ell,S)$ up to the $F-$wave that result from eqs.(8.4)–(8.7) are:

$$
\begin{aligned}
\mathcal{T}_{01}^{1\pi}(S,S,0) &= -\frac{g_A^2}{4f_\pi^2}\frac{1}{4p^2}\left[4p^2+m_\pi^2\ln\frac{m_\pi^2}{m_\pi^2+4p^2}\right] ,\\
\mathcal{T}_{10}^{1\pi}(S,S,1) &= -\frac{g_A^2}{4f_\pi^2}\frac{1}{4p^2}\left[4p^2+m_\pi^2\ln\frac{m_\pi^2}{m_\pi^2+4p^2}\right] ,\\
\mathcal{T}_{10}^{1\pi}(P,P,0) &= \frac{g_A^2}{4f_\pi^2}\frac{3m_\pi^2}{8p^4}\left[4p^2+(m_\pi^2+2p^2)\ln\frac{m_\pi^2}{m_\pi^2+4p^2}\right] ,\\
\mathcal{T}_{01}^{1\pi}(P,P,1) &= \frac{g_A^2}{4f_\pi^2}\frac{1}{4p^2}\left[4p^2+m_\pi^2\ln\frac{m_\pi^2}{m_\pi^2+4p^2}\right] ,\\
\mathcal{T}_{11}^{1\pi}(P,P,1) &= \frac{g_A^2}{4f_\pi^2}\frac{1}{16p^4}\left[4p^2(m_\pi^2-2p^2)+m_\pi^4\ln\frac{m_\pi^2}{m_\pi^2+4p^2}\right] ,\\
\mathcal{T}_{21}^{1\pi}(P,P,1) &= \frac{g_A^2}{4f_\pi^2}\frac{1}{80p^4}\left[4p^2(3m_\pi^2+2p^2)+m_\pi^2(3m_\pi^2+8p^2)\ln\frac{m_\pi^2}{m_\pi^2+4p^2}\right] ,\\
\mathcal{T}_{21}^{1\pi}(D,D,0) &= -\frac{g_A^2}{4f_\pi^2}\frac{m_\pi^2}{32p^6}\left[12p^2(m_\pi^2+2p^2)+(3m_\pi^4+12m_\pi^2p^2+8p^4)\ln\frac{m_\pi^2}{m_\pi^2+4p^2}\right] ,\\
\mathcal{T}_{10}^{1\pi}(D,D,1) &= -\frac{g_A^2}{4f_\pi^2}\frac{1}{16p^4}\left[4p^2(3m_\pi^2+2p^2)+m_\pi^2(3m_\pi^2+8p^2)\ln\frac{m_\pi^2}{m_\pi^2+4p^2}\right] ,\\
\mathcal{T}_{20}^{1\pi}(D,D,1) &= -\frac{g_A^2}{4f_\pi^2}\frac{1}{16p^6}\left[4p^2(3m_\pi^4+3m_\pi^2p^2-2p^4)+3(m_\pi^6+3m_\pi^4p^2)\ln\frac{m_\pi^2}{m_\pi^2+4p^2}\right] ,\\
\mathcal{T}_{30}^{1\pi}(D,D,1) &= -\frac{g_A^2}{4f_\pi^2}\frac{1}{224p^6}\left[4p^2(15m_\pi^4+42m_\pi^2p^2+8p^4)+3m_\pi^2(5m_\pi^4+24m_\pi^2p^2+24p^4)\ln\frac{m_\pi^2}{m_\pi^2+4p^2}\right] ,\\
\mathcal{T}_{30}^{1\pi}(F,F,0) &= \frac{g_A^2}{4f_\pi^2}\frac{m_\pi^2}{64p^8}\left[4p^2(15m_\pi^4+60m_\pi^2p^2+44p^4)+3(5m_\pi^6+30m_\pi^4p^2+48m_\pi^2p^4+16p^4)\ln\frac{m_\pi^2}{m_\pi^2+4p^2}\right] ,\\
\mathcal{T}_{21}^{1\pi}(F,F,1) &= \frac{g_A^2}{4f_\pi^2}\frac{1}{480p^6}\left[4p^2(15m_\pi^4+42m_\pi^2p^2+8p^4)+3m_\pi^2(5m_\pi^4+24m_\pi^2p^2+24p^4)\ln\frac{m_\pi^2}{m_\pi^2+4p^2}\right] ,\\
\mathcal{T}_{31}^{1\pi}(F,F,1) &= \frac{g_A^2}{4f_\pi^2}\frac{1}{768p^8}\left[4p^2(45m_\pi^6+150m_\pi^4p^2+48m_\pi^2p^4-16p^6)\right.\\
&\quad\left.+3m_\pi^4(15m_\pi^4+80m_\pi^2p^2+96p^4)\ln\frac{m_\pi^2}{m_\pi^2+4p^2}\right] ,\\
\mathcal{T}_{41}^{1\pi}(F,F,1) &= \frac{g_A^2}{4f_\pi^2}\frac{1}{6912p^8}\left[4p^2(105m_\pi^6+510m_\pi^4p^2+560m_\pi^2p^4+48p^6)\right.\\
&\quad\left.+3m_\pi^2(35m_\pi^6+240m_\pi^4p^2+480m_\pi^2p^4+256p^6)\ln\frac{m_\pi^2}{m_\pi^2+4p^2}\right] ,\\
\mathcal{T}_{30}^{1\pi}(G,G,1) &= -\frac{g_A^2}{4f_\pi^2}\frac{1}{1792p^8}\left[4p^2(105m_\pi^6+510m_\pi^4p^2+560m_\pi^2p^4+48p^6)\right.\\
&\quad\left.+3m_\pi^2(35m_\pi^6+240m_\pi^4p^2+480m_\pi^2p^4+256p^6)\ln\frac{m_\pi^2}{m_\pi^2+4p^2}\right]
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{T}_{10}^{1\pi}(S,D,1) &= -\frac{g_A^2}{4f_\pi^2}\frac{\sqrt{2}}{16p^4}\left[4p^2(3m_\pi^2-2p^2)+m_\pi^2(3m_\pi^2+4p^2)\ln\frac{m_\pi^2}{m_\pi^2+4p^2}\right] , \\
&= \mathcal{T}_{10}^{1\pi}(D,S,1) \\
\mathcal{T}_{21}^{1\pi}(P,F,1) &= \frac{g_A^2}{4f_\pi^2}\frac{\sqrt{6}}{480p^6}\left[4p^2(15m_\pi^4+24m_\pi^2p^2-4p^2)+3m_\pi^2(5m_\pi^4+18m_\pi^2p^2+8p^4)\ln\frac{m_\pi^2}{m_\pi^2+4p^2}\right] \\
&= \mathcal{T}_{21}^{1\pi}(F,P,1) \ , \\
\mathcal{T}_{30}^{1\pi}(D,G,1) &= -\frac{g_A^2}{4f_\pi^2}\frac{\sqrt{3}}{896p^8}\Big[4p^2(105m_\pi^6+390m_\pi^4p^2+224m_\pi^2p^4-16p^6) \\
&\quad + 3m_\pi^2(35m_\pi^6+200m_\pi^4p^2+288m_\pi^2p^4+64p^6)\ln\frac{m_\pi^2}{m_\pi^2+4p^2}\Big] \\
&= \mathcal{T}_{30}^{1\pi}(G,D,1) \ , \qquad \text{(D.1)}
\end{aligned}
$$

including those amplitudes coupling different ℓ and $\bar{\ell}$.

E Calculation of L_{10} function

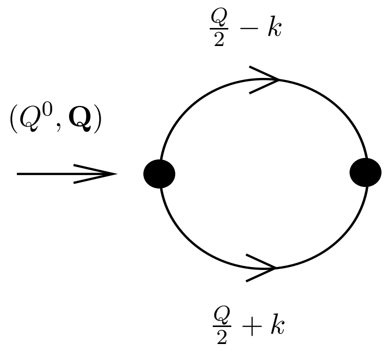

Figure 37: L_{10} function.

The function L_{10} is given by

$$
\begin{aligned}
L_{10} = i\int\frac{d^4k}{(2\pi)^4}&\left[\frac{1}{Q^0/2-k^0-w(\frac{\mathbf{Q}}{2}-\mathbf{k})+i\epsilon}+2\pi i\theta(\xi_1-|\frac{\mathbf{Q}}{2}-\mathbf{k}|)\delta(Q^0/2-k^0-w(\frac{\mathbf{Q}}{2}-\mathbf{k}))\right] \\
\times&\left[\frac{1}{Q^0/2+k^0-w(\frac{\mathbf{Q}}{2}+\mathbf{k})+i\epsilon}+2\pi i\theta(\xi_1-|\frac{\mathbf{Q}}{2}+\mathbf{k}|)\delta(Q^0/2+k^0-w(\frac{\mathbf{Q}}{2}+\mathbf{k}))\right] \ . \qquad \text{(E.1)}
\end{aligned}
$$

This integration corresponds to the loop in fig.37, where the four-momentum attached to each internal line is shown. In the following we define,

$$
\vec{\alpha} = \frac{1}{2}(\mathbf{p}_1+\mathbf{p}_2) = \frac{\mathbf{Q}}{2} \ . \qquad \text{(E.2)}
$$

The different contributions to L_{10} are calculated according to the number of in-medium insertions in the nucleon propagators, eq. (3.2).

E.1 Free part, $L_{10,f}$

We perform first the k^0 integration by applying the Cauchy's theorem,

$$
\begin{aligned}
L_{10,f} &= i\int \frac{d^4k}{(2\pi)^4}\frac{1}{Q^0/2-k^0-w(\vec{\alpha}-\mathbf{k})+i\epsilon}\frac{1}{Q^0/2+k^0-w(\vec{\alpha}+\mathbf{k})+i\epsilon} \\
&= \int \frac{d^3k}{(2\pi)^3}\frac{1}{Q^0-\frac{\mathbf{k}^2}{m}-\frac{\vec{\alpha}^2}{m}+i\epsilon} = -m\int\frac{d^3k}{(2\pi)^3}\frac{1}{\mathbf{k}^2-A} \\
&= -\frac{m}{2\pi^2}\int_0^\Lambda dk - \frac{mA}{4\pi^2}\int_{-\infty}^{\infty}\frac{dk}{k^2-A-i\epsilon} = -\frac{m\Lambda}{2\pi^2}-\frac{im\sqrt{A}}{4\pi}\ .
\end{aligned} \tag{E.3}
$$

with

$$
A = mQ^0 - \frac{\mathbf{Q}^2}{4}+i\epsilon = mQ^0-\alpha^2+i\epsilon\ , \tag{E.4}
$$

with $\alpha=|\vec{\alpha}|$. One has to keep in mind in the following the $+i\varepsilon$ prescription in the definition of A. In order to emphasize this, we will write explicitly the combination $A+i\varepsilon$ in many integrals, though the $+i\varepsilon$ is already contained in A according to eq.(E.4).

The result in eq.(E.3) corresponds to eq.(8.14), as it should because $g(A)=L_{10,f}(A)$. Note that here we have used a somewhat different scheme of calculation starting from four dimensions and removing the temporal component by explicit integration, so that we end with eq.(8.12) afterwards.

E.2 One-medium insertion, $L_{10,m}$

For the one-medium insertion, $L_{10,m}$ the k^0-integration is done by making use of the energy-conserving Dirac delta-function in the in-medium part of the nucleon propagator. We are then left with

$$
\begin{aligned}
L_{10,m} &= -\int\frac{d^3k}{(2\pi)^3}\frac{\theta(\xi_1-|\mathbf{k}-\vec{\alpha}|)+\theta(\xi_2-|\mathbf{k}+\vec{\alpha}|)}{Q^0-\frac{\mathbf{k}^2}{m}-\frac{\vec{\alpha}^2}{m}+i\epsilon} \\
&= m\int\frac{d^3k}{(2\pi)^3}\frac{\theta(\xi_1-|\mathbf{k}-\vec{\alpha}|)+\theta(\xi_2-|\mathbf{k}+\vec{\alpha}|)}{\mathbf{k}^2-A-i\epsilon}\ .
\end{aligned} \tag{E.5}
$$

Let us concentrate on the evaluation of the integral,

$$
\begin{aligned}
\ell_{10,m}(\xi_1,A,\alpha) &= m\int\frac{d^3k}{(2\pi)^3}\frac{\theta(\xi_1-|\mathbf{k}-\vec{\alpha}|)}{\mathbf{k}^2-A-i\epsilon} \\
&= \frac{m}{4\pi^2}\left\{\xi_1-\sqrt{A}\,\mathrm{arctanh}\frac{\xi_1-\alpha}{\sqrt{A}}-\sqrt{A}\,\mathrm{arctanh}\frac{\xi_1+\alpha}{\sqrt{A}}-\frac{A+\alpha^2-\xi_1^2}{4\alpha}\log\frac{(\alpha+\xi_1)^2-A}{(\alpha-\xi_1)^2-A}\right\}\ .
\end{aligned} \tag{E.6}
$$

Here we have taken into account that the Heaviside function in the numerator implies the conditions,

$$
\begin{aligned}
&\alpha\geq\xi_1\ , \\
&|\mathbf{k}|\in[\alpha-\xi_1,\alpha+\xi_1]\ ,\ \cos\theta\in\left[\frac{\mathbf{k}^2+\alpha^2-\xi_1^2}{2|\mathbf{k}|\alpha},1\right]\ . \\
&\alpha<\xi_1\ , \\
&|\mathbf{k}|\in[0,\xi_1-\alpha]\ ,\ \cos\theta\in[-1,1]\ , \\
&|\mathbf{k}|\in[\xi_1-\alpha,\xi_1+\alpha]\ ,\ \cos\theta\in\left[\frac{\mathbf{k}^2+\alpha^2-\xi_1^2}{2|\mathbf{k}|\alpha},1\right]\ .
\end{aligned} \tag{E.7}
$$

Despite the separation between the cases $\alpha \geq \xi_1$ and $\alpha < \xi_1$, both give rise to the same expression in eq.(E.6). In terms of the function $\ell_{10,m}(\xi_1, A, \alpha)$, eq.(E.5), one has

$$L_{10,m}(\xi_1, \xi_2, A, \alpha) = \ell_{10,m}(\xi_1, A, \alpha) + \ell_{10,m}(\xi_2, A, \alpha) \ . \tag{E.8}$$

E.3 Two-medium insertions, $L_{10,d}$

For the case with two medium insertions

$$\begin{aligned} L_{10,d} &= \frac{-i}{(2\pi)^2} \int d^4k\, \theta(\xi_1 - |\mathbf{k} - \vec{\alpha}|)\theta(\xi_2 - |\mathbf{k} + \vec{\alpha}|)\delta(\frac{Q^0}{2} - k^0 - w(|\mathbf{k} - \vec{\alpha}|))\delta(\frac{Q^0}{2} + k^0 - w(\mathbf{k} + \vec{\alpha})) \\ &= \frac{-im\sqrt{A}}{8\pi^2} \int d\hat{\mathbf{k}}\, \theta(\xi_1 - |\hat{\mathbf{k}}\sqrt{A} - \vec{\alpha}|)\theta(\xi_2 - |\hat{\mathbf{k}}\sqrt{A} + \vec{\alpha}|) \end{aligned} \tag{E.9}$$

Here we take that $\xi_2 \geq \xi_1$. If the opposite were true one can use the same expressions that we derive below but with the exchange $\xi_1 \leftrightarrow \xi_2$. This is clear after changing $\hat{\mathbf{k}} \to -\hat{\mathbf{k}}$ in the integral of eq.(E.9).

The two step functions can be easily solved. Denoting by θ the angle between $\hat{\mathbf{k}}$ and $\vec{\alpha}$, they imply

$$\begin{aligned} \cos\theta &\geq \frac{A + \alpha^2 - \xi_1^2}{2\alpha\sqrt{A}} \equiv y_1 \\ \cos\theta &\leq \frac{\xi_2^2 - A - \alpha^2}{2\alpha\sqrt{A}} \equiv y_2 \ . \end{aligned} \tag{E.10}$$

One has to require that $y_1 \leq 1$ and that $y_2 \geq -1$, otherwise $\cos\theta$ is out of the range $[-1, +1]$ from the conditions $(E.10)$. In addition, it is also necessary that $y_2 \geq y_1$.

$$\begin{aligned} y_1 \leq +1 &\to \alpha - \xi_1 \leq \sqrt{A} \leq \alpha + \xi_1 \ , \\ y_2 \geq -1 &\to \alpha - \xi_2 \leq \sqrt{A} \leq \alpha + \xi_2 \ , \\ y_1 \leq y_2 &\to A \leq \frac{\xi_1^2 + \xi_2^2}{2} - \alpha^2 \equiv A_{max} \ . \end{aligned} \tag{E.11}$$

For $\alpha \geq \xi_1$ in order that $(\alpha - \xi_1)^2 \leq A_{max}$, as the last of the three previous conditions requires, then

$$\alpha \leq \frac{\xi_1 + \xi_2}{2} \ . \tag{E.12}$$

Notice that because $\xi_2 \geq \xi_1$ the previous upper bound is larger than ξ_1. From eq.(E.12) it follows then that $\alpha - \xi_2 \leq 0$. In addition it is always the case that $(\alpha + \xi_2)^2 \geq A_{max}$. On the other hand,

$$\begin{aligned} \text{if} \ \ \alpha \geq \frac{\xi_2 - \xi_1}{2} &\to A_{max} \leq (\alpha + \xi_1)^2 \ , \\ \text{if} \ \ \alpha \leq \frac{\xi_2 - \xi_1}{2} &\to A_{max} \geq (\alpha + \xi_1)^2 \ . \end{aligned} \tag{E.13}$$

For the final form of $L_{10,d}$ one also has to take into account the conditions,

$$\begin{aligned} y_1 \geq -1 &\to \sqrt{A} \geq \xi_1 - \alpha, \\ y_2 \leq +1 &\to \sqrt{A} \geq \xi_2 - \alpha \ . \end{aligned} \tag{E.14}$$

Gathering together the conditions in eqs.(E.10)–(E.14) we have the following options,

$$y_1 \leq -1 \ , \ y_2 \leq +1 \ \rightarrow \ \xi_2 - \alpha \leq \sqrt{A} \leq \xi_1 - \alpha \ . \tag{E.15}$$

Which is not possible because $\xi_2 \geq \xi_1$.

$$y_1 \leq -1 \ , \ y_2 \geq +1 \ \rightarrow \ \sqrt{A} \leq \xi_1 - \alpha \ . \tag{E.16}$$

This only holds for $\alpha \leq \xi_1$. Then $\cos\theta \in [-1,+1]$ and $L_{10,d} = -i\, m\sqrt{A}/(2\pi)$.

$$-1 \leq y_1 \leq +1 \ , \ y_2 \geq +1 \ \rightarrow \ |\xi_1 - \alpha| \leq \sqrt{A} \leq \min(\xi_1 + \alpha, \xi_2 - \alpha) \ . \tag{E.17}$$

In this case, $\cos\theta \in [y_1,+1]$ and $L_{10,d} = -im(\xi_1^2 - (\sqrt{A}-\alpha)^2)/(8\pi\alpha)$. It follows that $\xi_1 + \alpha \leq \xi_2 - \alpha$ for $\alpha \leq (\xi_2 - \xi_1)/2$ and $\xi_1 + \alpha \geq \xi_2 - \alpha$ for $\alpha \geq (\xi_2 - \xi_1)/2$. In both cases $[min(\xi_1+\alpha, \xi_2-\alpha)]^2 \leq A_{max}$, as can be easily seen.

The last possibility is that

$$-1 \leq y_1 \leq +1 \ , \ y_2 \leq +1 \ \rightarrow \ \xi_2 - \alpha \leq \sqrt{A} \leq \xi_1 + \alpha \ . \tag{E.18}$$

For this case to hold, it is necessary that $\alpha \geq (\xi_2 - \xi_1)/2$. But then $A_{max} \leq (\xi_1 + \alpha)^2$ so that the allowed upper limit for $\sqrt{A}$ is $\sqrt{A_{max}}$ not $\xi_1 + \alpha$. In this case, $\cos\theta \in [y_1, y_2]$ and $L_{10,d} = -i\, m(\xi_1^2 + \xi_2^2 - 2A - 2\alpha^2)/(8\pi\alpha)$.

In summary,

$$L_{10,d} = \begin{cases} -\frac{im\sqrt{A}}{2\pi} \ , & \sqrt{A} \leq \xi_1 - \alpha \ , \ \alpha \leq \xi_1 \\ -\frac{im}{8\pi\alpha}(\xi_1^2 - (\sqrt{A}-\alpha)^2) \ , & |\xi_1 - \alpha| \leq \sqrt{A} \leq \xi_1 + \alpha \ , \ \alpha \leq \frac{\xi_2-\xi_1}{2} \\ -\frac{im}{8\pi\alpha}(\xi_1^2 - (\sqrt{A}-\alpha)^2) \ , & |\xi_1 - \alpha| \leq \sqrt{A} \leq \xi_2 - \alpha \ , \ \frac{\xi_2-\xi_1}{2} \leq \alpha \leq \frac{\xi_1+\xi_2}{2} \\ -\frac{im}{8\pi\alpha}(\xi_1^2 + \xi_2^2 - 2A - 2\alpha^2) \ , & \xi_2 - \alpha \leq \sqrt{A} \leq \sqrt{A_{max}} \ , \ \frac{\xi_2-\xi_1}{2} \leq \alpha \leq \frac{\xi_1+\xi_2}{2} \end{cases} \tag{E.19}$$

F Double Integration

For the final evaluation of Σ_7–Σ_{10}, one needs to calculate an integral involving two Fermi seas that is given by the multiple integral,

$$I_{F.S} = \int \frac{d^3p_m}{(2\pi)^3}\frac{d^3p_\ell}{(2\pi)^3}\theta(\xi_m - |\mathbf{p}_m|)\theta(\xi_\ell - |\mathbf{p}_\ell|) \ \langle N_m(\mathbf{p}_m)N_\ell(\mathbf{p}_\ell)|\mathcal{O}|N_m(\mathbf{p}_m)N_\ell(\mathbf{p}_\ell)\rangle \ , \tag{F.1}$$

with $\mathcal{O}$ a nucleon-nucleon operator corresponding to any of the amplitudes that are calculated. By definition $\xi_1 = min(\xi_m, \xi_\ell)$ and $\xi_2 = max(\xi_m, \xi_\ell)$, so that $\xi_2 \geq \xi_1$. Let us distinguish between the cases $\xi_m \geq \xi_\ell$ and $\xi_\ell \geq \xi_m$. For the latter case one has

$$I_{F.S} = \int \frac{d^3p_m}{(2\pi)^3}\frac{d^3p_\ell}{(2\pi)^3}\theta(\xi_1 - |\mathbf{p}_m|)\theta(\xi_2 - |\mathbf{p}_\ell|) \ \langle N_m(\mathbf{p}_m)N_\ell(\mathbf{p}_\ell)|\mathcal{O}|N_m(\mathbf{p}_m)N_\ell(\mathbf{p}_\ell)\rangle \ . \tag{F.2}$$

Due to the indistinguishable nature of the nucleons in a two-nucleon state within the isospin limit, we can write $\langle N_m(\mathbf{p}_m)N_\ell(\mathbf{p}_\ell)|\mathcal{O}|N_m(\mathbf{p}_m)N_\ell(\mathbf{p}_\ell)\rangle = \langle N_\ell(\mathbf{p}_\ell)N_m(\mathbf{p}_m)|\mathcal{O}|N_\ell(\mathbf{p}_\ell)N_m(\mathbf{p}_m)\rangle$. Exchanging also the integration vectors $\mathbf{p}_m \leftrightarrow \mathbf{p}_\ell$ between them, eq.(F.2) can be written as

$$I_{F.S} = \int \frac{d^3p_m}{(2\pi)^3}\frac{d^3p_\ell}{(2\pi)^3}\theta(\xi_1 - |\mathbf{p}_\ell|)\theta(\xi_2 - |\mathbf{p}_m|) \ \langle N_\ell(\mathbf{p}_m)N_m(\mathbf{p}_\ell)|\mathcal{O}|N_\ell(\mathbf{p}_m)N_m(\mathbf{p}_\ell)\rangle \ , \tag{F.3}$$

and the same expression as in eq.(F.1) when $\xi_m \geq \xi_\ell$ is obtained with regarding the integration variables, so that ξ_1 is associated with $\mathbf{p}_\ell$ and ξ_2 with $\mathbf{p}_m$. Furthermore, $\mathbf{p}_m$ is always the first three-momentum appearing from left to right in the matrix element and $\mathbf{p}_\ell$ the second. Let us define

$$\begin{aligned} \vec{\alpha} &= (\mathbf{p}_m + \mathbf{p}_\ell)/2 \ , \\ \mathbf{p} &= (\mathbf{p}_m - \mathbf{p}_\ell)/2 \ , \\ \mathbf{p}_m &= \vec{\alpha} + \mathbf{p} \ , \\ \mathbf{p}_\ell &= \vec{\alpha} - \mathbf{p} \ . \end{aligned} \tag{F.4}$$

Then

$$I_{F.S} = 8 \int \frac{d^3\alpha}{(2\pi)^3} \int \frac{d^3p}{(2\pi)^3} \theta(\xi_1 - |\vec{\alpha} - \mathbf{p}|)\theta(\xi_2 - |\vec{\alpha} + \mathbf{p}|) \ f(\vec{\alpha}, \mathbf{p}) \ . \tag{F.5}$$

The two step functions present in this integral are the same as discussed in section E for the calculation of $L_{10,d}$. The only difference is that now we have $|\mathbf{p}|$ instead of $\sqrt{A}$, see eq.(E.9). The analogy is not only formal because given the definition of A in eq.(E.4), one has now

$$A = \frac{\mathbf{p}_1^2}{2} + \frac{\mathbf{p}_2^2}{2} - \frac{(\mathbf{p}_1 + \mathbf{p}_2)^2}{4} = |\mathbf{p}|^2 \ , \tag{F.6}$$

since Q^0 in this case is the sum of the nucleon kinetic energies. In order to apply the conditions for α and $|\mathbf{p}| = \sqrt{A}$ in eq.(E.19), one has to distinguish two cases, corresponding to $\xi_1 \geq$ or $\leq$ than $(\xi_2 - \xi_1)/2$. The former case corresponds to $\xi_2 \leq 3\xi_1$ and the latter to $\xi_2 \geq 3\xi_1$. The following expressions follow directly from eq.(E.19) and the allowed interval of values for $\cos\theta$ according to the eqs.(E.16), (E.17) and (E.18) with y_1 and y_2 given in eq. (E.10) and A_{max} in eq.(E.11):

$a)$ $\xi_2 \leq 3\xi_1$,

$$\begin{aligned} I_{F.S} &= \frac{1}{\pi^4} \int_0^{\frac{\xi_2-\xi_1}{2}} d\alpha\, \alpha^2 \left[\int_0^{\xi_1-\alpha} dp\, p^2 \int_{-1}^{+1} d\cos\theta + \int_{\xi_1-\alpha}^{\xi_1+\alpha} dp\, p^2 \int_{y_1}^{+1} d\cos\theta \right] f(\vec{\alpha}, \mathbf{p}) \\ &+ \frac{1}{\pi^4} \int_{\frac{\xi_2-\xi_1}{2}}^{\xi_1} d\alpha\, \alpha^2 \left[\int_0^{\xi_1-\alpha} dp\, p^2 \int_{-1}^{+1} d\cos\theta + \int_{\xi_1-\alpha}^{\xi_2-\alpha} dp\, p^2 \int_{y_1}^{+1} d\cos\theta \right. \\ &+ \left. \int_{\xi_2-\alpha}^{\sqrt{A_{max}}} dp\, p^2 \int_{y_1}^{y_2} d\cos\theta \right] f(\vec{\alpha}, \mathbf{p}) \\ &+ \frac{1}{\pi^4} \int_{\xi_1}^{\frac{\xi_1+\xi_2}{2}} d\alpha\, \alpha^2 \left[\int_{\alpha-\xi_1}^{\xi_2-\alpha} dp\, p^2 \int_{y_1}^{+1} d\cos\theta + \int_{\xi_2-\alpha}^{\sqrt{A_{max}}} dp\, p^2 \int_{y_1}^{y_2} d\cos\theta \right] f(\vec{\alpha}, \mathbf{p}) \ . \end{aligned} \tag{F.7}$$

$b)$ $3\xi_1 \leq \xi_2$,

$$\begin{aligned} I_{F.S} &= \frac{1}{\pi^4} \int_0^{\xi_1} d\alpha\, \alpha^2 \left[\int_0^{\xi_1-\alpha} dp\, p^2 \int_{-1}^{+1} d\cos\theta + \int_{\xi_1-\alpha}^{\xi_1+\alpha} dp\, p^2 \int_{y_1}^{+1} d\cos\theta \right] f(\vec{\alpha}, \mathbf{p}) \\ &+ \frac{1}{\pi^4} \int_{\xi_1}^{\frac{\xi_2-\xi_1}{2}} d\alpha\, \alpha^2 \int_{\alpha-\xi_1}^{\xi_1+\alpha} dp\, p^2 \int_{y_1}^{+1} d\cos\theta \ f(\vec{\alpha}, \mathbf{p}) \\ &+ \frac{1}{\pi^4} \int_{\frac{\xi_2-\xi_1}{2}}^{\frac{\xi_1+\xi_2}{2}} d\alpha\, \alpha^2 \left[\int_{\alpha-\xi_1}^{\xi_2-\alpha} dp\, p^2 \int_{y_1}^{+1} d\cos\theta + \int_{\xi_2-\alpha}^{\sqrt{A_{max}}} dp\, p^2 \int_{y_1}^{y_2} d\cos\theta \right] f(\vec{\alpha}, \mathbf{p}) \ . \end{aligned} \tag{F.8}$$

These long expressions can be summarized in

$$I_{F.S} = \frac{1}{\pi^4}\int_0^{\frac{\xi_1+\xi_2}{2}} d\alpha\,\alpha^2 \int_0^{\sqrt{A_{max}}} dp\,p^2 \int_{t_1}^{t_2} d\cos\theta\, f(\alpha, p, \cos\theta)\ , \tag{F.9}$$

where $t_1 = \max(y_1, -1)$ and $t_2 = \min(y_2, +1)$.

G Calculation of L_{11}, L_{11}^a and L_{11}^{ab}.

Let us now proceed to the calculation of the integrals defined in eq.(13.17). As in section E we distinguish the free, one- and two-medium insertion parts.

G.1 L_{11}

L_{11} is defined in eq.(13.17) and we evaluate its different contributions according with the number of in-medium insertions.

$$\begin{aligned} L_{11} = i\int \frac{d^4k}{(2\pi)^4}\frac{1}{(\mathbf{k}+\mathbf{r})^2+m_\pi^2}&\left[\frac{\theta(\xi_m-|\vec{\alpha}-\mathbf{k}|)}{Q^0/2-k^0-w(\vec{\alpha}-\mathbf{k})-i\epsilon}+\frac{\theta(|\vec{\alpha}-\mathbf{k}|-\xi_m)}{Q^0/2-k^0-w(\vec{\alpha}-\mathbf{k})+i\epsilon}\right] \\ \times&\left[\frac{\theta(\xi_\ell-|\vec{\alpha}+\mathbf{k}|)}{Q^0/2+k^0-w(\vec{\alpha}+\mathbf{k})-i\epsilon}+\frac{\theta(|\vec{\alpha}+\mathbf{k}|-\xi_\ell)}{Q^0/2+k^0-w(\vec{\alpha}+\mathbf{k})+i\epsilon}\right]\ . \end{aligned} \tag{G.1}$$

G.1.1 Free part, $L_{11,f}$

After performing the k^0 integration by applying Cauchy's theorem we are left with

$$\begin{aligned} L_{11,f} &= -m\int\frac{d^3k}{(2\pi)^3}\frac{1}{(\mathbf{k}+\mathbf{p})^2+m_\pi^2}\frac{1}{\mathbf{k}^2-A-i\varepsilon} = -\frac{m}{8\pi}\int_0^1 dx\frac{1}{[\mathbf{p}^2x(1-x)+m_\pi^2x-A(1-x)-i\varepsilon]^{1/2}} \\ &= -\frac{im}{8\pi|\mathbf{p}|}\log\frac{A-(|\mathbf{p}|+im_\pi)^2}{m_\pi^2+(\sqrt{A}-|\mathbf{p}|)^2}\ . \end{aligned} \tag{G.2}$$

Where we have introduced a Feynman integration parameter $x\in[0,1]$ and $\sqrt{-a\pm i\varepsilon}=\pm i\sqrt{a}$ for $a>0$. One can also work out simply from eq.(G.2)

$$\frac{\partial L_{11,f}}{\partial A} = \frac{m/|\mathbf{p}|}{8\pi m_\pi(2|\mathbf{p}|+im_\pi)} \quad \text{for } A=p^2\ . \tag{G.3}$$

G.1.2 One-medium insertion, $L_{11,m}$

Once the k^0 integration is done by the presence of an energy Dirac function, we have

$$\begin{aligned} L_{11,m} &= m\int\frac{d^3k}{(2\pi)^3}\frac{\theta_m^-(\vec{\alpha}-\mathbf{k})+\theta_\ell^-(\vec{\alpha}+\mathbf{k})}{(\mathbf{k}^2-A-i\varepsilon)((\mathbf{k}+\mathbf{p})^2+m_\pi^2)} \\ &= m\int\frac{d^3k}{(2\pi)^3}\frac{\theta_m^-(\vec{\alpha}-\mathbf{k})}{(\mathbf{k}^2-A-i\varepsilon)((\mathbf{k}+\mathbf{p})^2+m_\pi^2)}+m\int\frac{d^3k}{(2\pi)^3}\frac{\theta_\ell^-(\vec{\alpha}-\mathbf{k})}{(\mathbf{k}^2-A-i\varepsilon)((\mathbf{k}-\mathbf{p})^2+m_\pi^2)}\ . \end{aligned} \tag{G.4}$$

Both terms in the sum can be obtained from the function

$$\ell_{11,m} = m \int \frac{d^3k}{(2\pi)^3} \frac{\theta(\xi_1 - |\vec{\alpha} - \mathbf{k}|)}{(\mathbf{k}^2 - A - i\varepsilon)((\mathbf{k}+\mathbf{p})^2 + m_\pi^2)} \ . \tag{G.5}$$

Let us work out the scalar product $\mathbf{k} \cdot \mathbf{p}$. For the integration, we introduce the reference frame

$$\begin{aligned}
\hat{\mathbf{z}} &= \hat{\alpha} \ , \\
\hat{\mathbf{x}} &= \frac{\vec{\alpha} \times \mathbf{p}}{\alpha p \sin\beta} \ , \\
\hat{\mathbf{y}} &= \hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\alpha}\,\mathrm{ctan}\beta - \hat{\mathbf{p}}\,\mathrm{csec}\beta \ .
\end{aligned} \tag{G.6}$$

From the last relation we have,

$$\hat{\mathbf{p}} = \hat{\alpha}\, \cos\beta - \hat{\mathbf{y}} \sin\beta \ . \tag{G.7}$$

Then,

$$\mathbf{k} \cdot \mathbf{p} = |\mathbf{k}||\mathbf{p}|(\cos\theta\cos\beta - \sin\theta\sin\phi\sin\beta) \ , \tag{G.8}$$

with θ and ϕ integration variables in eq.(G.5) and

$$\cos\beta = \hat{\mathbf{p}} \cdot \hat{\alpha} \ . \tag{G.9}$$

Let us perform the ϕ integration,

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi \frac{1}{\mathbf{p}^2 + \mathbf{k}^2 + 2\mathbf{k}\cdot\mathbf{p} + m_\pi^2} \ . \tag{G.10}$$

Notice that the denominator is always positive, $(\mathbf{p}+\mathbf{k})^2 + m_\pi^2$, so that the integrand is non-singular. This integration is of the type,

$$\int d\phi \frac{1}{a + b\sin\phi} = \frac{2}{\sqrt{a^2 - b^2}} \arctan \frac{b + a\tan(\phi/2)}{\sqrt{a^2 - b^2}} \ . \tag{G.11}$$

Since $\tan\phi/2$ is singular at $\phi = 2\pi$ the integration eq.(G.10) is split in two intervals

$$\frac{1}{2\pi} \int_0^{\pi^-} d\phi \frac{1}{\mathbf{p}^2 + \mathbf{k}^2 + 2\mathbf{k}\cdot\mathbf{p} + m_\pi^2} + \frac{1}{2\pi} \int_{\pi^+}^{2\pi} d\phi \frac{1}{\mathbf{p}^2 + \mathbf{k}^2 + 2\mathbf{k}\cdot\mathbf{p} + m_\pi^2} \ , \tag{G.12}$$

with $\pi^- = \pi - 0^+$ and $\pi^+ = \pi + 0^+$. Then, one obtains the result

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi \frac{1}{\mathbf{p}^2 + \mathbf{k}^2 + 2\mathbf{k}\cdot\mathbf{p} + m_\pi^2} = \frac{1}{\sqrt{a^2 - b^2}} \ , \tag{G.13}$$

that it has been checked also numerically. For eq.(G.10) we have

$$\begin{aligned}
a &= \mathbf{k}^2 + \mathbf{p}^2 + 2|\mathbf{k}||\mathbf{p}|\cos\beta\cos\theta + m_\pi^2 \ , \\
b &= -2|\mathbf{k}||\mathbf{p}|\sin\beta\sin\theta \ ,
\end{aligned} \tag{G.14}$$

where $\sin\beta = \sqrt{1 - \cos^2\beta}$. Next, we move to the $\cos\theta$ integration of eq.(G.5). For this integration one has to take into account the presence of the Heaviside function, which implies the conditions already worked out in eq.(E.7). These conditions determine an interval of integration for $\cos\theta \in [x_1(|\mathbf{k}|), x_2(|\mathbf{k}|)]$ for a given value of $|\mathbf{k}|$. Then, we can write,

$$\begin{aligned}\int_{x_1}^{x_2} d\cos\theta \frac{1}{\sqrt{a^2-b^2}} &= \int_{x_1}^{x_2} d\cos\theta \frac{1}{\sqrt{a'+b'\cos\theta+c'\cos^2\theta}} \\ &= \frac{1}{\sqrt{c'}}\left\{\log\left[\frac{b'+2c'\cos\theta}{\sqrt{c'}}+2\sqrt{a'+b'\cos\theta+c'\cos^2\theta}\right]\right\}_{x_1}^{x_2} ,\end{aligned} \tag{G.15}$$

with

$$\begin{aligned} a' &= \delta^2 - 4\mathbf{k}^2\mathbf{p}^2\sin^2\beta ~, \\ b' &= 4|\mathbf{k}||\mathbf{p}|\delta\cos\beta ~, \\ c' &= 4\mathbf{k}^2\mathbf{p}^2 ~, \\ \delta &= \mathbf{k}^2+\mathbf{p}^2+m_\pi^2 ~. \end{aligned} \tag{G.16}$$

Now, we consider the final integration on $|\mathbf{k}|$ in eq.(G.4) and define the auxiliary function,

$$f_{11,m}(|\mathbf{k}|) = \frac{m}{8\pi^2|\mathbf{p}|}\left\{\log\left[\frac{b'+2c'\cos\theta}{\sqrt{c'}}+2\sqrt{a'+b'\cos\theta+c'\cos^2\theta}\right]\right\}_{x_1(|\mathbf{k}|)}^{x_2(|\mathbf{k}|)} , \tag{G.17}$$

in terms of which eq.(G.5) reads

$$\begin{aligned} &\alpha \geq \xi_1 ~, \\ &\ell_{11,m}(\xi_1,\cos\beta) = \int_{\alpha-\xi_1}^{\alpha+\xi_1} d|\mathbf{k}| \frac{|\mathbf{k}|}{\mathbf{k}^2 - A - i\varepsilon} f_{11,m}(|\mathbf{k}|) ~. \\ &\alpha < \xi_1 ~, \\ &\ell_{11,m}(\xi_1,\cos\beta) = \left\{\int_0^{\xi_1-\alpha} + \int_{\xi_1-\alpha}^{\alpha+\xi_1}\right\} d|\mathbf{k}| \frac{|\mathbf{k}|}{\mathbf{k}^2 - A - i\varepsilon} f_{11,m}(|\mathbf{k}|) ~. \end{aligned} \tag{G.18}$$

Then from eq.(G.4) one has

$$L_{11,m} = \ell_{11,m}(\xi_m,\cos\beta) + \ell_{11,m}(\xi_\ell,-\cos\beta) ~. \tag{G.19}$$

Here, we have only indicated those arguments that change for each term in the sum of eq.(G.4) to calculate $L_{11,m}$. Indeed, $\ell_{11,m}$ depends furthermore on α and $|\mathbf{p}|$.

G.1.3 Two-medium insertions, $L_{11,d}$

The integration over $|\mathbf{k}|$ can be done straightforwardly done because of the presence of an additional Dirac delta function of energy, analogously as for $L_{10,d}$ in section E.3, which fixes $|\mathbf{k}| = \sqrt{A}$. Then,

$$L_{11,d}(\xi_1,\xi_2,\cos\beta) = \frac{-im\sqrt{A}}{8\pi^2}\int d\hat{\mathbf{k}} \frac{\theta(\xi_1 - |\hat{\mathbf{k}}\sqrt{A}-\vec{\alpha}|)\theta(\xi_2 - |\hat{\mathbf{k}}\sqrt{A}+\vec{\alpha}|)}{(\mathbf{k}+\mathbf{p})^2+m_\pi^2} ~. \tag{G.20}$$

We have the same ϕ integration as in eq.(G.10), with the same result but now with $|\mathbf{k}| = \sqrt{A}$. The integration over $\cos\theta$ is the same as that of eq.(G.15), though now the integration interval of $\cos\theta$ is different and it is fixed by the value of A and α, according to the results of section E.3 and that we collect here for $\xi_1 \leq \xi_2$,

$$\cos\theta \in [x_1,x_2]\ ,\ [x_1,x_2] \equiv \left\{ \begin{array}{ll} [-1,1]\ , & \sqrt{A} \leq \xi_1-\alpha\ ,\ \alpha \leq \xi_1 \\ [y_1,1]\ , & |\xi_1-\alpha| \leq \sqrt{A} \leq \xi_1+\alpha\ ,\ \alpha \leq \frac{\xi_2-\xi_1}{2} \\ [y_1,1]\ , & |\xi_1-\alpha| \leq \sqrt{A} \leq \xi_2-\alpha\ ,\ \frac{\xi_2-\xi_1}{2} \leq \alpha \leq \frac{\xi_1+\xi_2}{2} \\ [y_1,y_2]\ , & \xi_2-\alpha \leq \sqrt{A} \leq \sqrt{A_{max}}\ ,\ \frac{\xi_2-\xi_1}{2} \leq \alpha \leq \frac{\xi_1+\xi_2}{2} \end{array} \right. \tag{G.21}$$

with y_1 and y_2 defined in eq.(E.10). We also define, similar as done for $\ell_{11,m}$, the auxiliary function

$$f_{11,d}(\xi_1,\xi_2,\cos\beta) = \frac{m}{8\pi^2|\mathbf{p}|}\log\left[\frac{b'+2c'\cos\theta}{\sqrt{c'}}+2\sqrt{a'+b'\cos\theta+c'\cos^2\theta}\right]_{x_1}^{x_2}\ , \tag{G.22}$$

with $|\mathbf{k}|=\sqrt{A}$ for the values of a', b' and c' in eq.(G.16) and x_1 and x_2 according to eq.(G.21). In this way,

$$L_{11,d}(\xi_1,\xi_2,\cos\beta) = -i\pi f_{11,d}(\xi_1,\xi_2,\cos\beta)\ . \tag{G.23}$$

For the case $\xi_1 \geq \xi_2$ the change of variable $\mathbf{k}\to-\mathbf{k}$ is performed in eq.(G.20) and then for this case

$$L_{11,d} = -\,i\pi f_{11,d}(\xi_2,\xi_1,-\cos\beta)\ , \tag{G.24}$$

with $f_{11,d}$ calculated as above.

As a result,

$$L_{11} = L_{11,f}+L_{11,m}+L_{11,d}(\xi_m,\xi_\ell)\ . \tag{G.25}$$

G.2 L_{11}^a

$$\begin{aligned} L_{11}^a &= i\int\frac{d^4k}{(2\pi)^4}\frac{k^a}{(\mathbf{k}+\mathbf{r})^2+m_\pi^2}\left[\frac{\theta(\xi_m-|\vec{\alpha}-\mathbf{k}|)}{Q^0/2-k^0-w(\vec{\alpha}-\mathbf{k})-i\epsilon}+\frac{\theta(|\vec{\alpha}-\mathbf{k}|-\xi_m)}{Q^0/2-k^0-w(\vec{\alpha}-\mathbf{k})+i\epsilon}\right] \\ &\times\left[\frac{\theta(\xi_\ell-|\vec{\alpha}+\mathbf{k}|)}{Q^0/2+k^0-w(\vec{\alpha}+\mathbf{k})-i\epsilon}+\frac{\theta(|\vec{\alpha}+\mathbf{k}|-\xi_\ell)}{Q^0/2+k^0-w(\vec{\alpha}+\mathbf{k})+i\epsilon}\right] \\ &\equiv L_{11}^\alpha\alpha^a+L_{11}^p p^a\ . \end{aligned} \tag{G.26}$$

As usual the k^0 integration is firstly done either by applying the Cauchy's integration theorem or by using an energy Dirac delta function.

G.2.1 Free part, $L_{11,f}^a$

$$L_{11,f}^a = -m\int\frac{d^3q}{(2\pi)^3}\frac{k^a}{(\mathbf{k}+\mathbf{p})^2+m_\pi^2}\frac{1}{\mathbf{k}^2-A-i\varepsilon} = L_{11,f}^p p^a\ . \tag{G.27}$$

The function $L_{11,f}^p$ is given by

$$L_{11,f}^p = -\frac{m}{\mathbf{p}^2}\int\frac{d^3k}{(2\pi)^3}\frac{\mathbf{k}\cdot\mathbf{p}}{(\mathbf{k}+\mathbf{p})^2+m_\pi^2}\frac{1}{\mathbf{k}^2-A-i\varepsilon}\ . \tag{G.28}$$

Completing the squares of the denominator, $2\mathbf{k}\cdot\mathbf{p}=(2\mathbf{k}\cdot\mathbf{p}+\mathbf{k}^2+\mathbf{p}^2+m_\pi^2)-\mathbf{p}^2-m_\pi^2-(\mathbf{k}^2-A)-A$, one has

$$L^p_{11,f}=\frac{1}{2\mathbf{p}^2}\left[L_{10,f}-(\mathbf{p}^2+A+m_\pi^2)L_{11,f}+m\int\frac{d^3k}{(2\pi)^3}\frac{1}{\mathbf{k}^2+m_\pi^2}\right]\;. \tag{G.29}$$

The last integral diverges linearly. We perform its calculation as done for $L_{10,f}$ in section E.1,

$$L_{01,f}=\int\frac{d^3k}{(2\pi)^3}\frac{1}{\mathbf{k}^2+m_\pi^2}=\frac{1}{2\pi^2}\int_0^\Lambda d|\mathbf{k}|-\frac{m_\pi^2}{2\pi^2}\int_0^\infty d|\mathbf{k}|\frac{1}{\mathbf{k}^2+m_\pi^2}=\frac{\Lambda}{2\pi^2}-\frac{m_\pi}{4\pi}\;. \tag{G.30}$$

Note that the integral of eq.(G.28) is convergent and thus it cannot depend on Λ. This is accomplished because the dependence on Λ from eq.(G.30) is cancelled with that from $L_{10,f}$ in eq.(G.29), recall that $L_{11,f}$ is also finite. If $L_{01,f}$ were evaluated with dimensional regularization one would obtain only the last term in eq.(G.30). Then, eq.(G.29) reads

$$L^p_{11,f}=\frac{1}{2\mathbf{p}^2}\left[L_{10,f}-(\mathbf{p}^2+A+m_\pi^2)L_{11,f}+m\,L_{01,f}\right]\;. \tag{G.31}$$

G.2.2 One-medium insertion, $L^a_{11,m}$

$$\begin{aligned}L^a_{11,m}&=m\int\frac{d^3k}{(2\pi)^3}\frac{k^a}{(\mathbf{k}+\mathbf{p})^2+m_\pi^2}\frac{\theta^-_m(\vec{\alpha}-\mathbf{k})+\theta^-_\ell(\vec{\alpha}+\mathbf{k})}{\mathbf{k}^2-A-i\varepsilon}\\&=m\int\frac{d^3k}{(2\pi)^3}\frac{k^a}{(\mathbf{k}+\mathbf{p})^2+m_\pi^2}\frac{\theta^-_m(\vec{\alpha}-\mathbf{k})}{\mathbf{k}^2-A-i\varepsilon}-m\int\frac{d^3k}{(2\pi)^3}\frac{k^a}{(\mathbf{k}-\mathbf{p})^2+m_\pi^2}\frac{\theta^-_\ell(\vec{\alpha}-\mathbf{k})}{\mathbf{k}^2-A-i\varepsilon}\\&=L^\alpha_{11,m}\alpha^a+L^p_{11,m}p^a\;.\end{aligned} \tag{G.32}$$

We consider first the function

$$\ell^a_{11,m}=m\int\frac{d^3k}{(2\pi)^3}\frac{k^a}{(\mathbf{k}+\mathbf{p})^2+m_\pi^2}\frac{\theta^-_m(\vec{\alpha}-\mathbf{k})}{\mathbf{k}^2-A-i\varepsilon}=\ell^\alpha_{11,m}\,\alpha^a+\ell^p_{11,m}\,p^a\;. \tag{G.33}$$

Multiplying eq.(G.33) by p^a one has,

$$\begin{aligned}\ell^p_{11,m}\mathbf{p}^2+\ell_{11,m}\vec{\alpha}\cdot\mathbf{p}&=m\int\frac{d^3k}{(2\pi)^3}\frac{\mathbf{p}\cdot\mathbf{k}\theta^-_m(\vec{\alpha}-\mathbf{k})}{[(\mathbf{k}+\mathbf{p})^2+m_\pi^2](\mathbf{k}^2-A-i\varepsilon)}=\frac{1}{2}\ell_{10,m}-\frac{\mathbf{p}^2+A+m_\pi^2}{2}\ell_{11,m}\\&-\frac{m}{2}\int\frac{d^3k}{(2\pi)^3}\frac{\theta^-_m(\vec{\alpha}-\mathbf{k})}{(\mathbf{k}+\mathbf{p})^2+m_\pi^2}\;.\end{aligned} \tag{G.34}$$

As an intermediate step we have the integral

$$\begin{aligned}L_{01,m}(\xi_m,\cos\beta)&=\int\frac{d^3k}{(2\pi)^3}\frac{\theta^-_1(\vec{\alpha}-\mathbf{k})}{(\mathbf{k}+\mathbf{p})^2+m_\pi^2}=\frac{\xi_m}{(2\pi)^2}\left[1-\frac{m_\pi}{\xi_m}\arctan\frac{|\mathbf{p}_1|+\xi_m}{m_\pi}+\frac{m_\pi}{\xi_m}\arctan\frac{|\mathbf{p}_1|-\xi_m}{m_\pi}\right.\\&\left.+\frac{m_\pi^2-|\mathbf{p}_1|^2+\xi_m^2}{4|\mathbf{p}_1|\xi_m}\log\frac{(\xi_m+|\mathbf{p}_1|)^2+m_\pi^2}{(\xi_m-|\mathbf{p}_1|)^2+m_\pi^2}\right]\;,\\|\mathbf{p}_1|&=|\vec{\alpha}+\mathbf{p}|=\sqrt{\alpha^2+\mathbf{p}^2+2\alpha|\mathbf{p}|\cos\beta}\;.\end{aligned} \tag{G.35}$$

Then eq.(G.34) reads

$$\ell^p_{11,m}\mathbf{p}^2+\ell_{11,m}\vec{\alpha}\cdot\mathbf{p}=\frac{1}{2}\ell_{10,m}-\frac{\mathbf{p}^2+A+m_\pi^2}{2}\ell_{11,m}-\frac{m}{2}L_{01,m}\;. \tag{G.36}$$

Eq.(G.33) is now multiplied by α^a,

$$\ell^p_{11,m}\vec{\alpha}\cdot\mathbf{p}+\ell^\alpha_{11,m}\vec{\alpha}^2 = m\int\frac{d^3k}{(2\pi)^3}\frac{\vec{\alpha}\cdot\mathbf{k}\,\theta^-_m(\vec{\alpha}-\mathbf{k})}{[(\mathbf{k}+\mathbf{p})^2+m_\pi^2](\mathbf{k}^2-A-i\varepsilon)} = \frac{m\alpha}{(2\pi)^2}\int_0^\infty dk\frac{|\mathbf{k}|^3}{\mathbf{k}^2-A-i\varepsilon}$$
$$\times\int_{-1}^{+1}d\cos\theta\,\theta^-_m(\vec{\alpha}-\mathbf{k})\cos\theta\,\frac{1}{2\pi}\int_0^{2\pi}d\phi\frac{1}{\mathbf{k}^2+\mathbf{p}^2+m_\pi^2+2|\mathbf{k}||\mathbf{p}|(\cos\theta\cos\beta-\sin\theta\sin\beta\sin\phi)}\ . \quad \text{(G.37)}$$

Where we have followed the same steps as used in the calculation of $\ell_{11,m}$ in section G.1.2. As a result we are left with the integration on $\cos\theta$

$$\int_{x_1}^{x_2}d\cos\theta\frac{\cos\theta}{\sqrt{a'+b'\cos\theta+c'\cos^2\theta}} = \frac{1}{c'}\left\{\sqrt{\mathcal{C}}-\frac{b'}{2\sqrt{c'}}\log\left[\frac{b'+2c'\cos\theta}{\sqrt{c'}}+2\sqrt{\mathcal{C}}\right]\right\}_{x_1(|\mathbf{k}|)}^{x_2(|\mathbf{k}|)}\ , \quad \text{(G.38)}$$

with $\mathcal{C}=a'+b'\cos\theta+c'\cos^2$. See also eqs.(E.7) and (G.16) for the rest of elements in the previous expression. We define

$$f^{\alpha k}_{11,m}(|\mathbf{k}|,\cos\beta) = \frac{m\alpha}{16\pi^2|\mathbf{p}|^2}\left\{\sqrt{\mathcal{C}}-\frac{b'}{2\sqrt{c'}}\log\left[\frac{b'+2c'\cos\theta}{\sqrt{c'}}+2\sqrt{\mathcal{C}}\right]\right\}_{x_1(|\mathbf{k}|)}^{x_2(|\mathbf{k}|)}\ . \quad \text{(G.39)}$$

Then,

$$\begin{aligned}
&\alpha\geq\xi_1\ ,\\
&\ell^{\alpha k}_{11,m} = \int_{\alpha-\xi_1}^{\alpha+\xi_1}d|\mathbf{k}|\frac{|\mathbf{k}|}{\mathbf{k}^2-A-i\varepsilon}f^{\alpha k}_{11,m}(|\mathbf{k}|,\cos\beta)\ .\\
&\alpha<\xi_1\ ,\\
&\ell^{\alpha k}_{11,m} = \left\{\int_0^{\xi_1-\alpha}+\int_{\xi_1-\alpha}^{\alpha+\xi_1}\right\}d|\mathbf{k}|\frac{|\mathbf{k}|}{\mathbf{k}^2-A-i\varepsilon}f^{\alpha k}_{11,m}(|\mathbf{k}|,\cos\beta)\ .
\end{aligned} \quad \text{(G.40)}$$

with

$$\ell^{\alpha k}_{11,m} = \ell^p_{11,m}\vec{\alpha}\cdot\mathbf{p}+\ell^\alpha_{11,m}\vec{\alpha}^2\ . \quad \text{(G.41)}$$

From eqs.(G.36) and (G.41) it follows that,

$$\begin{aligned}
\ell^p_{11,m} &= \frac{1}{\mathbf{p}^2\alpha^2\sin^2\beta}\left[\frac{\alpha^2}{2}\left(\ell_{10,m}-(\mathbf{p}^2+A+m_\pi^2)\ell_{11,m}-mL_{01,m}\right)-|\mathbf{p}|\alpha\cos\beta\,\ell^{\alpha k}_{11,m}\right]\ ,\\
\ell^\alpha_{11,m} &= \frac{1}{\mathbf{p}^2\alpha^2\sin^2\beta}\left[\frac{-|\mathbf{p}|\alpha\cos\beta}{2}\left(\ell_{10,m}-(\mathbf{p}^2+A+m_\pi^2)\ell_{11,m}-mL_{01,m}\right)+\mathbf{p}^2\ell^{\alpha k}_{11,m}\right]\ .
\end{aligned} \quad \text{(G.42)}$$

In terms of these functions

$$L^a_{11,m} = \left[\ell^\alpha_{11,m}(\xi_m,\cos\beta)-\ell^\alpha_{11,m}(\xi_\ell,-\cos\beta)\right]\alpha^a+\left[\ell^p_{11,m}(\xi_m,\cos\beta)+\ell^p_{11,m}(\xi_\ell,-\cos\beta)\right]p^a\ . \quad \text{(G.43)}$$

G.2.3 Two-medium insertions, $L^a_{11,d}$

$$L^a_{11,d}(\xi_1,\xi_2,\cos\beta) = \frac{-im\sqrt{A}}{8\pi^2}\int d\hat{\mathbf{k}}\frac{\theta(\xi_1-|\vec{\alpha}-\hat{\mathbf{k}}\sqrt{A}|)\theta(\xi_2-|\vec{\alpha}+\hat{\mathbf{k}}\sqrt{A}|)}{(\mathbf{k}+\mathbf{p})^2+m_\pi^2}k^a = L^\alpha_{11,d}\alpha^a+L^p_{11,d}p^a\ . \quad \text{(G.44)}$$

Recall that because of the two energy Dirac delta functions $|\mathbf{k}| = \sqrt{A}$.

Multiplying by p^a one has,

$$L^p_{11,d}\mathbf{p}^2 + L^\alpha_{11,d}\vec{\alpha}\cdot\mathbf{p} = \frac{-im\sqrt{A}}{8\pi^2}\int d\hat{\mathbf{k}}\frac{\theta(\xi_1 - |\hat{\mathbf{k}}\sqrt{A}-\vec{\alpha}|)\theta(\xi_2 - |\hat{\mathbf{k}}\sqrt{A}+\vec{\alpha}|)}{(\mathbf{k}+\mathbf{p})^2+m_\pi^2}\mathbf{k}\cdot\mathbf{p} = \frac{1}{2}L_{10,d} - \frac{\mathbf{p}^2+A+m_\pi^2}{2}L_{11,d}\ . \tag{G.45}$$

Performing now the multiplication with α^a,

$$L^{\alpha k}_{11,d} = \frac{-im\sqrt{A}}{8\pi^2}\int d\hat{\mathbf{k}}\frac{\theta(\xi_1 - |\hat{\mathbf{k}}\sqrt{A}-\vec{\alpha}|)\theta(\xi_2 - |\hat{\mathbf{k}}\sqrt{A}+\vec{\alpha}|)}{(\mathbf{k}+\mathbf{p})^2+m_\pi^2}\mathbf{k}\cdot\vec{\alpha} \tag{G.46}$$

The angular integrations are the same as the ones calculated above in section G.2.2 for $\ell^{\alpha k}_{11,m}$. Though, the values of x_1 and x_2 as function of A are given now in eq.(G.21) assuming that $\xi_1 \leq \xi_2$. Defining the function

$$f^{\alpha k}_{11,d}(|\mathbf{k}|,\cos\beta) = \frac{m\alpha}{16\pi^2|\mathbf{p}|^2}\left\{\sqrt{\mathcal{C}} - \frac{b'}{2\sqrt{c'}}\log\left[\frac{b'+2c'\cos\theta}{\sqrt{c'}} + 2\sqrt{\mathcal{C}}\right]\right\}_{x_1}^{x_2}\ , \tag{G.47}$$

one simply has

$$L^{\alpha k}_{11,d} = -i\pi f^{\alpha k}_{11,d}\ . \tag{G.48}$$

From eqs.(G.45) and (G.48) it results

$$\begin{aligned}
L^p_{11,d} &= \frac{1}{\mathbf{p}^2\alpha^2\sin^2\beta}\left[\frac{\alpha^2}{2}\left(L_{10,d} - (\mathbf{p}^2+A+m_\pi^2)L_{11,d}\right) - |\mathbf{p}|\alpha\cos\beta\, L^{\alpha k}_{11,d}\right]\ ,\\
L^\alpha_{11,d} &= \frac{1}{\mathbf{p}^2\alpha^2\sin^2\beta}\left[\frac{-|\mathbf{p}|\alpha\cos\beta}{2}\left(L_{10,d} - (\mathbf{p}^2+A+m_\pi^2)L_{11,d}\right) + \mathbf{p}^2 L^{\alpha k}_{11,d}\right]\ .
\end{aligned} \tag{G.49}$$

For the case that $\xi_1 \geq \xi_2$ we perform first the change of variable $\mathbf{k}\to-\mathbf{k}$ in eq.(G.44) so that one has

$$\begin{aligned}
L^a_{11,d} &= +\frac{im\sqrt{A}}{8\pi^2}\int d\hat{\mathbf{k}}\frac{\theta(\xi_1 - |\vec{\alpha}+\hat{\mathbf{k}}\sqrt{A}|)\theta(\xi_2 - |\vec{\alpha}-\hat{\mathbf{k}}\sqrt{A}|)}{(\mathbf{k}-\mathbf{p})^2+m_\pi^2}k^a\\
&= -L^\alpha_{11,d}(\xi_2,\xi_1,-\cos\beta)\alpha^a + L^p_{11,d}(\xi_2,\xi_1,-\cos\beta)p^a\ ,
\end{aligned} \tag{G.50}$$

with the functions $L^\alpha_{11,d}$ and $L^p_{11,d}$ given in eq.(G.49).

The final result is

$$\begin{aligned}
L^\alpha_{11} &= L^\alpha_{11,m} + L^\alpha_{11,d}\ ,\\
L^p_{11} &= L^p_{11,f} + L^p_{11,m} + L^p_{11,d}\ .
\end{aligned} \tag{G.51}$$

G.3 L^{ab}_{11}

$$\begin{aligned}
L^{ab}_{11} &= i\int\frac{d^4k}{(2\pi)^4}\frac{k^a k^b}{(\mathbf{k}+\mathbf{r})^2+m_\pi^2}\left[\frac{\theta(\xi_m - |\vec{\alpha}-\mathbf{k}|)}{Q^0/2 - k^0 - w(\vec{\alpha}-\mathbf{k}) - i\epsilon} + \frac{\theta(|\vec{\alpha}-\mathbf{k}| - \xi_m)}{Q^0/2 - k^0 - w(\vec{\alpha}-\mathbf{k}) + i\epsilon}\right]\\
&\times\left[\frac{\theta(\xi_\ell - |\vec{\alpha}+\mathbf{k}|)}{Q^0/2 + k^0 - w(\vec{\alpha}+\mathbf{k}) - i\epsilon} + \frac{\theta(|\vec{\alpha}+\mathbf{k}| - \xi_\ell)}{Q^0/2 + k^0 - w(\vec{\alpha}+\mathbf{k}) + i\epsilon}\right]\\
&\equiv L^{Tg}_{11}\delta^{ab} + L^{T\alpha}_{11}\alpha^a\alpha^b + L^{Tp}_{11}p^a p^b + L^{T\alpha p}_{11}(\alpha^a p^b + \alpha^b p^a)\ .
\end{aligned} \tag{G.52}$$

G.3.1 Free part, $L^{ab}_{11,f}$

The k^0 integration is performed by applying the Cauchy's theorem as usual.

$$L^{ab}_{11,f} = -m \int \frac{d^3k}{(2\pi)^3} \frac{k^a k^b}{[(\mathbf{k}+\mathbf{p})^2 + m_\pi^2]\,(\mathbf{k}^2 - A - i\varepsilon)} = L^{Tg}_{11,f}\delta^{ab} + L^{Tp}_{11,f} p^a p^b \ . \tag{G.53}$$

In order to determine $L^{Tg}_{11,f}$ and $L^{Tp}_{11,f}$ we first contract eq.(G.53) with δ^{ab},

$$3L^{Tg}_{11,f} + \mathbf{p}^2 L^{Tp}_{11,f} = -mL_{01,f} + AL_{11,f} \ , \tag{G.54}$$

with $L_{10,f}$ given in eq.(G.30) and $L_{11,f}$ in eq.(G.2). Next, we multiply by p^b,

$$\begin{aligned} L^{Tg}_{11,f} p^a + L^{Tp}_{11,f} \mathbf{p}^2 p^a &= -m \int \frac{d^3k}{(2\pi)^3} \frac{k^a \mathbf{k}\cdot\mathbf{p}}{[(\mathbf{k}+\mathbf{p})^2 + m_\pi^2]\,(\mathbf{k}^2 - A - i\varepsilon)} \\ &= -\frac{m}{2} \int \frac{d^3k}{(2\pi)^3} k^a \left[\frac{1}{\mathbf{k}^2 - A - i\varepsilon} - \frac{1}{(\mathbf{k}+\mathbf{p})^2 + m_\pi^2} - \frac{\mathbf{p}^2 + A + m_\pi^2}{[(\mathbf{k}+\mathbf{p})^2 + m_\pi^2]\,(\mathbf{k}^2 - A - i\varepsilon)} \right] \\ &= \frac{m}{2} \int \frac{d^3k}{(2\pi)^3} \frac{k^a}{(\mathbf{k}+\mathbf{p})^2 + m_\pi^2} - \frac{\mathbf{p}^2 + A + m_\pi^2}{2} L^p_{11,f} p^a \ , \end{aligned} \tag{G.55}$$

where the function $L^p_{11,f}$ is given in eq.(G.31). We have the intermediate integral

$$\begin{aligned} L^a_{01,f} &= \int \frac{d^3k}{(2\pi)^3} \frac{k^a}{(\mathbf{k}+\mathbf{p})^2 + m_\pi^2} = p^a L^p_{01,f} \ , \\ L^p_{01,f} &= \frac{1}{\mathbf{p}^2} \int \frac{d^3k}{(2\pi)^3} \frac{\mathbf{k}\cdot\mathbf{p}}{(\mathbf{k}+\mathbf{p})^2 + m_\pi^2} = \frac{m_\pi}{4\pi} - \frac{\Lambda}{3\pi^2} \ . \end{aligned} \tag{G.56}$$

We should stress that the result for $L^a_{01,f}$ is not the same as the one obtained by performing the shift of variables $\mathbf{k}+\mathbf{p} \to \mathbf{k}$ in the previous integral. In this (wrong) way we would obtain $L^p_{01,f} = -L_{01,f}$ and the coefficient multiplying Λ would be different, $-\Lambda/2\pi^2$. We have performed the integral for $L^p_{01,f}$ exactly keeping those terms that do not vanish for $\Lambda \to \infty$.

Eq.(G.55) then reads

$$L^{Tg}_{11,f} + L^{Tp}_{11,f}\mathbf{p}^2 = \frac{m}{2} L^p_{01,f} - \frac{\mathbf{p}^2 + A + m_\pi^2}{2} L^p_{11,f} \ . \tag{G.57}$$

Solving eqs.(G.54) and (G.57) for $L^{Tg}_{11,f}$ and $L^{Tp}_{11,f}$ it results

$$\begin{aligned} L^{Tg}_{11,f} &= \frac{A}{2} L_{11,f} + \frac{\mathbf{p}^2 + A + m_\pi^2}{4} L^p_{11,f} + \frac{m m_\pi}{16\pi} - \frac{m\Lambda}{6\pi^2} \ , \\ L^{Tp}_{11,f} &= -\frac{1}{2\mathbf{p}^2} \left(AL_{11,f} + \frac{3}{2}(\mathbf{p}^2 + A + m_\pi^2) L^p_{11,f} - \frac{m m_\pi}{8\pi} \right) \ . \end{aligned} \tag{G.58}$$

G.3.2 One-medium insertion, $L^{ab}_{11,m}$

Once the integration over k^0 is done with the energy Dirac delta function,

$$\begin{aligned} L^{ab}_{11,m} &= m \int \frac{d^3k}{(2\pi)^3} \frac{k^a k^b}{(\mathbf{k}+\mathbf{p})^2 + m_\pi^2} \frac{\theta^-_m(\vec{\alpha}-\mathbf{k}) + \theta^-_\ell(\vec{\alpha}+\mathbf{k})}{\mathbf{k}^2 - A - i\varepsilon} \\ &= m \int \frac{d^3k}{(2\pi)^3} \frac{k^a k^b}{(\mathbf{k}+\mathbf{p})^2 + m_\pi^2} \frac{\theta^-_m(\vec{\alpha}-\mathbf{k})}{\mathbf{k}^2 - A - i\varepsilon} + m \int \frac{d^3k}{(2\pi)^3} \frac{k^a k^b}{(\mathbf{k}-\mathbf{p})^2 + m_\pi^2} \frac{\theta^-_\ell(\vec{\alpha}-\mathbf{k})}{\mathbf{k}^2 - A - i\varepsilon} \\ &= L^{Tg}_{11,m}\delta^{ab} + L^{T\alpha}_{11,m}\alpha^a\alpha^b + L^{Tp}_{11,m} p^a p^b + L^{T\alpha p}_{11,m}(\alpha^a p^b + \alpha^b p^a) \ . \end{aligned} \tag{G.59}$$

As usual, we first consider the integral

$$\begin{aligned}\ell^{ab}_{11,m} &= m\int \frac{d^3k}{(2\pi)^3}\frac{k^a k^b}{(\mathbf{k}+\mathbf{p})^2+m_\pi^2}\frac{\theta^-_m(\vec{\alpha}-\mathbf{k})}{\mathbf{k}^2-A-i\varepsilon}\\ &= \ell^{Tg}_{11,m}\delta^{ab}+\ell^{T\alpha}_{11,m}\alpha^a\alpha^b+\ell^{Tp}_{11,m}p^ap^b+\ell^{T\alpha p}_{11,m}(\alpha^ap^b+\alpha^bp^a)\ . \end{aligned} \tag{G.60}$$

We contract with δ^{ab},

$$3\ell^{Tg}_{11,m}+\ell^{T\alpha}_{11,m}\alpha^2+\ell^{Tp}_{11,m}|\mathbf{p}|^2+2\ell^{T\alpha p}_{11,m}\alpha|\mathbf{p}|\cos\beta = m\int\frac{d^3k}{(2\pi)^3}\frac{\mathbf{k}^2}{(\mathbf{k}+\mathbf{p})^2+m_\pi^2}\frac{\theta^-_m(\vec{\alpha}-\mathbf{k})}{\mathbf{k}^2-A-i\varepsilon} = mL_{01,m}+A\ell_{11,m}\ , \tag{G.61}$$

with $\vec{\alpha}\cdot\mathbf{p}=\alpha|\mathbf{p}|\cos\beta$. The integral $\ell_{11,m}$ is given in eq.(G.18) and $L_{01,m}$ in eq.(G.35).

Contracting with $\alpha^a\alpha^b$ it results

$$\ell^{Tg}_{11,m}\alpha^2+\ell^{T\alpha}_{11,m}\alpha^4+\ell^{Tp}_{11,m}(\vec{\alpha}\cdot\mathbf{p})^2+2\ell^{T\alpha p}_{11,m}\alpha^2\vec{\alpha}\cdot\mathbf{p} = m\int\frac{d^3k}{(2\pi)^3}\frac{\mathbf{k}^2\alpha^2\cos^2\theta}{(\mathbf{k}+\mathbf{p})^2+m_\pi^2}\frac{\theta^-_m(\vec{\alpha}-\mathbf{k})}{\mathbf{k}^2-A-i\varepsilon} = \ell^{T2\alpha k}_{11,m} \tag{G.62}$$

We calculate $\ell^{T2\alpha k}_{11,m}$ similarly as done above for $\ell_{11,m}$ and $\ell^{\alpha k}_{11,m}$. One has,

$$\ell^{T2\alpha k}_{11,m} = \frac{m\alpha^2}{(2\pi)^2}\int_0^\infty d|\mathbf{k}|\frac{|\mathbf{k}|^4}{\mathbf{k}^2-A-i\varepsilon}\int_{-1}^{+1}d\cos\theta\,\cos^2\theta\frac{1}{2\pi}\int_0^{2\pi}\frac{d\phi}{\mathbf{k}^2+\mathbf{p}^2+2|\mathbf{k}||\mathbf{p}|(\cos\theta\cos\beta-\sin\theta\sin\phi\sin\beta)+m_\pi^2} \tag{G.63}$$

We have the new angular integration on $\cos\theta$

$$\int_{x_1}^{x_2} d\cos\theta\,\frac{\cos^2\theta}{\sqrt{a'+b'\cos\theta+c'\cos^2\theta}} = \frac{1}{4c'^2}\left\{(-3b'+2c'\cos\theta)\sqrt{\mathcal{C}}+\frac{3b'^2-4a'c'}{2\sqrt{c'}}\log\left(\frac{b'+2c'\cos\theta}{\sqrt{c'}}+2\sqrt{\mathcal{C}}\right)\right\}_{x_1}^{x_2}\ , \tag{G.64}$$

following the notation of eq.(G.38). We also define the function

$$f^{T2\alpha k}_{11,m} = \frac{m\alpha^2}{256\pi^2|\mathbf{p}|^4|\mathbf{k}|}\left\{(-3b'+2c'\cos\theta)\sqrt{\mathcal{C}}+\frac{3b'^2-4a'c'}{2\sqrt{c'}}\log\left(\frac{b'+2c'\cos\theta}{\sqrt{c'}}+2\sqrt{\mathcal{C}}\right)\right\}_{x_1}^{x_2}\ , \tag{G.65}$$

and then

$$\begin{aligned}&\alpha\geq\xi_1\ ,\\ &\ell^{T2\alpha k}_{11,m} = \int_{\alpha-\xi_1}^{\alpha+\xi_1}d|\mathbf{k}|\frac{|\mathbf{k}|}{\mathbf{k}^2-A-i\varepsilon}f^{T2\alpha k}_{11,m}(|\mathbf{k}|,\cos\beta)\ .\\ &\alpha<\xi_1\ ,\\ &\ell^{T2\alpha k}_{11,m} = \left\{\int_0^{\xi_1-\alpha}+\int_{\xi_1-\alpha}^{\alpha+\xi_1}\right\}d|\mathbf{k}|\frac{|\mathbf{k}|}{\mathbf{k}^2-A-i\varepsilon}f^{T2\alpha k}_{11,m}(|\mathbf{k}|,\cos\beta)\ .\end{aligned} \tag{G.66}$$

When contracting eq.(G.60) with p^ap^b one has in the numerator $(\mathbf{k}\cdot\mathbf{p})^2$ that is rewriting as $[(\mathbf{k}+\mathbf{p})^2-\mathbf{k}^2-\mathbf{p}^2](\mathbf{k}\cdot\mathbf{p})/2$. Then, it reads

$$\begin{aligned}&\ell^{Tg}_{11,m}\mathbf{p}^2+\ell^{T\alpha}_{11,m}(\vec{\alpha}\cdot\mathbf{p})^2+\ell^{Tp}_{11,m}|\mathbf{p}|^4+2\ell^{T\alpha p}_{11,m}|\mathbf{p}|^2\vec{\alpha}\cdot\mathbf{p} = \frac{m}{2}\int\frac{d^3k}{(2\pi)^3}\frac{\mathbf{k}\cdot\mathbf{p}\theta^-_m(\vec{\alpha}-\mathbf{k})}{\mathbf{k}^2-A-i\varepsilon}\\ &-\frac{m}{2}\int\frac{d^3k}{(2\pi)^3}\frac{\mathbf{k}\cdot\mathbf{p}\theta^-_m(\vec{\alpha}-\mathbf{k})}{(\mathbf{k}+\mathbf{p})^2+m_\pi^2}-\frac{\mathbf{p}^2+A+m_\pi^2}{2}m\int\frac{d^3k}{(2\pi)^3}\frac{\mathbf{k}\cdot\mathbf{p}\theta^-_m(\vec{\alpha}-\mathbf{k})}{[(\mathbf{k}+\mathbf{p})^2+m_\pi^2](\mathbf{k}^2-A-i\varepsilon)}\ .\end{aligned} \tag{G.67}$$

The last integral was already evaluated in eq.(G.36) while the first two integrals have not been calculated yet. The first one on the right hand side of the previous equation is the scalar product of $\mathbf{p}$ with

$$\ell^a_{10,m} = m\int \frac{d^3k}{(2\pi)^3}\frac{k^a\theta^-(\vec{\alpha}-\mathbf{k})}{\mathbf{k}^2-A-i\varepsilon} = \ell^\alpha_{10,m}\alpha^a \ . \tag{G.68}$$

Performing the shift

$$\mathbf{k} = \vec{\alpha}+\mathbf{v} \tag{G.69}$$

and integrating over $\mathbf{v}$,

$$\ell^\alpha_{10,m}\alpha^a = m\int_{v\leq\xi_m}\frac{d^3v}{(2\pi)^3}\frac{\vec{\alpha}+\mathbf{v}}{(\vec{\alpha}+\mathbf{v})^2-A-i\varepsilon} = \vec{\alpha}\ell_{10,m} + m\int_{v\leq\xi_m}\frac{d^3v}{(2\pi)^3}\frac{\mathbf{v}}{(\vec{\alpha}+\mathbf{v})^2-A-i\varepsilon} \ . \tag{G.70}$$

Multiplying by α^a and completing squares,

$$\ell^\alpha_{10,m}\alpha^2 = \frac{A+\alpha^2}{2}\ell_{10,m} + \frac{m\xi_m^3}{12\pi^2} - \frac{1}{2}m\int_{v\leq\xi_m}\frac{d^3v}{(2\pi)^3}\frac{\mathbf{v}^2}{(\vec{\alpha}+\mathbf{v})^2-A-i\varepsilon} \ . \tag{G.71}$$

The evaluation of the last integral is straightforward with the result,

$$\ell^{v^2}_{10,m} = m\int_{v\leq\xi_m}\frac{d^3v}{(2\pi)^3}\frac{\mathbf{v}^2}{(\vec{\alpha}+\mathbf{v})^2-A-i\varepsilon} = \frac{m}{32\pi^2\alpha}\left[\xi_m^4\log\frac{(\xi_m+\alpha)^2-A}{(\xi_m-\alpha)^2-A} + \frac{4}{3}\alpha\xi_m^3 + 4\alpha\xi_m(3A+\alpha^2)\right.$$
$$\left. -(\sqrt{A}-\alpha)^4\log\frac{\alpha+\xi_m-\sqrt{A}}{\alpha-\xi_m-\sqrt{A}} - (\sqrt{A}+\alpha)^4\log\frac{\alpha+\xi_m+\sqrt{A}}{\alpha-\xi_m+\sqrt{A}}\right] \ . \tag{G.72}$$

Thus,

$$\ell^\alpha_{10,m} = \frac{1}{2\alpha^2}\left((A+\alpha^2)\ell_{10,m} + \frac{m\xi_m^3}{6\pi^2} - \ell^{v^2}_{10,m}\right) \ . \tag{G.73}$$

We now consider the second integral in eq.(G.67). Taking the shift of eq.(G.69) one has,

$$\int\frac{d^3k}{(2\pi)^3}\frac{\mathbf{k}\cdot\mathbf{p}\,\theta_m^-(\vec{\alpha}-\mathbf{k})}{(\mathbf{k}+\mathbf{p})^2+m_\pi^2} = \int_{v\leq\xi_m}\frac{d^3v}{(2\pi)^3}\frac{\vec{\alpha}\cdot\mathbf{p}+\mathbf{p}\cdot\mathbf{v}}{(\mathbf{v}+\mathbf{p}_1)^2+m_\pi^2} = (\vec{\alpha}\cdot\mathbf{p})L_{01,m} + (\mathbf{p}\cdot\mathbf{p}_1)L^p_{01,m} \ , \tag{G.74}$$

with $\mathbf{p}_1 = \vec{\alpha}+\mathbf{p}$ and

$$L^a_{01,m} = \int_{v\leq\xi_m}\frac{d^3v}{(2\pi)^3}\frac{v^a}{(\mathbf{v}+\mathbf{p}_1)^2+m_\pi^2} = L^p_{01,m}p_1^a \ ,$$
$$L^p_{01,m} = \frac{1}{2|\mathbf{p}_1|^2}\left(\frac{\xi_m^3}{6\pi^2} - (m_\pi^2+\mathbf{p}_1^2)L_{01,m} - L^{v^2}_{01,m}\right) \ . \tag{G.75}$$

The only new integral is

$$L^{v^2}_{01,m} = \int_{v\leq\xi_m}\frac{d^3v}{(2\pi)^3}\frac{\mathbf{v}^2}{(\mathbf{v}+\mathbf{p}_1)^2+m_\pi^2} = \frac{1}{8\pi^2|\mathbf{p}_1|}\left(-3m_\pi^2|\mathbf{p}_1|\xi_m + |\mathbf{p}_1|^3\xi_m + 2m_\pi|\mathbf{p}_1|(m_\pi^2-|\mathbf{p}_1|^2)\right.$$
$$\times\left\{\arctan\frac{\xi_m+|\mathbf{p}_1|}{m_\pi} + \arctan\frac{\xi_m-|\mathbf{p}_1|}{m_\pi}\right\} + |\mathbf{p}_1|\frac{\xi_m^3}{3} - \frac{1}{4}\left[(m_\pi^2-|\mathbf{p}_1|^2)^2 - 4m_\pi^2|\mathbf{p}_1|^2 - \xi_m^4\right]$$
$$\left.\times\log\frac{m_\pi^2+(\xi_m+|\mathbf{p}_1|)^2}{m_\pi^2+(\xi_m-|\mathbf{p}_1|)^2}\right) \ . \tag{G.76}$$

In terms of integrals already evaluated eq.(G.67) can be written as

$$\ell^{Tg}_{11,m}\mathbf{p}^2+\ell^{T\alpha}_{11,m}(\vec{\alpha}\cdot\mathbf{p})^2+\ell^{Tp}_{11,m}|\mathbf{p}|^4+2\ell^{T\alpha p}_{11,m}|\mathbf{p}|^2\vec{\alpha}\cdot\mathbf{p}=\ell_{10,m}\frac{1}{4}\left(\vec{\alpha}\cdot\mathbf{p}\left[1+\frac{A}{\alpha^2}\right]-\delta_A\right)$$
$$+\frac{m(\vec{\alpha}\cdot\mathbf{p})\xi_m^3}{24\pi^2\alpha^2}-\frac{\vec{\alpha}\cdot\mathbf{p}}{4\alpha^2}\ell^{v^2}_{10,m}-\frac{m(\vec{\alpha}\cdot\mathbf{p}+\mathbf{p}^2)}{2}L^p_{01,m}+\frac{(\mathbf{p}^2+A+m_\pi^2)^2}{4}\ell_{11,m}-\frac{m}{2}\left(\vec{\alpha}\cdot\mathbf{p}-\frac{\delta_A}{2}\right)L_{01,m}\ , \tag{G.77}$$

with $\delta_A=\mathbf{p}^2+A+m_\pi^2$.

We proceed now by contracting eq.(G.60) with $\alpha^a p^b$,

$$\ell^{Tg}_{11,m}\vec{\alpha}\cdot\mathbf{p}+\ell^{T\alpha}_{11,m}\alpha^2\vec{\alpha}\cdot\mathbf{p}+\ell^{Tp}_{11,m}\mathbf{p}^2\vec{\alpha}\cdot\mathbf{p}+\ell^{T\alpha p}_{11,m}(\alpha^2\mathbf{p}^2+(\vec{\alpha}\cdot\mathbf{p})^2)=\frac{m}{2}\int\frac{d^3k}{(2\pi)^3}\frac{\vec{\alpha}\cdot\mathbf{k}\,\theta^-_m(\vec{\alpha}-\mathbf{k})}{\mathbf{k}^2-A-i\varepsilon}$$
$$-\frac{m}{2}\int\frac{d^3k}{(2\pi)^3}\frac{\vec{\alpha}\cdot\mathbf{k}\,\theta^-_m(\vec{\alpha}-\mathbf{k})}{(\mathbf{p}+\mathbf{k})^2+m_\pi^2}-\frac{\mathbf{p}^2+A+m_\pi^2}{2}m\int\frac{d^3k}{(2\pi)^3}\frac{\vec{\alpha}\cdot\mathbf{k}\,\theta^-_m(\vec{\alpha}-\mathbf{k})}{[(\mathbf{p}+\mathbf{k})^2+m_\pi^2]\,(\mathbf{k}^2-A-i\varepsilon)}\ . \tag{G.78}$$

The first integral can be written as $\ell^\alpha_{10,m}\alpha^2$, with $\ell^\alpha_{10,m}$ given in eq.(G.73). The last integral corresponds to $\ell^{\alpha k}_{11,m}$ calculated in eq.(G.40). The remaining integral is

$$\int\frac{d^3k}{(2\pi)^3}\frac{\vec{\alpha}\cdot\mathbf{k}\theta^-_m(\vec{\alpha}-\mathbf{k})}{(\mathbf{k}+\mathbf{p})^2+m_\pi^2}=\int_{v\le\xi_m}\frac{d^3v}{(2\pi)^3}\frac{\vec{\alpha}\cdot\mathbf{v}+\alpha^2}{(\mathbf{v}+\mathbf{p}_1)^2+m_\pi^2}=\alpha^2L_{01,m}+(\alpha^2+\vec{\alpha}\cdot\mathbf{p})L^p_{01,m}\ , \tag{G.79}$$

with $L_{01,m}$ and $L^p_{01,m}$ given in eqs.(G.35) and (G.75), respectively. Then eq.(G.78) reads,

$$\ell^{Tg}_{11,m}\vec{\alpha}\cdot\mathbf{p}+\ell^{T\alpha}_{11,m}\alpha^2\vec{\alpha}\cdot\mathbf{p}+\ell^{Tp}_{11,m}\mathbf{p}^2\vec{\alpha}\cdot\mathbf{p}+\ell^{T\alpha p}_{11,m}(\alpha^2\mathbf{p}^2+(\vec{\alpha}\cdot\mathbf{p})^2)$$
$$=\frac{\alpha^2}{2}\ell^\alpha_{10,m}-\frac{m}{2}\left[\alpha^2L_{01,m}+(\alpha^2+\vec{\alpha}\cdot\mathbf{p})L^p_{01,m}\right]-\frac{\delta_A}{2}\ell^{\alpha k}_{11,m}\ . \tag{G.80}$$

We are then left with the system of equations

$$\begin{pmatrix}3 & \alpha^2 & \mathbf{p}^2 & 2\vec{\alpha}\cdot\mathbf{p}\\ \alpha^2 & \alpha^4 & (\vec{\alpha}\cdot\mathbf{p})^2 & 2\alpha^2(\vec{\alpha}\cdot\mathbf{p})\\ \mathbf{p}^2 & (\vec{\alpha}\cdot\mathbf{p})^2 & |\mathbf{p}|^4 & 2\mathbf{p}^2(\vec{\alpha}\cdot\mathbf{p})\\ \vec{\alpha}\cdot\mathbf{p} & \alpha^2(\vec{\alpha}\cdot\mathbf{p}) & \mathbf{p}^2(\vec{\alpha}\cdot\mathbf{p}) & \alpha^2\mathbf{p}^2+(\vec{\alpha}\cdot\mathbf{p})^2\end{pmatrix}\cdot\begin{pmatrix}\ell^{Tg}_{11,m}\\ \ell^{T\alpha}_{11,m}\\ \ell^{Tp}_{11,m}\\ \ell^{T\alpha p}_{11,m}\end{pmatrix}=\begin{pmatrix}A\ell_{11,m}+mL_{01,m}\\ \ell^{T2\alpha k}_{11,m}\\ \ell^{T2pk}_{11,m}\\ \ell^{T\alpha pk}_{11,m}\end{pmatrix}\ , \tag{G.81}$$

with $\ell^{T2\alpha k}_{11,m}$ given in eq.(G.66). On the other hand, $\ell^{T2pk}_{11,m}$ and $\ell^{T\alpha pk}_{11,m}$ corresponds to the right hand side of eqs.(G.77) and (G.80), respectively.The inversion of this system of equations is straightforward and then the function $\ell^{ab}_{11,m}$ is calculated. Thus, from eq.(G.59) one has

$$L^{ab}_{11,m}=\left[\ell^{Tg}_{11,m}(\xi_m,c\beta)+\ell^{Tg}_{11,m}(\xi_\ell,-c\beta)\right]\delta^{ab}+\left[\ell^{T\alpha}_{11,m}(\xi_m,c\beta)+\ell^{T\alpha}_{11,m}(\xi_\ell,-c\beta)\right]\alpha^a\alpha^b$$
$$+\left[\ell^{Tp}_{11,m}(\xi_m,c\beta)+\ell^{Tp}_{11,m}(\xi_\ell,-c\beta)\right]p^ap^b+\left[\ell^{T\alpha p}_{11,m}(\xi_m,c\beta)-\ell^{T\alpha p}_{11,m}(\xi_\ell,-c\beta)\right]\left(\alpha^ap^b+\alpha^bp^a\right)\ , \tag{G.82}$$

with $c\beta=\cos\beta$.

G.3.3 Two-medium insertions, $L_{11,d}^{ab}$

$$
\begin{aligned}
L_{11,d}^{ab} &= \frac{-im\sqrt{A}}{8\pi^2}\int d\hat{\mathbf{k}}\frac{\theta(\xi_1-|\hat{\mathbf{k}}\sqrt{A}-\vec{\alpha}|)\theta(\xi_2-|\hat{\mathbf{k}}\sqrt{A}+\vec{\alpha}|)}{(\mathbf{k}+\mathbf{p})^2+m_\pi^2}k^a k^b \\
&= L_{11,d}^{Tg}\delta^{ab}+L_{11,d}^{T\alpha}\alpha^a\alpha^b+L_{11,d}^{Tp}p^a p^b \\
&+ L_{11,d}^{T\alpha p}(\alpha^a p^b+\alpha^b p^a)\ .
\end{aligned} \tag{G.83}
$$

We proceed similarly as in the calculation of the previous section for $L_{11,m}^{ab}$, though the process is somewhat simpler because $\mathbf{k}^2=A$ is fixed due to the presence of the two energy Dirac delta functions. As usual we take $\xi_1\leq\xi_2$. First we contract with δ^{ab} resulting the equation

$$
3L_{11,d}^{Tg}+L_{11,d}^{T\alpha}\alpha^2+L_{11,d}^{Tp}\mathbf{p}^2+L_{11,d}^{T\alpha p}2\vec{\alpha}\cdot\mathbf{p}=AL_{11,d}\ , \tag{G.84}
$$

with $L_{11,d}$ given in eq.(G.20). Contracting eq.(G.83) with $\alpha^a\alpha^b$

$$
L_{11,d}^{Tg}\alpha^2+L_{11,d}^{T\alpha}\alpha^4+L_{11,d}^{Tp}(\vec{\alpha}\cdot\mathbf{p})^2+L_{11,d}^{T\alpha p}2\alpha^2(\vec{\alpha}\cdot\mathbf{p})=L_{11,d}^{T2\alpha k}\ . \tag{G.85}
$$

One has the same angular integrations for $L_{11,d}^{T2\alpha k}$ as before for $\ell_{11,m}^{T2\alpha k}$, eq.(G.62). Then, in terms of the function $f_{11,d}^{T2\alpha k}$,

$$
\begin{aligned}
f_{11,d}^{T2\alpha k} &= \frac{m\alpha^2}{256\pi^2|\mathbf{p}|^4\sqrt{A}}\left\{(-3b'+2c'\cos\theta)\sqrt{\mathcal{C}}+\frac{3b'^2-4a'c'}{2\sqrt{c'}}\log\left(\frac{b'+2c'\cos\theta}{\sqrt{c'}}+2\sqrt{\mathcal{C}}\right)\right\}_{x_1}^{x_2}\ , \\
L_{11,d}^{T2\alpha k} &= -i\pi f_{11,d}^{T2\alpha k}\ .
\end{aligned} \tag{G.86}
$$

The limits x_1 and x_2 are fixed according to eq.(G.21). We now contract with $p^a p^b$ with the result

$$
L_{11,d}^{Tg}\mathbf{p}^2+L_{11,d}^{T\alpha}(\vec{\alpha}\cdot\mathbf{p})^2+L_{11,d}^{Tp}|\mathbf{p}|^4+L_{11,d}^{T\alpha p}2\mathbf{p}^2(\vec{\alpha}\cdot\mathbf{p})=\frac{1}{2}L_{10,d}^{Tpk}-\frac{\mathbf{p}^2+A+m_\pi^2}{4}L_{10,d}+\frac{(\mathbf{p}^2+A+m_\pi^2)^2}{4}L_{11,d}\ . \tag{G.87}
$$

The only integral not yet evaluated is

$$
\begin{aligned}
L_{10,d}^{Tpk} &= \frac{-im\sqrt{A}}{8\pi^2}\int d\hat{\mathbf{k}}\theta_1^-(\vec{\alpha}-\mathbf{k})\theta_2^-(\vec{\alpha}+\mathbf{k})\mathbf{k}\cdot\mathbf{p} \\
&= -\frac{imA|\mathbf{p}|}{8\pi^2}\int d\hat{\mathbf{k}}\theta_1^-(\vec{\alpha}-\mathbf{k})\theta_2^-(\vec{\alpha}+\mathbf{k})(\cos\theta\cos\beta-\sin\theta\sin\beta\sin\phi)\ .
\end{aligned} \tag{G.88}
$$

The term proportional to $\sin\phi$ vanishes after the ϕ integration because the Heaviside functions do not depend on it. Then,

$$
L_{10,d}^{Tpk}=\frac{-imA|\mathbf{p}|\cos\beta}{4\pi}\int_{-1}^{+1}d\cos\theta\ \cos\theta\,\theta_1^-(\vec{\alpha}-\mathbf{k})\theta_2^-(\vec{\alpha}+\mathbf{k})=\frac{-imA|\mathbf{p}|\cos\beta}{8\pi}(x_2^2-x_1^2)\ , \tag{G.89}
$$

with x_1 and x_2 given in eq.(G.21). Next, we contract with $\alpha^a p^b$

$$
L_{11,d}^{Tg}\vec{\alpha}\cdot\mathbf{p}+L_{11,d}^{T\alpha}\alpha^2(\vec{\alpha}\cdot\mathbf{p})+L_{11,d}^{Tp}\mathbf{p}^2(\vec{\alpha}\cdot\mathbf{p})+L_{11,d}^{T\alpha p}(\alpha^2\mathbf{p}^2+(\vec{\alpha}\cdot\mathbf{p})^2)=\frac{1}{2}L_{10,d}^{T\alpha k}-\frac{\mathbf{p}^2+A+m_\pi^2}{2}L_{11,d}^{\alpha k}\ . \tag{G.90}
$$

Where $L^{\alpha k}_{11,d}$ was already evaluated in eq.(G.48) and

$$L^{T\alpha k}_{10,d} = \frac{-im\sqrt{A}}{8\pi^2}\int d\hat{\mathbf{k}}\theta_1^-(\vec{\alpha}-\mathbf{k})\theta_2^-(\vec{\alpha}+\mathbf{k})\vec{\alpha}\cdot\mathbf{k} = \frac{-imA\alpha}{8\pi}(x_2^2-x_1^2) = \frac{\alpha}{|\mathbf{p}|\cos\beta}L^{Tpk}_{10,d} \,. \tag{G.91}$$

We have the system of equations

$$\begin{pmatrix} 3 & \alpha^2 & \mathbf{p}^2 & 2\vec{\alpha}\cdot\mathbf{p} \\ \alpha^2 & \alpha^4 & (\vec{\alpha}\cdot\mathbf{p})^2 & 2\alpha^2(\vec{\alpha}\cdot\mathbf{p}) \\ \mathbf{p}^2 & (\vec{\alpha}\cdot\mathbf{p})^2 & |\mathbf{p}|^4 & 2\mathbf{p}^2(\vec{\alpha}\cdot\mathbf{p}) \\ \vec{\alpha}\cdot\mathbf{p} & \alpha^2(\vec{\alpha}\cdot\mathbf{p}) & \mathbf{p}^2(\vec{\alpha}\cdot\mathbf{p}) & \alpha^2\mathbf{p}^2+(\vec{\alpha}\cdot\mathbf{p})^2 \end{pmatrix} \cdot \begin{pmatrix} L^{Tg}_{11,d} \\ L^{T\alpha}_{11,d} \\ L^{Tp}_{11,d} \\ L^{T\alpha p}_{11,d} \end{pmatrix}$$
$$= \begin{pmatrix} AL_{11,d} \\ L^{T2\alpha k}_{11,d} \\ \frac{1}{2}L^{Tpk}_{10,d} - \frac{\mathbf{p}^2+A+m_\pi{}^2}{4}L_{10,d} + \frac{(\mathbf{p}^2+A+m_\pi^2)^2}{4}L_{11,d} \\ \frac{1}{2}L^{T\alpha k}_{10,d} - \frac{\mathbf{p}^2+A+m_\pi{}^2}{2}L^{\alpha k}_{11,d} \end{pmatrix} \,. \tag{G.92}$$

Its inversion gives $L^{ab}_{11,d}$. For the case $\xi_1 \geq \xi_2$, similarly as done already in section G.2.3, one performs first the exchange $\mathbf{k} \to -\mathbf{k}$ in eq.(G.83) which implies

$$\begin{aligned} L^{ab}_{11,d} &= L^{Tg}_{11,d}(\xi_2,\xi_1,-c\beta)\delta^{ab} + L^{T\alpha}_{11,d}(\xi_2,\xi_1,-c\beta)\alpha^a\alpha^b + L^{Tp}_{11,d}(\xi_2,\xi_1,-c\beta)p^a p^b \\ &\quad - L^{T\alpha p}_{11,d}(\xi_2,\xi_1,-c\beta)(\alpha^a p^b + \alpha^b p^a) \,, \end{aligned} \tag{G.93}$$

with the different coefficient functions calculated as in eq.(G.92).

Thus,

$$\begin{aligned} L^{Tg}_{11} &= L^{Tg}_{11,f} + L^{Tg}_{11,m} + L^{Tg}_{11,d} \,, \\ L^{T\alpha}_{11} &= L^{T\alpha}_{11,m} + L^{T\alpha}_{11,d} \,, \\ L^{Tp}_{11} &= L^{Tp}_{11,f} + L^{Tp}_{11,m} + L^{Tp}_{11,d} \,, \\ L^{T\alpha p}_{11} &= L^{T\alpha p}_{11,m} + L^{T\alpha p}_{11,d} \,. \end{aligned} \tag{G.94}$$

H Calculation of L_{12}, L^a_{12} and L^{ab}_{12}.

The different integrals involved in the evaluation of Σ_{14} and Σ_{15} can be expressed in terms of a set of scalar integrals. The tensor structure of these integrals is determined by the matrix elements in eq.(14.11). We also perform here the shift of integration variable $k \to \frac{p_1-p_2}{2} + k = p + k$, as in eq.(13.12) and rewrite the matrix elements of eq.(14.11) accordingly, as well as the rest of terms in eqs.(14.3) and (14.14). The integrals necessary for the calculation of the latter equations are evaluated along this section. In the expressions that follow it is always assumed that the k^0 integration has been done either by using Cauchy's theorem or employing the energy Dirac delta functions from the in-medium insertions.

On the other hand, since now two pion propagators are involved we join them in one introducing an integration Feynman parameter

$$\frac{1}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]} = \int_0^1 dy \frac{1}{\left[(\mathbf{k}+\vec{\lambda})^2+M^2\right]^2} \,, \tag{H.1}$$

with

$$\vec{\lambda} = \mathbf{p}' + (\mathbf{p} - \mathbf{p}')y \ ,$$
$$M^2 = m_\pi^2 + 2y(1-y)(\mathbf{p}^2 - \mathbf{p}\cdot\mathbf{p}') = m_\pi^2 + 2y(1-y)(1-\cos\varphi)\mathbf{p}^2 \ , \tag{H.2}$$

with

$$\mathbf{p}\cdot\mathbf{p}' = \mathbf{p}^2\cos\varphi \ . \tag{H.3}$$

In order to apply the results already derived in Appendix G, where only one pion propagator was involved, we take into account that

$$\frac{1}{\left[(\mathbf{k}+\vec{\lambda})^2 + M^2\right]^2} = -\frac{\partial}{\partial m_\pi^2}\frac{1}{(\mathbf{k}+\vec{\lambda})^2 + M^2} \ , \tag{H.4}$$

as follows from the definition of M^2 in eq.(H.3). In this way, the calculation of the integrals involved in Σ_{14} and Σ_{15} can be done following similar steps as already done in Appendix G, taking finally the derivative with respect to m_π^2 and then the y-integration, with the latter done numerically.

H.1 L_{12}

The scalar function L_{12} is defined by

$$\begin{aligned}
L_{12} = i\int \frac{d^4k}{(2\pi)^4}\frac{1}{[(\mathbf{k}+\mathbf{p}')^2 + m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2 + m_\pi^2]}\left[\frac{\theta(\xi_m - |\vec{\alpha}-\mathbf{k}|)}{Q^0/2 - k^0 - w(\vec{\alpha}-\mathbf{k}) - i\epsilon} + \frac{\theta(|\vec{\alpha}-\mathbf{k}| - \xi_m)}{Q^0/2 - k^0 - w(\vec{\alpha}-\mathbf{k}) + i\epsilon}\right] \\
\times \left[\frac{\theta(\xi_\ell - |\vec{\alpha}+\mathbf{k}|)}{Q^0/2 + k^0 - w(\vec{\alpha}+\mathbf{k}) - i\epsilon} + \frac{\theta(|\vec{\alpha}+\mathbf{k}| - \xi_\ell)}{Q^0/2 + k^0 - w(\vec{\alpha}+\mathbf{k}) + i\epsilon}\right]
\end{aligned} \tag{H.5}$$

H.1.1 Free part, $L_{12,f}$

For the free part

$$\begin{aligned}
L_{12,f} &= -m\int \frac{d^3k}{(2\pi)^3}\frac{1}{[(\mathbf{k}+\mathbf{p}')^2 + m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2 + m_\pi^2]\,(\mathbf{k}^2 - A - i\varepsilon)} \\
&= m\frac{\partial}{\partial m_\pi^2}\int_0^1 dy \int \frac{d^3k}{(2\pi)^3}\frac{1}{[(\mathbf{k}+\vec{\lambda})^2 + M^2](\mathbf{k}^2 - A - i\varepsilon)} \ .
\end{aligned} \tag{H.6}$$

The integration over $\mathbf{k}$ was already done in eq.(G.2). Making use of this result one has

$$L_{12,f} = -\frac{m}{8\pi}\int_0^1 dy \frac{1}{M\left(\mathbf{p}^2 + m_\pi^2 - A - 2iM\sqrt{A}\right)} \ . \tag{H.7}$$

with $M = \sqrt{M^2}$ of eq.(H.2).

H.1.2 One-medium insertion, $L_{12,m}$

For the part with one medium insertion

$$
\begin{aligned}
L_{12,m} &= m\int \frac{d^3k}{(2\pi)^3}\frac{\theta_m^-(\vec{\alpha}-\mathbf{k})+\theta_\ell^-(\vec{\alpha}+\mathbf{k})}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]\,(\mathbf{k}^2-A-i\varepsilon)} \\
&= m\int \frac{d^3k}{(2\pi)^3}\frac{\theta_m^-(\vec{\alpha}-\mathbf{k})}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]\,(\mathbf{k}^2-A-i\varepsilon)} \\
&+ m\int \frac{d^3k}{(2\pi)^3}\frac{\theta_\ell^-(\vec{\alpha}-\mathbf{k})}{[(\mathbf{k}-\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}-\mathbf{p})^2+m_\pi^2]\,(\mathbf{k}^2-A-i\varepsilon)}
\end{aligned}
\tag{H.8}
$$

The two terms in the sum can be obtained from the function

$$
\begin{aligned}
\ell_{12,m} &= m\int \frac{d^3k}{(2\pi)^3}\frac{\theta_m^-(\vec{\alpha}-\mathbf{k})}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]\,(\mathbf{k}^2-A-i\varepsilon)} \\
&= m\int_0^1 dy\int \frac{d^3k}{(2\pi)^3}\frac{\theta_m^-(\vec{\alpha}-\mathbf{k})}{(\mathbf{k}^2-A-i\varepsilon)[(\mathbf{k}+\vec{\lambda})^2+M^2]^2}\ .
\end{aligned}
\tag{H.9}
$$

This integral is very similar to $\ell_{11,m}$ in eq.(G.5), but with $\mathbf{p}$ in this equation replaced by $\vec{\lambda}$ and furthermore one of the factors in the denominator is squared. Following the calculation of $\ell_{11,m}$ we adopt the reference system

$$
\begin{aligned}
\hat{\mathbf{z}} &= \hat{\alpha}\ , \\
\hat{\mathbf{x}} &= \frac{\vec{\alpha}\times\vec{\lambda}}{\alpha|\vec{\lambda}|\sin\eta}\ , \\
\hat{\mathbf{y}} &= \hat{\mathbf{z}}\times\hat{\mathbf{x}} = \hat{\alpha}\,\mathrm{ctan}\eta - \hat{\lambda}\,\mathrm{csec}\eta\ ,
\end{aligned}
\tag{H.10}
$$

with

$$
\begin{aligned}
\vec{\alpha}\cdot\vec{\lambda} &= |\mathbf{p}|\alpha\left[(1-y)\cos\beta'+y\cos\beta\right] = \alpha|\vec{\lambda}|\cos\eta\ , \\
\cos\beta' &= \hat{\mathbf{p}}'\cdot\vec{\alpha}\ ,
\end{aligned}
\tag{H.11}
$$

So that the scalar product

$$
\mathbf{k}\cdot\vec{\lambda} = |\mathbf{k}||\vec{\lambda}|(\cos\theta\cos\eta-\sin\theta\sin\phi\sin\eta)\ ,
\tag{H.12}
$$

where θ and ϕ are the polar and azimuthal angles of $\mathbf{k}$. Let us perform the ϕ integration in eq.(H.9)

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} d\phi\frac{1}{\left[(\mathbf{k}+\vec{\lambda})^2+M^2\right]^2} = \frac{1}{2\pi}\int_0^{2\pi} d\phi\frac{1}{\left[\mathbf{k}^2+\vec{\lambda}^2+M^2+2|\mathbf{k}||\vec{\lambda}|(\cos\theta\cos\eta-\sin\theta\sin\phi\sin\eta)\right]^2} \\
&= \frac{-1}{2|\lambda||\mathbf{k}|\cos\theta}\frac{\partial}{\partial\cos\eta}\frac{1}{2\pi}\int_0^{2\pi} d\phi\frac{1}{\mathbf{k}^2+\vec{\lambda}^2+M^2+2|\mathbf{k}||\vec{\lambda}|(\cos\theta\cos\eta-\sin\theta\sin\phi\sin\eta)}\ .
\end{aligned}
\tag{H.13}
$$

The last integral is of the type already evaluated in eq.(G.13) where now

$$
\begin{aligned}
a &= \delta + 2|\vec{\lambda}||\mathbf{k}|\cos\theta\cos\eta\ , \\
b &= -2|\vec{\lambda}||\mathbf{k}|\sin\theta\sin\eta\ .
\end{aligned}
\tag{H.14}
$$

It is convenient to remark that

$$\delta = \mathbf{k}^2 + \vec{\lambda}^2 + M^2 = \mathbf{k}^2 + \mathbf{p}^2 + m_\pi^2 \ , \tag{H.15}$$

as already defined in eq.(G.16). Then, eq.(H.13) reads

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi \frac{1}{\left[(\mathbf{k}+\vec{\lambda})^2 + M^2\right]^2} = \frac{\delta + 2|\vec{\lambda}||\mathbf{k}|\cos\eta\cos\theta}{\left[4\vec{\lambda}^2\mathbf{k}^2(\cos^2\theta - \sin^2\eta) + 4|\vec{\lambda}||\mathbf{k}|\delta\cos\eta\cos\theta + \delta^2\right]^{3/2}} \ . \tag{H.16}$$

The $\cos\theta$ integration of the previous result is sensitive to the presence of the Heaviside function of eq.(H.9) and is given by

$$\begin{aligned}&\int_{x_1}^{x_2} d\cos\theta \frac{\delta + 2|\vec{\lambda}||\mathbf{k}|\cos\eta\cos\theta}{\left[4|\vec{\lambda}|^2|\mathbf{k}|^2(\cos^2\theta - \sin^2\eta) + 4|\vec{\lambda}||\mathbf{k}|\delta\cos\theta\cos\eta + \delta^2\right]^{3/2}} \\ &= \left.\frac{2|\vec{\lambda}||\mathbf{k}|\cos\eta + \delta\cos\theta}{(\delta^2 - 4|\vec{\lambda}|^2|\mathbf{k}|^2)\sqrt{4\vec{\lambda}^2\mathbf{k}^2(\cos^2\theta - \sin^2\eta) + 4|\vec{\lambda}||\mathbf{k}|\delta\cos\eta\cos\theta + \delta^2}}\right|_{x_1}^{x_2} \ .\end{aligned} \tag{H.17}$$

The limits of integration x_1 and x_2 are worked out in eq.(E.7) to guarantee that the Heaviside function is fulfilled. Our result for $\ell_{12,m}$ is

$$\begin{aligned}&\alpha \geq \xi_1 \ , \\ &\ell_{12,m} = \int_{\alpha-\xi_1}^{\alpha+\xi_1} d|\mathbf{k}| \frac{|\mathbf{k}|}{\mathbf{k}^2 - A - i\varepsilon} f_{12,m}(|\mathbf{k}|) \ . \\ &\alpha < \xi_1 \ , \\ &\ell_{12,m} = \left\{\int_0^{\xi_1-\alpha} + \int_{\xi_1-\alpha}^{\xi_1+\alpha}\right\} d|\mathbf{k}| \frac{|\mathbf{k}|}{\mathbf{k}^2 - A - i\varepsilon} f_{12,m}(|\mathbf{k}|) \ .\end{aligned} \tag{H.18}$$

Where we have used the function

$$f_{12,m}(|\mathbf{k}|) = \frac{m|\mathbf{k}|}{(2\pi)^2}\int_0^1 dy \left.\frac{2|\vec{\lambda}||\mathbf{k}|\cos\eta + \delta\cos\theta}{(\delta^2 - 4|\vec{\lambda}|^2|\mathbf{k}|^2)\sqrt{4\vec{\lambda}^2\mathbf{k}^2(\cos^2\theta - \sin^2\eta) + 4|\vec{\lambda}||\mathbf{k}|\delta\cos\eta\cos\theta + \delta^2}}\right|_{x_1(|\mathbf{k}|)}^{x_2(|\mathbf{k}|)} \ . \tag{H.19}$$

For the function $L_{12,m}$ of eq.(H.8) we have

$$L_{12,m} = \ell_{12,m}(\xi_m, c\beta, c\beta') + \ell_{12,m}(\xi_\ell, -c\beta, -c\beta') \ , \tag{H.20}$$

with $c\beta' = \cos\beta'$.

H.1.3 Two-medium insertions, $L_{12,d}$

For the part with two medium insertions

$$L_{12,d} = \frac{-im\sqrt{A}}{8\pi^2}\int d\hat{\mathbf{k}} \frac{\theta(\xi_1 - |\hat{\mathbf{k}}\sqrt{A} - \vec{\alpha}|)\theta(\xi_2 - |\hat{\mathbf{k}}\sqrt{A} + \vec{\alpha}|)}{[(\mathbf{k}+\mathbf{p}')^2 + m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2 + m_\pi^2]} \tag{H.21}$$

The angular integrations are of the same type as already developed for the case of one-medium insertion and, hence, we define

$$f_{12,d}(\sqrt{A}) = \frac{m\sqrt{A}}{(2\pi)^2}\int_0^1 dy\, \left.\frac{2|\vec{\lambda}|\sqrt{A}\cos\eta + \delta\cos\theta}{(\delta^2 - 4|\vec{\lambda}|^2 A)\sqrt{4\vec{\lambda}^2 A(\cos^2\theta - \sin^2\eta) + 4|\vec{\lambda}|\sqrt{A}\delta\cos\eta\cos\theta + \delta^2}}\right|_{x_1(\sqrt{A})}^{x_2(\sqrt{A})}, \tag{H.22}$$

with the integration limits given in eq.(G.21), where it is assumed that $\xi_1 \le \xi_2$. In terms of this function,

$$L_{12,d} = -i\pi f_{12,d}(\xi_1, \xi_2, c\beta, c\beta')\ . \tag{H.23}$$

When $\xi_1 \ge \xi_2$ we perform, as usual, the change of variables $\mathbf{k} \to -\mathbf{k}$ in eq.(H.21) so that

$$L_{12,d} = -i\pi f_{12,d}(\xi_2, \xi_1, -c\beta, -c\beta')\ , \tag{H.24}$$

H.2 $L_{12}^{(2)}$

$$L_{12}^{(2)} = i\int \frac{d^4k}{(2\pi)^4}\frac{\mathbf{k}^2}{[(\mathbf{k}+\mathbf{p}')^2 + m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2 + m_\pi^2]}\left[\frac{\theta(\xi_m - |\vec{\alpha}-\mathbf{k}|)}{Q^0/2 - k^0 - w(\vec{\alpha}-\mathbf{k}) - i\epsilon} + \frac{\theta(|\vec{\alpha}-\mathbf{k}| - \xi_m)}{Q^0/2 - k^0 - w(\vec{\alpha}-\mathbf{k}) + i\epsilon}\right]$$
$$\times\left[\frac{\theta(\xi_\ell - |\vec{\alpha}+\mathbf{k}|)}{Q^0/2 + k^0 - w(\vec{\alpha}+\mathbf{k}) - i\epsilon} + \frac{\theta(|\vec{\alpha}+\mathbf{k}| - \xi_\ell)}{Q^0/2 + k^0 - w(\vec{\alpha}+\mathbf{k}) + i\epsilon}\right] \tag{H.25}$$

H.2.1 Free part, $L_{12,f}^{(2)}$

$$\begin{aligned}L_{12,f}^{(2)} &= -m\int \frac{d^3k}{(2\pi)^3}\frac{\mathbf{k}^2}{[(\mathbf{k}+\mathbf{p}')^2 + m_\pi^2][(\mathbf{k}+\mathbf{p})^2 + m_\pi^2](\mathbf{k}^2 - A - i\varepsilon)}\\ &= -m\int \frac{d^3k}{(2\pi)^3}\frac{1}{[(\mathbf{k}+\mathbf{p}')^2 + m_\pi^2][(\mathbf{k}+\mathbf{p})^2 + m_\pi^2]}\\ &- Am\int \frac{d^3k}{(2\pi)^3}\frac{1}{[(\mathbf{k}+\mathbf{p}')^2 + m_\pi^2][(\mathbf{k}+\mathbf{p})^2 + m_\pi^2]}\frac{1}{\mathbf{k}^2 - A - i\varepsilon} = -mL_{02,f} + AL_{12,f}\ .\end{aligned} \tag{H.26}$$

With

$$\begin{aligned}L_{02,f} &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{[(\mathbf{k}+\mathbf{p}')^2 + m_\pi^2][(\mathbf{k}+\mathbf{p})^2 + m_\pi^2]} = -\frac{\partial}{\partial m_\pi^2}\int_0^1 dy\int \frac{d^3k}{(2\pi)^3}\frac{1}{(\mathbf{k}+\vec{\lambda})^2 + M^2}\\ &= \frac{1}{8\pi}\int_0^1 dy\frac{1}{M}\ .\end{aligned} \tag{H.27}$$

Here we have made use of the calculation of $L_{01,f}$ in eq.(G.30) as an intermediate step.

H.2.2 One-medium insertion, $L_{12,m}^{(2)}$

$$
\begin{aligned}
L_{12,m} &= m\int \frac{d^3k}{(2\pi)^3}\mathbf{k}^2 \frac{\theta_m^-(\vec{\alpha}-\mathbf{k})+\theta_\ell^-(\vec{\alpha}+\mathbf{k})}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]\,(\mathbf{k}^2-A-i\varepsilon)} \\
&= m\int \frac{d^3k}{(2\pi)^3}\mathbf{k}^2 \frac{\theta_m^-(\vec{\alpha}-\mathbf{k})}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]\,(\mathbf{k}^2-A-i\varepsilon)} \\
&+ m\int \frac{d^3k}{(2\pi)^3}\mathbf{k}^2 \frac{\theta_\ell^-(\vec{\alpha}-\mathbf{k})}{[(\mathbf{k}-\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}-\mathbf{p})^2+m_\pi^2]\,(\mathbf{k}^2-A-i\varepsilon)}
\end{aligned} \tag{H.28}
$$

The two terms in the sum can be obtained from the function

$$
\begin{aligned}
\ell_{12,m}^{(2)} &= m\int \frac{d^3k}{(2\pi)^3} \frac{|\mathbf{k}|^2\theta_m^-(\vec{\alpha}-\mathbf{k})}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]\,(\mathbf{k}^2-A-i\varepsilon)} \\
&= m\int_0^1 dy \int \frac{d^3k}{(2\pi)^3} \frac{|\mathbf{k}|^2\theta_m^-(\vec{\alpha}-\mathbf{k})}{(\mathbf{k}^2-A-i\varepsilon)[(\mathbf{k}+\vec{\lambda})^2+M^2]^2} \ .
\end{aligned} \tag{H.29}
$$

This integral can be expressed in terms of the function $f_{12,m}$ of eq.(H.19) and it is given by

$$
\begin{aligned}
&\alpha \geq \xi_1 \ , \\
&\ell_{12,m}^{(2)} = \int_{\alpha-\xi_1}^{\alpha+\xi_1} d|\mathbf{k}| \frac{|\mathbf{k}|^3}{\mathbf{k}^2-A-i\varepsilon} f_{12,m}(|\mathbf{k}|) \ . \\
&\alpha < \xi_1 \ , \\
&\ell_{12,m}^{(2)} = \left\{ \int_0^{\xi_1-\alpha} + \int_{\xi_1-\alpha}^{\xi_1+\alpha} \right\} d|\mathbf{k}| \frac{|\mathbf{k}|^3}{\mathbf{k}^2-A-i\varepsilon} f_{12,m}(|\mathbf{k}|) \ .
\end{aligned} \tag{H.30}
$$

As a result,

$$
L_{12,m}^{(2)} = \ell_{12,m}^{(2)}(\xi_m, c\beta, c\beta') + \ell_{12,m}^{(2)}(\xi_\ell, -c\beta, -c\beta') \ . \tag{H.31}
$$

H.2.3 Two-medium insertions, $L_{12,d}^{(2)}$

Since for this part $\mathbf{k}^2 = A$ then

$$
\begin{aligned}
L_{12,d}^{(2)} &= \frac{-imA^{3/2}}{8\pi^2} \int d\hat{\mathbf{k}} \frac{\theta(\xi_1 - |\hat{\mathbf{k}}\sqrt{A}-\vec{\alpha}|)\theta(\xi_2 - |\hat{\mathbf{k}}\sqrt{A}+\vec{\alpha}|)}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]} \\
&= AL_{12,d} \ .
\end{aligned} \tag{H.32}
$$

H.3 $L_{12}^{(4)}$

$$
\begin{aligned}
L_{12}^{(4)} = i\int \frac{d^4k}{(2\pi)^4} &\frac{|\mathbf{k}|^4}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]} \left[\frac{\theta(\xi_m - |\vec{\alpha}-\mathbf{k}|)}{Q^0/2-k^0-w(\vec{\alpha}-\mathbf{k})-i\epsilon} + \frac{\theta(|\vec{\alpha}-\mathbf{k}|-\xi_m)}{Q^0/2-k^0-w(\vec{\alpha}-\mathbf{k})+i\epsilon} \right] \\
\times &\left[\frac{\theta(\xi_\ell - |\vec{\alpha}+\mathbf{k}|)}{Q^0/2+k^0-w(\vec{\alpha}+\mathbf{k})-i\epsilon} + \frac{\theta(|\vec{\alpha}+\mathbf{k}|-\xi_\ell)}{Q^0/2+k^0-w(\vec{\alpha}+\mathbf{k})+i\epsilon} \right]
\end{aligned} \tag{H.33}
$$

H.3.1 Free part, $L^{(4)}_{12,f}$

$$L^{(4)}_{12,f} = -m \int \frac{d^3k}{(2\pi)^3} \frac{|\mathbf{k}|^4}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]} \frac{1}{\mathbf{k}^2-A-i\varepsilon} = -mL^{(2)}_{02,f} - mAL_{02,f} + A^2 L_{12,f} \tag{H.34}$$

with

$$L^{(2)}_{02,f} = \int \frac{d^3k}{(2\pi)^3} \frac{\mathbf{k}^2}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]} = -\frac{\partial}{\partial m_\pi^2} \int_0^1 dy \int \frac{d^3k}{(2\pi)^3} \frac{\mathbf{k}^2}{(\mathbf{k}+\vec{\lambda})^2+m_\pi^2} \ . \tag{H.35}$$

The integral over $\mathbf{k}$ can be calculated straightforwardly and neglecting those terms that vanish in the limit $\Lambda \to \infty$ we are left with

$$L^{(2)}_{02,f} = \frac{1}{8\pi} \int_0^1 dy \frac{-3m_\pi^2 + \mathbf{p}^2(1-8y(1-y)(1-\cos\varphi))}{M} - \frac{g_0}{m} \ . \tag{H.36}$$

H.3.2 One-medium insertion, $L^{(4)}_{12,m}$

$$\begin{aligned}
L^{(4)}_{12,m} &= m \int \frac{d^3k}{(2\pi)^3} |\mathbf{k}|^4 \frac{\theta^-_m(\vec{\alpha}-\mathbf{k}) + \theta^-_\ell(\vec{\alpha}+\mathbf{k})}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]\,(\mathbf{k}^2-A-i\varepsilon)} \\
&= m \int \frac{d^3k}{(2\pi)^3} |\mathbf{k}|^4 \frac{\theta^-_m(\vec{\alpha}-\mathbf{k})}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]\,(\mathbf{k}^2-A-i\varepsilon)} \\
&+ m \int \frac{d^3k}{(2\pi)^3} |\mathbf{k}|^4 \frac{\theta^-_\ell(\vec{\alpha}-\mathbf{k})}{[(\mathbf{k}-\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}-\mathbf{p})^2+m_\pi^2]\,(\mathbf{k}^2-A-i\varepsilon)}
\end{aligned} \tag{H.37}$$

The two terms in the sum can be obtained from the function

$$\begin{aligned}
\ell^{(4)}_{12,m} &= m \int \frac{d^3k}{(2\pi)^3} \frac{|\mathbf{k}|^4 \theta^-_m(\vec{\alpha}-\mathbf{k})}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]\,(\mathbf{k}^2-A-i\varepsilon)} \\
&= m \int_0^1 dy \int \frac{d^3k}{(2\pi)^3} \frac{|\mathbf{k}|^4 \theta^-_m(\vec{\alpha}-\mathbf{k})}{(\mathbf{k}^2-A-i\varepsilon)[(\mathbf{k}+\vec{\lambda})^2+M^2]^2} \ .
\end{aligned} \tag{H.38}$$

In terms of the function $f_{12,m}$ of eq.(H.19) this function is given by

$$\begin{aligned}
&\alpha \geq \xi_1 \ , \\
&\ell^{(4)}_{12,m} = \int_{\alpha-\xi_1}^{\alpha+\xi_1} d|\mathbf{k}| \frac{|\mathbf{k}|^5}{\mathbf{k}^2-A-i\varepsilon} f_{12,m}(|\mathbf{k}|) \ . \\
&\alpha < \xi_1 \ , \\
&\ell^{(4)}_{12,m} = \left\{ \int_0^{\xi_1-\alpha} + \int_{\xi_1-\alpha}^{\xi_1+\alpha} \right\} d|\mathbf{k}| \frac{|\mathbf{k}|^5}{\mathbf{k}^2-A-i\varepsilon} f_{12,m}(|\mathbf{k}|) \ .
\end{aligned} \tag{H.39}$$

Thus,

$$L^{(4)}_{12,m} = \ell^{(4)}_{12,m}(\xi_m, c\beta, c\beta') + \ell^{(4)}_{12,m}(\xi_\ell, -c\beta, -c\beta') \ . \tag{H.40}$$

H.3.3 Two-medium insertions, $L^{(4)}_{12,d}$

Since for this part $\mathbf{k}^2 = A$ then

$$L^{(4)}_{12,d} = \frac{-imA^{5/2}}{8\pi^2}\int d\hat{\mathbf{k}}\frac{\theta(\xi_1 - |\hat{\mathbf{k}}\sqrt{A} - \vec{\alpha}|)\theta(\xi_2 - |\hat{\mathbf{k}}\sqrt{A} + \vec{\alpha}|)}{[(\mathbf{k}+\mathbf{p}')^2 + m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2 + m_\pi^2]}$$
$$= A^2 L_{12,d}\ . \tag{H.41}$$

H.4 L^a_{12}

$$L^a_{12} = i\int \frac{d^4k}{(2\pi)^4}\frac{k^a}{[(\mathbf{k}+\mathbf{p}')^2 + m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2 + m_\pi^2]}\left[\frac{\theta(\xi_m - |\vec{\alpha}-\mathbf{k}|)}{Q^0/2 - k^0 - w(\vec{\alpha}-\mathbf{k}) - i\epsilon} + \frac{\theta(|\vec{\alpha}-\mathbf{k}| - \xi_m)}{Q^0/2 - k^0 - w(\vec{\alpha}-\mathbf{k}) + i\epsilon}\right]$$
$$\times\left[\frac{\theta(\xi_\ell - |\vec{\alpha}+\mathbf{k}|)}{Q^0/2 + k^0 - w(\vec{\alpha}+\mathbf{k}) - i\epsilon} + \frac{\theta(|\vec{\alpha}+\mathbf{k}| - \xi_\ell)}{Q^0/2 + k^0 - w(\vec{\alpha}+\mathbf{k}) + i\epsilon}\right]$$
$$= L^\alpha_{12}\,\alpha^a + L^p_{12}\,p^a + L^{p'}_{12}\,p'^a\ . \tag{H.42}$$

H.4.1 Free part, $L^a_{12,f}$

$$L^a_{12,f} = -m\int \frac{d^3q}{(2\pi)^3}\frac{k^a}{[(\mathbf{k}+\mathbf{p}')^2 + m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2 + m_\pi^2]\,(\mathbf{k}^2 - A - i\varepsilon)} = L^p_{12,f}(p+p')^a\ . \tag{H.43}$$

Note that in this case $\mathbf{p}$ and $\mathbf{p}'$ appears only in a symmetric way because it is the free part and there is no vector of reference that could introduce any asymmetry between $\mathbf{p}$ and $\mathbf{p}'$.

$$L^a_{12,f} = \int_0^1 dy\left(-\frac{\partial}{\partial m_\pi^2}\right)(-m)\int \frac{d^3k}{(2\pi)^3}\frac{k^a}{(\mathbf{k}^2 - A - i\varepsilon)[(\mathbf{k}+\vec{\lambda})^2 + M^2]}\ . \tag{H.44}$$

Using the result of eq.(G.31), but now in terms of $\vec{\lambda}$ and M. Then

$$L^a_{12,f} = \int_0^1 dy\left(-\frac{\partial}{\partial m_\pi^2}\right)\frac{\lambda^a}{2|\vec{\lambda}|^2}\left(L_{10,f} - (\vec{\lambda}^2 + M^2 + A)L_{11,f} + mL_{01,f}\right)$$
$$= \frac{m}{16\pi}\int_0^1 dy\frac{\lambda^a}{|\vec{\lambda}|^2}\left\{\frac{2}{M}\frac{\mathbf{p}^2 + m_\pi^2 - iM\sqrt{A}}{\mathbf{p}^2 + m_\pi^2 - A - 2iM\sqrt{A}} - \frac{i}{|\vec{\lambda}|}\log\frac{A - (|\vec{\lambda}| + iM)^2}{M^2 + (\sqrt{A} - |\vec{\lambda}|)^2}\right\}\ . \tag{H.45}$$

From eq.(H.2) $\vec{\lambda} = y\mathbf{p} + (1-y)\mathbf{p}'$ and since $M(y)$ is symmetric under the exchange $y \leftrightarrow 1-y$, as follows from its definition in eq.(H.2), one can write

$$L^a_{12,f} = (p+p')^a\frac{m}{16\pi}\int_0^1 dy\frac{y}{|\vec{\lambda}|^2}\left\{\frac{2}{M}\frac{\mathbf{p}^2 + m_\pi^2 - iM\sqrt{A}}{\mathbf{p}^2 + m_\pi^2 - A - 2iM\sqrt{A}} - \frac{i}{|\vec{\lambda}|}\log\frac{A - (|\vec{\lambda}| + iM)^2}{M^2 + (\sqrt{A} - |\vec{\lambda}|)^2}\right\}\ . \tag{H.46}$$

So that,

$$L^p_{12,f} = \frac{m}{16\pi}\int_0^1 dy\frac{y}{|\vec{\lambda}|^2}\left\{\frac{2}{M}\frac{\mathbf{p}^2 + m_\pi^2 - iM\sqrt{A}}{\mathbf{p}^2 + m_\pi^2 - A - 2iM\sqrt{A}} - \frac{i}{|\vec{\lambda}|}\log\frac{A - (|\vec{\lambda}| + iM)^2}{M^2 + (\sqrt{A} - |\vec{\lambda}|)^2}\right\}\ . \tag{H.47}$$

H.4.2 One-medium insertion, $L^a_{12,m}$

$$
\begin{aligned}
L^a_{12,m} &= m\int \frac{d^3k}{(2\pi)^3}\frac{k^a}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]}\frac{\theta^-_m(\vec{\alpha}-\mathbf{k})+\theta^-_\ell(\vec{\alpha}+\mathbf{k})}{\mathbf{k}^2-A-i\varepsilon} \\
&= m\int \frac{d^3k}{(2\pi)^3}\frac{k^a}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]}\frac{\theta^-_m(\vec{\alpha}-\mathbf{k})}{\mathbf{k}^2-A-i\varepsilon} \\
&- m\int \frac{d^3k}{(2\pi)^3}\frac{k^a}{[(\mathbf{k}-\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}-\mathbf{p})^2+m_\pi^2]}\frac{\theta^-_\ell(\vec{\alpha}-\mathbf{k})}{\mathbf{k}^2-A-i\varepsilon} \\
&= L^\alpha_{12,m}\,\alpha^a + L^p_{12,m}\,p^a + L^{p'}_{12,m}\,p'^a \ .
\end{aligned} \tag{H.48}
$$

As usual, the previous integral can be put in terms of

$$
\begin{aligned}
\ell^a_{12,m} &= m\int \frac{d^3k}{(2\pi)^3}\frac{k^a}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]}\frac{\theta^-_m(\vec{\alpha}-\mathbf{k})}{\mathbf{k}^2-A-i\varepsilon} \\
&= \ell^\alpha_{12,m}\,\alpha^a + \ell^p_{12,m}\,p^a + \ell^{p'}_{12,m}\,p'^a \ .
\end{aligned} \tag{H.49}
$$

Eq.(H.49) is multiplied by p^a and then we proceed by completing the square $2\mathbf{p}\cdot\mathbf{k} = (\mathbf{k}+\mathbf{p})^2+m_\pi^2-(\mathbf{k}^2+\mathbf{p}^2+m_\pi^2)$ appearing in the denominator. The resulting integral is called $\ell^{pk}_{12,m}$ and is given by

$$
\begin{aligned}
\ell^{pk}_{12,m} &= \ell^\alpha_{12,m}\vec{\alpha}\cdot\mathbf{p} + \ell^p_{12,m}\mathbf{p}^2 + \ell^{p'}_{12,m}\mathbf{p}\cdot\mathbf{p}' \\
&= \frac{1}{2}\ell_{11,m}(\xi_m,\cos\beta') - \frac{\mathbf{p}^2+m_\pi^2+A}{2}\ell_{12,m} - \frac{m}{2}L_{02,m} \ ,
\end{aligned} \tag{H.50}
$$

with the new integral

$$
L_{02,m} = \int \frac{d^3k}{(2\pi)^3}\frac{\theta^-_m(\vec{\alpha}-\mathbf{k})}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]} = \int_0^1 dy\frac{-\partial}{\partial m_\pi^2}\int\frac{d^3k}{(2\pi)^3}\frac{\theta^-_m(\vec{\alpha}-\mathbf{k})}{(\mathbf{k}+\vec{\lambda})^2+M^2} \ . \tag{H.51}
$$

This last integral corresponds to $L_{01,m}$ evaluated in eq.(G.35), but in terms of $\vec{\lambda}$, so that instead of $\mathbf{p}_1$ we have now

$$
\begin{aligned}
\widetilde{\mathbf{p}}_1 &= \vec{\alpha}+\vec{\lambda} \ , \\
\widetilde{\mathbf{p}}_1^2 &= \alpha^2+\vec{\lambda}^2+2\alpha|\vec{\lambda}|\cos\eta \ ,
\end{aligned} \tag{H.52}
$$

with $\cos\eta$ defined in eq.(H.11). After taking the derivative one ends with

$$
L_{02,m} = \frac{1}{8\pi^2}\int_0^1\frac{dy}{M|\widetilde{\mathbf{p}}_1|}\left\{|\widetilde{\mathbf{p}}_1|\arctan\frac{|\widetilde{\mathbf{p}}_1|+\xi_m}{M} - |\widetilde{\mathbf{p}}_1|\arctan\frac{|\widetilde{\mathbf{p}}_1|-\xi_m}{M} - \frac{M}{2}\log\frac{(\xi_m+|\widetilde{\mathbf{p}}_1|)^2+M^2}{(|\xi_m-\widetilde{\mathbf{p}}_1|)^2+M^2}\right\} \ . \tag{H.53}
$$

We now contract eq.(H.49) with p'^a and call the resulting integral by $\ell^{p'k}_{12,m}$. Since eq.(H.49) is symmetric under the exchange $\mathbf{p}\leftrightarrow\mathbf{p}'$ we are left with the same expression as eq.(H.50) but exchanging $\mathbf{p}\leftrightarrow\mathbf{p}'$

$$
\begin{aligned}
\ell^{p'k}_{12,m} &= \ell^\alpha_{12,m}\vec{\alpha}\cdot\mathbf{p}' + \ell^p_{12,m}\mathbf{p}\cdot\mathbf{p}' + \ell^{p'}_{12,m}\mathbf{p}^2 \\
&= \frac{1}{2}\ell_{11,m}(\xi_m,\cos\beta) - \frac{\mathbf{p}^2+m_\pi^2+A}{2}\ell_{12,m} - \frac{m}{2}L_{02,m} \ .
\end{aligned} \tag{H.54}
$$

Eq.(H.49) is contracted with α^a so that

$$\begin{aligned}
\ell^{\alpha k}_{12,m} &= \ell^{\alpha}_{12,m}\alpha^2 + \ell^{p}_{12,m}\vec{\alpha}\cdot\mathbf{p} + \ell^{p'}_{12,m}\vec{\alpha}\cdot\mathbf{p}' \\
&= m\int \frac{d^3k}{(2\pi)^3}\frac{\theta^-_m(\mathbf{k}-\vec{\alpha})\vec{\alpha}\cdot\mathbf{k}}{(\mathbf{k}^2-A-i\varepsilon)\,[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]} \\
&= \frac{m\alpha}{(2\pi)^2}\int\frac{d|\mathbf{k}|\,|\mathbf{k}|^3}{\mathbf{k}^2-A-i\varepsilon}\int_0^1 dy\int_{x_1}^{x_2} d\cos\theta\cos\theta\frac{1}{2\pi}\int_0^{2\pi}d\phi\frac{1}{\left[(\mathbf{k}+\vec{\lambda})^2+M^2\right]^2}\ . \qquad \text{(H.55)}
\end{aligned}$$

The same integral in ϕ was already evaluated in eq.(H.16). Next, from the integration in $\cos\theta$ we define

$$\begin{aligned}
f^{\alpha k}_{12,m} &= \frac{m\alpha\mathbf{k}^2}{4\pi^2}\int_0^1 dy\int_{x_1}^{x_2} d\cos\theta\, cos\theta\,\frac{\delta+2|\vec{\lambda}||\mathbf{k}|\cos\eta\cos\theta}{\left[\delta^2+4|\vec{\lambda}|^2|\mathbf{k}|^2(\cos^2\theta-\sin^2\eta)+4\delta|\vec{\lambda}||\mathbf{k}|\cos\theta\cos\eta\right]^{3/2}} \\
&= \frac{m\alpha}{16\pi^2}\int_0^1 dy\left\{\frac{b_2(b_2^2\cos\theta+a_2(b_2-2c_2\cos\theta))-2\delta^2c_2(2a_2+b_2\cos\theta)}{|\lambda|^2(4a_2c_2-b_2^2)\delta\sqrt{a_2+b_2\cos\theta+c_2\cos^2\theta}}\right. \\
&\left.+\frac{b_2}{2|\vec{\lambda}|^2\delta\sqrt{c_2}}\log\left(\frac{b_2+2c_2\cos\theta}{\sqrt{c_2}}+2\sqrt{a_2+b_2\cos\theta+c_2\cos^2\theta}\right)\right\}^{x_2(|\mathbf{k}|)}_{x_1(|\mathbf{k}|)}\ , \qquad \text{(H.56)}
\end{aligned}$$

with the coefficients

$$\begin{aligned}
a_2 &= \delta^2-4|\vec{\lambda}|^2|\mathbf{k}|^2\sin^2\eta\ , \\
b_2 &= 4|\vec{\lambda}||\mathbf{k}|\delta\cos\eta\ , \\
c_2 &= 4|\vec{\lambda}|^2|\mathbf{k}|^2\ . \qquad \text{(H.57)}
\end{aligned}$$

On the other hand, $x_1(|\mathbf{k}|)$ and $x_2(\mathbf{k})$ are given in eq.(E.7).

In this way,

$$\begin{aligned}
&\alpha\geq\xi_1\ , \\
&\ell^{\alpha k}_{12,m}(\xi_1) = \int_{\alpha-\xi_1}^{\alpha+\xi_1} d|\mathbf{k}|\frac{|\mathbf{k}|}{\mathbf{k}^2-A-i\varepsilon}f^{\alpha k}_{12,m}(|\mathbf{k}|)\ . \\
&\alpha<\xi_1\ , \\
&\ell^{\alpha k}_{12,m}(\xi_1) = \left\{\int_0^{\xi_1-\alpha}+\int_{\xi_1-\alpha}^{\xi_1+\alpha}\right\}d|\mathbf{k}|\frac{|\mathbf{k}|}{\mathbf{k}^2-A-i\varepsilon}f^{\alpha k}_{12,m}(|\mathbf{k}|)\ . \qquad \text{(H.58)}
\end{aligned}$$

Taking together eqs.(H.50), (H.54) and (H.55) we have the system of equations

$$\begin{pmatrix} \vec{\alpha}\cdot\mathbf{p} & \mathbf{p}^2 & \mathbf{p}\cdot\mathbf{p}' \\ \vec{\alpha}\cdot\mathbf{p}' & \mathbf{p}\cdot\mathbf{p}' & \mathbf{p}^2 \\ \alpha^2 & \vec{\alpha}\cdot\mathbf{p} & \vec{\alpha}\cdot\mathbf{p}' \end{pmatrix}\cdot\begin{pmatrix} \ell^{\alpha}_{12,m} \\ \ell^{p}_{12,m} \\ \ell^{p'}_{12,m}\end{pmatrix} = \begin{pmatrix} \ell^{pk}_{12,m} \\ \ell^{p'k}_{12,m} \\ \ell^{\alpha k}_{12,m}\end{pmatrix} \qquad \text{(H.59)}$$

with $\vec{\alpha}\cdot\mathbf{p}=\alpha|\mathbf{p}|\cos\beta$, $\vec{\alpha}\cdot\mathbf{p}'=\alpha|\mathbf{p}|\cos\beta'$, $\mathbf{p}\cdot\mathbf{p}'=|\mathbf{p}|^2\cos\varphi$. From eq.(H.59) the functions $\ell^{\alpha}_{12,m}$, $\ell^{p}_{12,m}$ and $\ell^{p'}_{12,m}$ are determined. In terms of them $L^a_{12,m}$, eq.(H.48), reads

$$\begin{aligned}
L^a_{12,m} &= \left[\ell^{\alpha}_{12,m}(\xi_m,c\beta,c\beta')-\ell^{\alpha}_{12,m}(\xi_\ell,-c\beta,-c\beta')\right]\alpha^a+\left[\ell^{p}_{12,m}(\xi_m,c\beta,c\beta')+\ell^{p}_{12,m}(\xi_\ell,-c\beta,-c\beta')\right]p^a \\
&+\left[\ell^{p'}_{12,m}(\xi_m,c\beta,c\beta')+\ell^{p'}_{12,m}(\xi_\ell,-c\beta,-c\beta')\right]p'^a\ . \qquad \text{(H.60)}
\end{aligned}$$

H.4.3 Two-medium insertions, $L^a_{12,d}$

$$
\begin{aligned}
L^a_{12,d} &= \frac{-im\sqrt{A}}{8\pi^2}\int d\hat{\mathbf{k}}\frac{\theta(\xi_1-|\hat{\mathbf{k}}\sqrt{A}-\vec{\alpha}|)\theta(\xi_2-|\hat{\mathbf{k}}\sqrt{A}+\vec{\alpha}|)}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]}k^a \\
&= L^\alpha_{12,d}\,\alpha^a + L^p_{12,d}\,p^a + L^{p'}_{12,d}\,p'^a \ . \qquad \text{(H.61)}
\end{aligned}
$$

We are driven to the same angular integrations as before for the case of one-medium insertion so that many of the previous results can be used. In particular, the contraction with p^a and p'^a gives, respectively,

$$
\begin{aligned}
L^{pk}_{12,d} &= \frac{1}{2}L_{11,d}(\cos\beta') - \frac{A+\mathbf{p}^2+m_\pi^2}{2}L_{12,d}\ , \\
L^{p'k}_{12,d} &= \frac{1}{2}L_{11,d}(\cos\beta) - \frac{A+\mathbf{p}^2+m_\pi^2}{2}L_{12,d}\ . \qquad \text{(H.62)}
\end{aligned}
$$

Regarding the contraction with α^a we can write

$$
\begin{aligned}
L^{\alpha k}_{12,d} &= -i\pi f^{\alpha k}_{12m,d}(\sqrt{A})\ , \\
f^{\alpha k}_{12,d} &= \frac{m\alpha}{16\pi^2}\int_0^1 dy\left\{\frac{b_2(b_2^2\cos\theta + a_2(b_2-2c_2\cos\theta)) - 2\delta^2 c_2(2a_2+b_2\cos\theta)}{|\lambda|^2(4a_2c_2-b_2^2)\delta\sqrt{a_2+b_2\cos\theta+c_2\cos^2\theta}}\right. \\
&\left. +\frac{b_2}{2|\vec{\lambda}|^2\delta\sqrt{c_2}}\log\left(\frac{b_2+2c_2\cos\theta}{\sqrt{c_2}}+2\sqrt{a_2+b_2\cos\theta+c_2\cos^2\theta}\right)\right\}^{x_2(|\sqrt{A}|)}_{x_1(|\sqrt{A}|)}\ , \qquad \text{(H.63)}
\end{aligned}
$$

where a_2, b_2 and c_2 are given in eq.(H.57) with $|\mathbf{k}|=\sqrt{A}$. The integration limits are given in eq.(G.21), where it is assumed that $\xi_1 \le \xi_2$.

With the previous results for $\ell^{pk}_{12,d}$, $\ell^{p'k}_{12,d}$ and $\ell^{\alpha k}_{12,d}$ we are driven to the analogous equations as in eq.(H.59) above,

$$
\begin{pmatrix} \vec{\alpha}\cdot\mathbf{p} & \mathbf{p}^2 & \mathbf{p}\cdot\mathbf{p}' \\ \vec{\alpha}\cdot\mathbf{p}' & \mathbf{p}\cdot\mathbf{p}' & \mathbf{p}^2 \\ \alpha^2 & \vec{\alpha}\cdot\mathbf{p} & \vec{\alpha}\cdot\mathbf{p}' \end{pmatrix}\cdot\begin{pmatrix} L^\alpha_{12,d} \\ L^p_{12,d} \\ L^{p'}_{12,d}\end{pmatrix} = \begin{pmatrix} L^{pk}_{12,d} \\ L^{p'k}_{12,d} \\ L^{\alpha sk}_{12,d}\end{pmatrix} \qquad \text{(H.64)}
$$

in terms of which $L^{pk}_{12,d}$, $L^{p'k}_{12,d}$ and $L^{\alpha k}_{12,d}$ are calculated for $\xi_1 \le \xi_2$.

For $\xi_1 \ge \xi_2$ the change of variable $\mathbf{k}\to-\mathbf{k}$ is performed in eq.(H.61) and then one has

$$
L^a_{12,d} = -L^\alpha_{12,d}(\xi_2,\xi_1,-c\beta,-c\beta')\alpha^a + L^p_{12,d}(\xi_2,\xi_1,-c\beta,-c\beta')p^a + L^{p'}_{12,d}(\xi_2,\xi_1,-c\beta,-c\beta')p'^a\ , \qquad \text{(H.65)}
$$

with the different coefficients functions calculated as above.

H.5 L^{2ka}_{12}

$$
\begin{aligned}
L^{2ka}_{12} &= i\int\frac{d^4k}{(2\pi)^4}\frac{|\mathbf{k}|^2k^a}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]}\left[\frac{\theta(\xi_m-|\vec{\alpha}-\mathbf{k}|)}{Q^0/2-k^0-w(\vec{\alpha}-\mathbf{k})-i\epsilon}+\frac{\theta(|\vec{\alpha}-\mathbf{k}|-\xi_m)}{Q^0/2-k^0-w(\vec{\alpha}-\mathbf{k})+i\epsilon}\right] \\
&\times\left[\frac{\theta(\xi_\ell-|\vec{\alpha}+\mathbf{k}|)}{Q^0/2+k^0-w(\vec{\alpha}+\mathbf{k})-i\epsilon}+\frac{\theta(|\vec{\alpha}+\mathbf{k}|-\xi_\ell)}{Q^0/2+k^0-w(\vec{\alpha}+\mathbf{k})+i\epsilon}\right] \\
&= L^{2k\alpha}_{12}\,\alpha^a + L^{2kp}_{12}\,p^a + L^{2kp'}_{12}\,p'^a\ . \qquad \text{(H.66)}
\end{aligned}
$$

H.5.1 Free part, $L^{2ka}_{12,f}$

$$
\begin{aligned}
L^{2ka}_{12,f} &= L^{2kp}_{12,f}(p+p')^a \\
&= -m\int \frac{d^3q}{(2\pi)^3}\frac{|\mathbf{k}|^2k^a}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]\,(\mathbf{k}^2-A-i\varepsilon)} \\
&= AL^a_{12,f} - m\int \frac{d^3k}{(2\pi)^3}\frac{k^a}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]}\ .
\end{aligned}
\tag{H.67}
$$

We need to evaluate the integral

$$
\begin{aligned}
&\int \frac{d^3k}{(2\pi)^3}\frac{k^a}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]} = \int_0^1 dy\int\frac{d^3k}{(2\pi)^3}\frac{k^a}{\left[(\mathbf{k}+\vec{\lambda})^2+M^2\right]^2} \\
&= -\frac{\partial}{\partial m_\pi^2}\int_0^1 dy\int\frac{d^3k}{(2\pi)^3}\frac{k^a}{(\mathbf{k}+\vec{\lambda})^2+M^2}\ .
\end{aligned}
\tag{H.68}
$$

The last integral was already evaluated in eq.(G.56), but now $\mathbf{p}\to\vec{\lambda}$. Then,

$$
L^{2ka}_{12,f} = AL^a_{12,f} + \frac{m(p^a+p'^a)}{16\pi p}\sqrt{\frac{2}{1-\cos\varphi}}\arcsin\frac{p\sqrt{1-\cos\varphi}}{\sqrt{2m_\pi^2+p^2(1-\cos\varphi)}}\ .
\tag{H.69}
$$

H.5.2 One-medium insertion, $L^{2ka}_{12,m}$

$$
\begin{aligned}
L^{2ka}_{12,m} &= m\int\frac{d^3k}{(2\pi)^3}\frac{|\mathbf{k}|^2k^a}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]}\frac{\theta^-_m(\vec{\alpha}-\mathbf{k})+\theta^-_\ell(\vec{\alpha}+\mathbf{k})}{\mathbf{k}^2-A-i\varepsilon} \\
&= m\int\frac{d^3k}{(2\pi)^3}\frac{|\mathbf{k}|^2k^a}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]}\frac{\theta^-_m(\vec{\alpha}-\mathbf{k})}{\mathbf{k}^2-A-i\varepsilon} \\
&- m\int\frac{d^3k}{(2\pi)^3}\frac{\mathbf{k}|^2k^a}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]}\frac{\theta^-_\ell(\vec{\alpha}-\mathbf{k})}{\mathbf{k}^2-A-i\varepsilon} \\
&= L^{2k\alpha}_{12,m}\,\alpha^a + L^{2kp}_{12,m}\,p^a + L^{2kp'}_{12,m}\,p'^a\ .
\end{aligned}
\tag{H.70}
$$

We proceed along similar steps as those used in section H.4.2 for the evaluation of $L^a_{12,m}$ and define the function

$$
\begin{aligned}
\ell^{2ka}_{12,m} &= m\int\frac{d^3k}{(2\pi)^3}\frac{|\mathbf{k}|^2k^a}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]}\frac{\theta^-_m(\vec{\alpha}-\mathbf{k})}{\mathbf{k}^2-A-i\varepsilon} \\
&= \ell^{2k\alpha}_{12,m}\,\alpha^a + \ell^{2kp}_{12,m}\,p^a + \ell^{2kp'}_{12,m}\,p'^a\ .
\end{aligned}
\tag{H.71}
$$

The contraction with p^a gives

$$
\begin{aligned}
\ell^{p2k}_{12,m} &= \ell^{2k\alpha}_{12,m}\vec{\alpha}\cdot\mathbf{p} + \ell^{2kp}_{12,m}\mathbf{p}^2 + \ell^{2kp'}_{12,m}\mathbf{p}\cdot\mathbf{p}' \\
&= A\ell^{pk}_{12,m} + m\int\frac{d^3k}{(2\pi)^3}\frac{\theta^-_m(\vec{\alpha}-\mathbf{k})\mathbf{p}\cdot\mathbf{k}}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]}\ .
\end{aligned}
\tag{H.72}
$$

We need to evaluate the last integral

$$m\int\frac{d^3k}{(2\pi)^3}\frac{\theta_m^-(\vec{\alpha}-\mathbf{k})\mathbf{p}\cdot\mathbf{k}}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]}=\frac{m}{2}L_{01,m}(\cos\beta')-\frac{m(m_\pi^2+|\mathbf{p}|^2)}{2}L_{02,m}-\frac{m}{2}L_{02,m}^{2k}\ . \tag{H.73}$$

with $L_{01,m}$ and $L_{02,m}$ given in eqs.(G.35) and (H.53), respectively. On the other hand,

$$L_{02,m}^{2k}=\int\frac{d^3k}{(2\pi)^3}\frac{\theta_m^-(\vec{\alpha}-\mathbf{k})|\mathbf{k}|^2}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]}=\int_0^1 dy\int\frac{d^3k}{(2\pi)^3}\frac{\theta_m^-(\vec{\alpha}-\mathbf{k})|\mathbf{k}|^2}{\left[(\mathbf{k}+\vec{\lambda})^2+M^2\right]^2}\ . \tag{H.74}$$

The resulting integral can be written in terms of $f_{12,m}$, eq.(H.19), and one has

$$\begin{aligned}
&\alpha\geq\xi_1\ ,\\
&L_{02,m}^{2k}=\frac{1}{m}\int_{\alpha-\xi_1}^{\alpha+\xi_1}d|\mathbf{k}||\mathbf{k}|^3\,f_{12,m}(|\mathbf{k}|)\ .\\
&\alpha<\xi_1\ ,\\
&L_{02,m}^{2k}=\frac{1}{m}\left\{\int_0^{\xi_1-\alpha}+\int_{\xi_1-\alpha}^{\xi_1+\alpha}\right\}d|\mathbf{k}||\mathbf{k}|^3\,f_{12,m}(|\mathbf{k}|)\ .
\end{aligned}\tag{H.75}$$

We can then write for eq.(H.72)

$$\ell_{12,m}^{p2k}=A\ell_{12,m}^{pk}+\frac{m}{2}L_{01,m}(\cos\beta')-\frac{m(m_\pi^2+|\mathbf{p}|^2)}{2}L_{02,m}-\frac{m}{2}L_{02,m}^{2k} \tag{H.76}$$

Similarly for the contraction with p'^a of eq.(H.71)

$$\begin{aligned}
\ell_{12,m}^{p'2k}&=\ell_{12,m}^{2k\alpha}\vec{\alpha}\cdot\mathbf{p}'+\ell_{12,m}^{2kp}\mathbf{p}\cdot\mathbf{p}'+\ell_{12,m}^{2kp'}\mathbf{p}^2\\
&=A\ell_{12,m}^{p'k}+\frac{m}{2}L_{01,m}(\cos\beta)-\frac{m(m_\pi^2+|\mathbf{p}|^2)}{2}L_{02,m}-\frac{m}{2}L_{02,m}^{2k}
\end{aligned}\tag{H.77}$$

When contracting eq.(H.71) with α^a we have the same angular integrations as in $\ell_{12,m}^{\alpha k}$ and the only difference is in the $|\mathbf{k}|$ integration in which we have now an extra power of $|\mathbf{k}|^2$. Modifying accordingly eq.(H.58) one has

$$\begin{aligned}
&\alpha\geq\xi_1\ ,\\
&\ell_{12,m}^{\alpha 2k}(\xi_1)=\int_{\alpha-\xi_1}^{\alpha+\xi_1}d|\mathbf{k}|\frac{|\mathbf{k}|^3}{\mathbf{k}^2-A-i\varepsilon}f_{12,m}^{\alpha k}(|\mathbf{k}|)\ .\\
&\alpha<\xi_1\ ,\\
&\ell_{12,m}^{\alpha 2k}(\xi_1)=\left\{\int_0^{\xi_1-\alpha}+\int_{\xi_1-\alpha}^{\xi_1+\alpha}\right\}d|\mathbf{k}|\frac{|\mathbf{k}|^3}{\mathbf{k}^2-A-i\varepsilon}f_{12,m}^{\alpha k}(|\mathbf{k}|)\ ,
\end{aligned}\tag{H.78}$$

with $f_{12,m}^{\alpha k}(|\mathbf{k}|)$ given in eq.(H.56).

We then have the analogous set of equations as above for $\ell_{12,m}^a$ that now reads

$$\begin{pmatrix}\vec{\alpha}\cdot\mathbf{p} & \mathbf{p}^2 & \mathbf{p}\cdot\mathbf{p}'\\ \vec{\alpha}\cdot\mathbf{p}' & \mathbf{p}\cdot\mathbf{p}' & \mathbf{p}^2\\ \alpha^2 & \vec{\alpha}\cdot\mathbf{p} & \vec{\alpha}\cdot\mathbf{p}'\end{pmatrix}\cdot\begin{pmatrix}\ell_{12,m}^{2k\alpha}\\ \ell_{12,m}^{2kp}\\ \ell_{12,m}^{2kp'}\end{pmatrix}=\begin{pmatrix}\ell_{12,m}^{p2k}\\ \ell_{12,m}^{p'2k}\\ \ell_{12,m}^{\alpha 2k}\end{pmatrix}\ . \tag{H.79}$$

We can then write

$$L^{2ka}_{12,m} = \left[\ell^{2k\alpha}_{12,m}(\xi_m, c\beta, c\beta') - \ell^{2k\alpha}_{12,m}(\xi_\ell, -c\beta, -c\beta')\right]\alpha^a + \left[\ell^{2kp}_{12,m}(\xi_m, c\beta, c\beta') + \ell^{2kp}_{12,m}(\xi_\ell, -c\beta, -c\beta')\right]p^a$$
$$+ \left[\ell^{2kp'}_{12,m}(\xi_m, c\beta, c\beta') + \ell^{2kp'}_{12,m}(\xi_\ell, -c\beta, -c\beta')\right]p'^a \ . \tag{H.80}$$

H.5.3 Two-medium insertion, $L^{2ka}_{12,d}$

$$L^{2ka}_{12,d} = \frac{-im\sqrt{A}}{8\pi^2}\int d\hat{\mathbf{k}}|\mathbf{k}|^2 \frac{\theta(\xi_1 - |\hat{\mathbf{k}}\sqrt{A} - \vec{\alpha}|)\theta(\xi_2 - |\hat{\mathbf{k}}\sqrt{A} + \vec{\alpha}|)}{(\mathbf{k}+\mathbf{p})^2 + m_\pi^2}k^a$$
$$= L^{2k\alpha}_{12,d}\,\alpha^a + L^{2kp}_{12,d}\,p^a + L^{2kp'}_{12,d}\,p'^a \ . \tag{H.81}$$

Due to the extra energy Dirac delta function the additional factor $|\mathbf{k}|^2$ is fixed to A so that we have simply

$$L^{2ka}_{12,d} = A L^a_{12,d} \ . \tag{H.82}$$

H.6 L^{ab}_{12}

$$L^{ab}_{12} = i\int \frac{d^4k}{(2\pi)^4}\frac{k^a k^b}{[(\mathbf{k}+\mathbf{p}')^2 + m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2 + m_\pi^2]}\left[\frac{\theta(\xi_m - |\vec{\alpha} - \mathbf{k}|)}{Q^0/2 - k^0 - w(\vec{\alpha} - \mathbf{k}) - i\epsilon} + \frac{\theta(|\vec{\alpha} - \mathbf{k}| - \xi_m)}{Q^0/2 - k^0 - w(\vec{\alpha} - \mathbf{k}) + i\epsilon}\right]$$
$$\times\left[\frac{\theta(\xi_\ell - |\vec{\alpha} + \mathbf{k}|)}{Q^0/2 + k^0 - w(\vec{\alpha} + \mathbf{k}) - i\epsilon} + \frac{\theta(|\vec{\alpha} + \mathbf{k}| - \xi_\ell)}{Q^0/2 + k^0 - w(\vec{\alpha} + \mathbf{k}) + i\epsilon}\right]$$
$$\equiv L^{T\alpha}_{12}\,\alpha^a\alpha^b + L^{Tp}_{12}\,p^a p^b + L^{Tp'}_{12}\,p'^a p'^b + L^{Tpp'}_{12}(p^a p'^b + p^b p'^a) + L^{T\alpha p}_{12}(\alpha^a p^b + \alpha^b p^a) + L^{T\alpha p'}_{12}(\alpha^a p'^b + \alpha^b p'^a) \ . \tag{H.83}$$

In the tensor decomposition we have taken into account the exchange symmetry between the indices a and b in eq.(H.83).

H.6.1 Free part, $L^{ab}_{12,f}$

$$L^{ab}_{12,f} = -m\int \frac{d^3q}{(2\pi)^3}\frac{k^a k^b}{[(\mathbf{k}+\mathbf{p}')^2 + m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2 + m_\pi^2]\,(\mathbf{k}^2 - A - i\varepsilon)}$$
$$= -\int_0^1 dy \frac{\partial}{\partial m_\pi^2}(-m)\int \frac{d^3k}{(2\pi)^3}\frac{k^a k^b}{(\mathbf{k}^2 - A - i\varepsilon)\left[(\mathbf{k}+\vec{\lambda})^2 + M^2\right]} \ . \tag{H.84}$$

The last integral corresponds to $L^{ab}_{11,f}$ evaluated in section G.3 so that

$$L^{ab}_{12,f} = -\int_0^1 dy\left(\frac{L^{Tg}_{11,f}}{\partial m_\pi^2}\delta^{ab} + \frac{\partial L^p_{11,f}}{\partial m_\pi^2}\lambda^a\lambda^b\right) \ , \tag{H.85}$$

with $L^{Tg}_{11,f}$ and $L^{Tp}_{11,f}$ given in eq.(G.58), where m_π and $|\mathbf{p}|$ are replaced now by M and $|\vec{\lambda}|$, respectively. One can simplify further the previous result.

$$\lambda^a\lambda^b = y^2 p^a p^b + (1-y)^2 p'^a p'^b + (1-y)y(p^a p'^b + p^b p'^a) \ , \tag{H.86}$$

Taking into account that in the integral of eq.(H.85) we can exchange $y \leftrightarrow (1-y)$ without changing the result, because $|\vec{\lambda}|$ and M depend only on the product $y(1-y)$, we can rewrite eq.(H.85) as

$$L_{12,f}^{ab} = -\delta^{ab}\int_0^1 dy \frac{L_{11,f}^{Tg}}{\partial m_\pi^2} - (p^a p^b + p'^a p'^b)\int_0^1 dy\, y^2 \frac{\partial L_{11,f}^{Tp}}{\partial m_\pi^2} - (p^a p'^b + p^b p'^a)\int_0^1 dy\, y(1-y)\frac{\partial L_{11,f}^{Tp}}{\partial m_\pi^2} \,. \tag{H.87}$$

We have still to rewrite the tensor δ^{ab} in terms of the tensor basis employed in eq.(H.83). For that let us note that given a Cartesian basis of vectors $\hat{\mathbf{u}}_1$, $\hat{\mathbf{u}}_2$ and $\hat{\mathbf{u}}_3$ we can set

$$\delta^{ab} = \hat{\mathbf{u}}_1^a \hat{\mathbf{u}}_1^b + \hat{\mathbf{u}}_2^a \hat{\mathbf{u}}_2^b + \hat{\mathbf{u}}_3^a \hat{\mathbf{u}}_3^b \,. \tag{H.88}$$

We take as as these unitary vectors,

$$\begin{aligned}
\hat{\mathbf{u}}_1 &= \frac{p^a}{|\mathbf{p}|} \,, \\
\hat{\mathbf{u}}_2 &= N_2(\hat{\alpha} - \hat{\alpha}\cdot\hat{\mathbf{u}}_1\,\hat{\mathbf{u}}_1) \,, \\
\hat{\mathbf{u}}_3 &= N_3(\hat{\mathbf{p}'} - \hat{\mathbf{p}'}\cdot\hat{\mathbf{u}}_1\,\hat{\mathbf{u}}_1 - \hat{\mathbf{p}'}\cdot\hat{\mathbf{u}}_2\,\hat{\mathbf{u}}_2) \,,
\end{aligned} \tag{H.89}$$

with the normalization constants N_2 and N_3 fixed such that $\hat{\mathbf{u}}_1^2 = \hat{\mathbf{u}}_2^2 = 1$. With these expressions it is then straightforward to rewrite eq.(H.87) in the basis of eq.(H.83).

H.6.2 One-medium insertion, $L_{12,m}^{ab}$

$$\begin{aligned}
L_{12,m}^{ab} &= m\int \frac{d^3k}{(2\pi)^3}\frac{k^a k^b}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]}\frac{\theta_m^-(\vec{\alpha}-\mathbf{k})+\theta_\ell^-(\vec{\alpha}+\mathbf{k})}{\mathbf{k}^2-A-i\varepsilon} \\
&= m\int \frac{d^3k}{(2\pi)^3}\frac{k^a k^b}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]}\frac{\theta_m^-(\vec{\alpha}-\mathbf{k})}{\mathbf{k}^2-A-i\varepsilon} \\
&+ m\int \frac{d^3k}{(2\pi)^3}\frac{k^a k^b}{[(\mathbf{k}-\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}-\mathbf{p})^2+m_\pi^2]}\frac{\theta_\ell^-(\vec{\alpha}-\mathbf{k})}{\mathbf{k}^2-A-i\varepsilon} \\
&= L_{12,m}^{T\alpha}\,\alpha^a\alpha^b + L_{12,m}^{Tp}\,p^a p^b + L_{12,m}^{Tp'}\,p'^a p'^b + L_{12,m}^{Tpp'}(p^a p'^b + p^b p'^a) + L_{12,m}^{T\alpha p}(\alpha^a p^b + \alpha^b p^a) \\
&+ L_{12,m}^{T\alpha p'}(\alpha^a p'^b + \alpha^b p'^a) \,.
\end{aligned} \tag{H.90}$$

As usual, we consider first the tensor integral

$$\begin{aligned}
\ell_{12,m}^{ab} &= m\int \frac{d^3k}{(2\pi)^3}\frac{k^a k^b}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]}\frac{\theta_m^-(\vec{\alpha}-\mathbf{k})}{\mathbf{k}^2-A-i\varepsilon} \\
&= \ell_{12,m}^{T\alpha}\,\alpha^a\alpha^b + \ell_{12,m}^{Tp}\,p^a p^b + \ell_{12,m}^{Tp'}\,p'^a p'^b + \ell_{12,m}^{Tpp'}(p^a p'^b + p^b p'^a) + \ell_{12,m}^{T\alpha p}(\alpha^a p^b + \alpha^b p^a) \\
&+ \ell_{12,m}^{T\alpha p'}(\alpha^a p'^b + \alpha^b p'^a) \,.
\end{aligned} \tag{H.91}$$

The first equation to determine the coefficient functions is obtained by contracting with δ^{ab} the previous integral with the result

$$\begin{aligned}
\ell_{12,m}^{2k} &= m\int \frac{d^3k}{(2\pi)^3}\frac{|\mathbf{k}|^2}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]}\frac{\theta_m^-(\vec{\alpha}-\mathbf{k})}{\mathbf{k}^2-A-i\varepsilon} \\
&= A\ell_{12,m} + mL_{02,m} \\
&= \ell_{12,m}^{T\alpha}\alpha^2 + \ell_{12,m}^{Tp}|\mathbf{p}|^2 + \ell_{12,m}^{Tp'}|\mathbf{p}|^2 + \ell_{12,m}^{Tpp'}2\mathbf{p}\cdot\mathbf{p}' + \ell_{12,m}^{T\alpha p}2\vec{\alpha}\cdot\mathbf{p} + \ell_{12,m}^{T\alpha p'}2\vec{\alpha}\cdot\mathbf{p}' \,,
\end{aligned} \tag{H.92}$$

with $\ell_{12,m}$ given in eq.(H.18) and $L_{02,m}$ in eq.(H.53).

Now, we contract with $\alpha^a\alpha^b$ and have

$$\ell^{2\alpha k}_{12,m} = m\int \frac{d^3k}{(2\pi)^3}\frac{(\vec{\alpha}\cdot\mathbf{k})^2}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]}\frac{\theta^-_m(\vec{\alpha}-\mathbf{k})}{\mathbf{k}^2-A-i\varepsilon}\ . \tag{H.93}$$

This integral is similar to $\ell^{\alpha k}_{12,m}$, eq.(H.55). We have now an extra factor of $\vec{\alpha}\cdot\mathbf{k}$. Then, we introduce

$$\begin{aligned}
f^{2\alpha k}_{12,m} &= \frac{m\alpha^2|\mathbf{k}|^3}{4\pi^2}\int_0^1 dy\int_{x_1}^{x_2} d\cos\theta\, cos^2\theta\,\frac{\delta+2|\vec{\lambda}||\mathbf{k}|\cos\eta\cos\theta}{\left[\delta^2+4|\vec{\lambda}|^2|\mathbf{k}|^2(\cos^2\theta-\sin^2\eta)+4\delta|\vec{\lambda}||\mathbf{k}|\cos\theta\cos\eta\right]^{3/2}}\\
&= -\frac{m\alpha^2|\mathbf{k}|^3}{4\pi^2}\int_0^1 dy\frac{1}{4\,c_2^{\frac{5}{2}}\,\left(b_2^2-4\,a_2\,c_2\right)\,\delta\,\sqrt{a_2+\cos\theta\,\left(b_2+c_2\,\cos\theta\right)}}\Big\{2\,\sqrt{c_2}\,\big[8\,a_2^2\,b_2\,c_2\\
&-b_2^2\,\cos\theta\,\left(3\,b_2^2-4\,c_2\,\delta^2+b_2\,c_2\,\cos\theta\right)+a_2\,\left(-3\,b_2^3+10\,b_2^2\,c_2\,\cos\theta-8\,c_2^2\,\delta^2\,\cos\theta\right.\\
&\left.+4\,b_2\,c_2\,\left(\delta^2+c_2\,\cos^2\theta\right)\right)\big]+\left(b_2^2-4\,a_2\,c_2\right)\,\left(3\,b_2^2-4\,c_2\,\delta^2\right)\,\sqrt{a_2+\cos\theta\,\left(b_2+c_2\,\cos\theta\right)}\\
&\times\log\left[\frac{b_2+2\,c_2\,\cos\theta}{\sqrt{c_2}}+2\,\sqrt{a_2+\cos\theta\,\left(b_2+c_2\,\cos\theta\right)}\right]\Big\}_{x_1(|\mathbf{k}|)}^{x_2(|\mathbf{k}|)}\ .
\end{aligned}\tag{H.94}$$

The coefficients a_2, b_2, c_2 are given in eq.(H.57) and $x_1(|\mathbf{k}|)$, $x_2(|\mathbf{k}|)$ in eq.(E.7). On the other hand, δ was already defined in eq.(H.15). In terms of $f^{2\alpha k}_{12,m}$ we can write

$$\begin{aligned}
&\alpha\geq\xi_1\ ,\\
&\ell^{2\alpha k}_{12,m}(\xi_1) = \int_{\alpha-\xi_1}^{\alpha+\xi_1} d|\mathbf{k}|\frac{|\mathbf{k}|}{\mathbf{k}^2-A-i\varepsilon}f^{2\alpha k}_{12,m}(|\mathbf{k}|)\ .\\
&\alpha<\xi_1\ ,\\
&\ell^{2\alpha k}_{12,m}(\xi_1) = \left\{\int_0^{\xi_1-\alpha}+\int_{\xi_1-\alpha}^{\xi_1+\alpha}\right\}d|\mathbf{k}|\frac{|\mathbf{k}|}{\mathbf{k}^2-A-i\varepsilon}f^{2\alpha k}_{12,m}(|\mathbf{k}|)\ .
\end{aligned}\tag{H.95}$$

Another equation then results

$$\ell^{2\alpha k}_{12,m} = \ell^{T\alpha}_{12,m}\alpha^4+\ell^{Tp}_{12,m}(\vec{\alpha}\cdot\mathbf{p})^2+\ell^{Tp'}_{12,m}(\vec{\alpha}\cdot\mathbf{p}')^2+\ell^{Tpp'}_{12,m}2\vec{\alpha}\cdot\mathbf{p}\,\vec{\alpha}\cdot\mathbf{p}'+\ell^{T\alpha p}_{12,m}2\alpha^2\vec{\alpha}\cdot\mathbf{p}+\ell^{T\alpha p'}_{12,m}2\alpha^2\vec{\alpha}\cdot\mathbf{p}'\ . \tag{H.96}$$

We contract now eq.(H.91) with $\alpha^a p^b$

$$\begin{aligned}
\ell^{\alpha pk}_{12,m} &= m\int\frac{d^3k}{(2\pi)^3}\frac{(\vec{\alpha}\cdot\mathbf{k})(\mathbf{p}\cdot\mathbf{k})}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]}\frac{\theta^-_m(\vec{\alpha}-\mathbf{k})}{\mathbf{k}^2-A-i\varepsilon}\\
&= \frac{1}{2}\ell^{\alpha k}_{11,m}(\cos\beta')-\frac{A+\mathbf{p}^2+m_\pi^2}{2}\ell^{\alpha k}_{12,m}-\frac{m}{2}L^{\alpha k}_{02,m}\ .
\end{aligned}\tag{H.97}$$

The only new integrals is the last one

$$L^{\alpha k}_{02,m} = \int\frac{d^3k}{(2\pi)^3}\frac{\vec{\alpha}\cdot\mathbf{k}\,\theta^-_m(\vec{\alpha}-\mathbf{k})}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]} = \int_0^1 dy\int\frac{d^3k}{(2\pi)^3}\frac{\vec{\alpha}\cdot\mathbf{k}\,\theta^-_m(\vec{\alpha}-\mathbf{k})}{\left[(\vec{\lambda}+\mathbf{k})^2+M^2\right]^2}\ . \tag{H.98}$$

We perform now the shift of eq.(G.69) and have

$$L^{\alpha k}_{02,m} = \int_0^1 dy \int_{v\leq \xi_m} \frac{d^3v}{(2\pi)^3} \frac{\vec{\alpha}\cdot(\vec{\alpha}+\mathbf{v})}{\left[(\mathbf{v}+\vec{\alpha}+\vec{\lambda})^2+M^2\right]^2} = \alpha^2 L_{02,m} - \int_0^1 dy \widetilde{\mathbf{p}}_1\cdot\vec{\alpha}\frac{\partial}{\partial m_\pi^2} L^p_{01,m}(\widetilde{\mathbf{p}}_1, M) \ , \quad \text{(H.99)}$$

with $\widetilde{\mathbf{p}}_1$ defined in eq.(H.52). The function $L^p_{01,m}$ was already calculated in eq.(G.75), but now $\mathbf{p}_1$ and m_π must be replaced by $\widetilde{\mathbf{p}}_1$ and M, respectively. Thus, we have another equation

$$\begin{aligned}\ell^{\alpha pk}_{12,m} &= \ell^{T\alpha}_{12,m}\alpha^2\vec{\alpha}\cdot\mathbf{p} + \ell^{Tp}_{12,m}|\mathbf{p}|^2\vec{\alpha}\cdot\mathbf{p} + \ell^{Tp'}_{12,m}\vec{\alpha}\cdot\mathbf{p}'\,\mathbf{p}\cdot\mathbf{p}' + \ell^{Tpp'}_{12,m}(\vec{\alpha}\cdot\mathbf{p}\,\mathbf{p}\cdot\mathbf{p}' + \vec{\alpha}\cdot\mathbf{p}'\,\mathbf{p}^2) \\ &+ \ell^{T\alpha p}_{12,m}(\alpha^2\mathbf{p}^2 + (\vec{\alpha}\cdot\mathbf{p})^2) + \ell^{T\alpha p'}_{12,m}(\alpha^2\,\mathbf{p}\cdot\mathbf{p}' + \mathbf{p}\cdot\vec{\alpha}\,\vec{\alpha}\cdot\mathbf{p}') \ . \end{aligned} \quad \text{(H.100)}$$

Given the exchange symmetry between $\mathbf{p}$ and $\mathbf{p}'$ in eq.(H.91) when contracting with $\alpha^a p'^b$ one has from eq.(H.97)

$$\ell^{\alpha p' k}_{12,m} = \frac{1}{2}\ell^{\alpha k}_{11,m}(\cos\beta) - \frac{A+\mathbf{p}^2+m_\pi^2}{2}\ell^{\alpha k}_{12,m} - \frac{m}{2}L^{\alpha k}_{02} \ . \quad \text{(H.101)}$$

And have the new equation

$$\begin{aligned}\ell^{\alpha p' k}_{12,m} &= \ell^{T\alpha}_{12,m}\alpha^2\vec{\alpha}\cdot\mathbf{p}' + \ell^{Tp}_{12,m}\vec{\alpha}\cdot\mathbf{p}\,\mathbf{p}'\cdot\mathbf{p} + \ell^{Tp'}_{12,m}|\mathbf{p}|^2\vec{\alpha}\cdot\mathbf{p}' + \ell^{Tpp'}_{12,m}(\vec{\alpha}\cdot\mathbf{p}\,\mathbf{p}^2 + \vec{\alpha}\cdot\mathbf{p}'\,\mathbf{p}'\cdot\mathbf{p}) \\ &+ \ell^{T\alpha p}_{12,m}(\alpha^2\mathbf{p}'\cdot\mathbf{p} + \mathbf{p}'\cdot\vec{\alpha}\,\vec{\alpha}\cdot\mathbf{p}) + \ell^{T\alpha p'}_{12,m}(\alpha^2\,\mathbf{p}^2 + (\mathbf{p}'\cdot\vec{\alpha})^2) \ . \end{aligned} \quad \text{(H.102)}$$

We now contract eq.(H.91) with $p^a p^b$ giving rise to the integral

$$\begin{aligned}\ell^{2pk}_{12,m} &= m\int\frac{d^3k}{(2\pi)^3}\frac{(\mathbf{p}\cdot\mathbf{k})^2\,\theta^-_m(\vec{\alpha}-\mathbf{k})}{(\mathbf{k}^2-A-i\varepsilon)\,[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]} \\ &= -\frac{\mathbf{p}^2+m_\pi^2+A}{2}\ell^{pk}_{12,m} + \frac{m}{2}\int\frac{d^3k}{(2\pi)^3}\frac{\mathbf{p}\cdot\mathbf{k}\,\theta^-_m(\vec{\alpha}-\mathbf{k})}{(\mathbf{k}^2-A-i\varepsilon)\,[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]} \\ &- \frac{m}{2}\int\frac{d^3k}{(2\pi)^3}\frac{\mathbf{p}\cdot\mathbf{k}\,\theta^-_m(\vec{\alpha}-\mathbf{k})}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]} \ , \end{aligned} \quad \text{(H.103)}$$

with $\ell^{pk}_{12,m}$ given in eq.(H.50). On the other hand,

$$\frac{m}{2}\int\frac{d^3k}{(2\pi)^3}\frac{\mathbf{p}\cdot\mathbf{k}\,\theta^-_m(\vec{\alpha}-\mathbf{k})}{(\mathbf{k}^2-A-i\varepsilon)\,[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]} = \frac{m}{2}\left[\ell^{\alpha}_{11,m}(c\beta')\vec{\alpha}\cdot\mathbf{p} + \ell^{p}_{11,m}(c\beta')\mathbf{p}'\cdot\mathbf{p}\right] \ , \quad \text{(H.104)}$$

employing the results of section G.1.2. The last integral in eq.(H.103) has not been calculated yet. One has,

$$\begin{aligned}L^{pk}_{02,m} &= \int\frac{d^3k}{(2\pi)^3}\frac{\mathbf{p}\cdot\mathbf{k}\,\theta^-_m(\vec{\alpha}-\mathbf{k})}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]} = \frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\frac{\theta^-_m(\vec{\alpha}-\mathbf{k})}{(\mathbf{k}+\mathbf{p}')^2+m_\pi^2} \\ &- \frac{\mathbf{p}^2+m_\pi^2}{2}\int\frac{d^3k}{(2\pi)^3}\frac{\theta^-_m(\vec{\alpha}-\mathbf{k})}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]} - \frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\frac{\mathbf{k}^2\,\theta(\vec{\alpha}-\mathbf{k})}{[(\mathbf{k}+\mathbf{p}')^2+m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2+m_\pi^2]} \\ &= \frac{1}{2}L_{01,m}(c\beta') - \frac{\mathbf{p}^2+m_\pi^2}{2}L_{02,m} - \frac{1}{2}L^{2k}_{02,m} \ , \end{aligned} \quad \text{(H.105)}$$

with $L_{01,m}$, $L_{02,m}$ and $L^{2k}_{02,m}$ given in eqs.(G.35), (H.53) and (H.75), respectively. Taking together eqs.(H.103), (H.104) and (H.105) we can write the new equation

$$\begin{aligned}\ell^{2pk}_{12,m} &= -\frac{\mathbf{p}^2+m_\pi^2+A}{2}\ell^{pk}_{12,m} + \frac{1}{2}\left\{\ell^{\alpha}_{11,m}(c\beta')\vec{\alpha}\cdot\mathbf{p} + \ell^{p}_{11,m}(c\beta')\mathbf{p}\cdot\mathbf{p}'\right\} \\ &- \frac{m}{4}\left\{L_{01,m}(c\beta') - (\mathbf{p}^2+m_\pi^2)L_{02,m} - L^{2k}_{02,m}\right\} \\ &= \ell^{T\alpha}_{12,m}(\vec{\alpha}\cdot\mathbf{p})^2 + \ell^{Tp}_{12,m}|\mathbf{p}|^4 + \ell^{Tp'}_{12,m}(\mathbf{p}\cdot\mathbf{p}')^2 + \ell^{Tpp'}_{12,m}2\mathbf{p}^2\,\mathbf{p}\cdot\mathbf{p}' + \ell^{T\alpha p}_{12,m}2\mathbf{p}^2\,\vec{\alpha}\cdot\mathbf{p} + \ell^{T\alpha p'}_{12,m}2\vec{\alpha}\cdot\mathbf{p}\,\mathbf{p}\cdot\mathbf{p}'\ .\end{aligned} \tag{H.106}$$

The contraction with $p'^a p'^b$, given the symmetry between $\mathbf{p}$ and $\mathbf{p}'$ in eq.(G.59), can be directly worked out from the previous equation resulting

$$\begin{aligned}\ell^{2p'k}_{12,m} &= -\frac{\mathbf{p}^2+m_\pi^2+A}{2}\ell^{p'k}_{12,m} + \frac{1}{2}\left\{\ell^{\alpha}_{11,m}(c\beta)\vec{\alpha}\cdot\mathbf{p}' + \ell^{p}_{11,m}(c\beta)\mathbf{p}\cdot\mathbf{p}'\right\} \\ &- \frac{m}{4}\left\{L_{01,m}(c\beta) - (\mathbf{p}^2+m_\pi^2)L_{02,m} - L^{2k}_{02,m}\right\} \\ &= \ell^{T\alpha}_{12,m}(\vec{\alpha}\cdot\mathbf{p}')^2 + \ell^{Tp}_{12,m}(\mathbf{p}\cdot\mathbf{p}')^2 + \ell^{Tp'}_{12,m}|\mathbf{p}|^4 + \ell^{Tpp'}_{12,m}2\mathbf{p}^2\,\mathbf{p}\cdot\mathbf{p}' + \ell^{T\alpha p}_{12,m}2\mathbf{p}\cdot\mathbf{p}'\,\vec{\alpha}\cdot\mathbf{p}' + \ell^{T\alpha p'}_{12,m}2|\mathbf{p}|^2\,\vec{\alpha}\cdot\mathbf{p}'\ ,\end{aligned} \tag{H.107}$$

with $\ell^{p'k}_{12,m}$ given in eq.(H.54).

We are then provided with 6 equations that are used to determine the coefficients functions that appear in the tensor decomposition of $\ell^{ab}_{12,m}$, eq.(H.90). This system of equations can be written in more compact form

$$\begin{pmatrix} \alpha^2 & |\mathbf{p}|^2 & |\mathbf{p}|^2 & 2\mathbf{p}\cdot\mathbf{p}' & 2\vec{\alpha}\cdot\mathbf{p} & 2\vec{\alpha}\cdot\mathbf{p}' \\ \alpha^4 & (\vec{\alpha}\cdot\mathbf{p})^2 & (\vec{\alpha}\cdot\mathbf{p}')^2 & 2\vec{\alpha}\cdot\mathbf{p}\,\vec{\alpha}\cdot\mathbf{p}' & 2\alpha^2\vec{\alpha}\cdot\mathbf{p} & 2\alpha^2\vec{\alpha}\cdot\mathbf{p}' \\ \alpha^2\vec{\alpha}\cdot\mathbf{p} & |\mathbf{p}|^2\vec{\alpha}\cdot\mathbf{p} & \vec{\alpha}\cdot\mathbf{p}'\,\mathbf{p}\cdot\mathbf{p}' & (\vec{\alpha}\cdot\mathbf{p}\,\mathbf{p}\cdot\mathbf{p}' + \vec{\alpha}\cdot\mathbf{p}'\,\mathbf{p}^2) & (\alpha^2\mathbf{p}^2 + (\vec{\alpha}\cdot\mathbf{p})^2) & (\alpha^2\,\mathbf{p}\cdot\mathbf{p}' + \mathbf{p}\cdot\vec{\alpha}\,\vec{\alpha}\cdot\mathbf{p}') \\ \alpha^2\vec{\alpha}\cdot\mathbf{p}' & \vec{\alpha}\cdot\mathbf{p}\,\mathbf{p}'\cdot\mathbf{p} & |\mathbf{p}|^2\vec{\alpha}\cdot\mathbf{p}' & (\vec{\alpha}\cdot\mathbf{p}\,\mathbf{p}^2 + \vec{\alpha}\cdot\mathbf{p}'\,\mathbf{p}'\cdot\mathbf{p}) & (\alpha^2\mathbf{p}'\cdot\mathbf{p} + \mathbf{p}'\cdot\vec{\alpha}\,\vec{\alpha}\cdot\mathbf{p}) & (\alpha^2\,\mathbf{p}^2 + (\mathbf{p}'\cdot\vec{\alpha})^2) \\ (\vec{\alpha}\cdot\mathbf{p})^2 & |\mathbf{p}|^4 & (\mathbf{p}\cdot\mathbf{p}')^2 & 2\mathbf{p}^2\,\mathbf{p}\cdot\mathbf{p}' & 2\mathbf{p}^2\,\vec{\alpha}\cdot\mathbf{p}' & 2\mathbf{p}\cdot\mathbf{p}'\,\vec{\alpha}\cdot\mathbf{p}' \\ (\vec{\alpha}\cdot\mathbf{p}')^2 & (\mathbf{p}\cdot\mathbf{p}')^2 & |\mathbf{p}|^4 & 2\mathbf{p}^2\,\mathbf{p}\cdot\mathbf{p}' & 2\mathbf{p}\cdot\mathbf{p}'\,\vec{\alpha}\cdot\mathbf{p}' & 2|\mathbf{p}|^2\,\vec{\alpha}\cdot\mathbf{p}' \end{pmatrix}$$

$$\cdot\begin{pmatrix} \ell^{T\alpha}_{12,m} \\ \ell^{Tp}_{12,m} \\ \ell^{Tp'}_{12,m} \\ \ell^{Tpp'}_{12,m} \\ \ell^{T\alpha p}_{12,m} \\ \ell^{T\alpha p'}_{12,m} \end{pmatrix} = \begin{pmatrix} \ell^{2k}_{12,m} \\ \ell^{2\alpha k}_{12,m} \\ \ell^{\alpha pk}_{12,m} \\ \ell^{\alpha p'k}_{12,m} \\ \ell^{2pk}_{12,m} \\ \ell^{2p'k}_{12,m} \end{pmatrix}\ . \tag{H.108}$$

Inverting the previous matrix one obtains the required functions. In terms of which we have

$$\begin{aligned}L^{ab}_{12,m} &= \left(\ell^{T\alpha}_{12,m}(\xi_m,c\beta,c\beta') + \ell^{T\alpha}_{12,m}(\xi_\ell,-c\beta,-c\beta')\right)\alpha^a\alpha^b + \left(\ell^{Tp}_{12,m}(\xi_m,c\beta,c\beta') + \ell^{Tp}_{12,m}(\xi_\ell,-c\beta,-c\beta')\right)p^a p^b \\ &+ \left(\ell^{Tp'}_{12,m}(\xi_m,c\beta,c\beta') + \ell^{Tp'}_{12,m}(\xi_\ell,-c\beta,-c\beta')\right)p'^a p'^b \\ &+ \left(\ell^{Tpp'}_{12,m}(\xi_m,c\beta,c\beta') + \ell^{Tpp'}_{12,m}(\xi_\ell,-c\beta,-c\beta')\right)(p^a p'^b + p^b p'^a) \\ &+ \left(\ell^{T\alpha p}_{12,m}(\xi_m,c\beta,c\beta') - \ell^{T\alpha p}_{12,m}(\xi_\ell,-c\beta,-c\beta')\right)(\alpha^a p^b + \alpha^b p^a) \\ &+ \left(\ell^{T\alpha p'}_{12,m}(\xi_m,c\beta,c\beta') - \ell^{T\alpha p'}_{12,m}(\xi_\ell,-c\beta,-c\beta')\right)(\alpha^a p'^b + \alpha^b p'^a)\ .\end{aligned} \tag{H.109}$$

H.6.3 Two-medium insertions, $L^{ab}_{12,d}$

$$L^{ab}_{12,d} = \frac{-im\sqrt{A}}{8\pi^2}\int d\hat{\mathbf{k}} \frac{\theta(\xi_1 - |\hat{\mathbf{k}}\sqrt{A} - \vec{\alpha}|)\theta(\xi_2 - |\hat{\mathbf{k}}\sqrt{A} + \vec{\alpha}|)}{[(\mathbf{k}+\mathbf{p}')^2 + m_\pi^2]\,[(\mathbf{k}+\mathbf{p})^2 + m_\pi^2]} k^a k^b$$
$$= L^{T\alpha}_{12,d}\,\alpha^a\alpha^b + L^{Tp}_{12,d}\,p^a p^b + L^{Tp'}_{12,d}\,p'^a p'^b + L^{Tpp'}_{12,d}\,(p^a p'^b + p^b p'^a) + L^{T\alpha p}_{12,d}(\alpha^a p^b + \alpha^b p^a)$$
$$+ L^{T\alpha p'}_{12,d}\,(\alpha^a p'^b + \alpha^b p'^a) \ . \tag{H.110}$$

We proceed similarly as for the calculation of $L^{ab}_{12,m}$ above but taking into account that $\mathbf{k}^2 = A$. First, the contraction with δ^{ab} gives

$$AL_{12,d} = L^{T\alpha}_{12,d}\,\alpha^2 + L^{Tp}_{12,d}\,|\mathbf{p}|^2 + L^{Tp'}_{12,d}\,|\mathbf{p}|^2 + \ell^{Tpp'}_{12,d} 2\mathbf{p}\cdot\mathbf{p}' + \ell^{T\alpha p}_{12,d}\,2\vec{\alpha}\cdot\mathbf{p} + \ell^{T\alpha p'}_{12,d}\,2\vec{\alpha}\cdot\mathbf{p}' \ , \tag{H.111}$$

We now take the contraction with $\alpha^a\alpha^b$. The angular integrations are the same as for $\ell^{2\alpha k}_{12,m}$ so that we can write in short

$$L^{2\alpha k}_{12,d} = -i\pi f^{2\alpha k}_{12,m}(\sqrt{A}) \ . \tag{H.112}$$

Here, $f^{2\alpha k}_{12,m}(\sqrt{A})$ has to be evaluated according to eq.(H.94) but with the integration limits given according to eq.(G.21), for two medium insertions and $\xi_1 \le \xi_2$. We then have the equation,

$$L^{2\alpha k}_{12,d} = L^{T\alpha}_{12,d}\alpha^4 + L^{Tp}_{12,d}(\vec{\alpha}\cdot\mathbf{p})^2 + L^{Tp'}_{12,d}(\vec{\alpha}\cdot\mathbf{p}')^2 + L^{Tpp'}_{12,d} 2\vec{\alpha}\cdot\mathbf{p}\,\vec{\alpha}\cdot\mathbf{p}' + L^{T\alpha p}_{12,d} 2\alpha^2\vec{\alpha}\cdot\mathbf{p} + L^{T\alpha p'}_{12,d}\,2\alpha^2\vec{\alpha}\cdot\mathbf{p}' \ . \tag{H.113}$$

The contraction with $\alpha^a p^b$ can be readily worked out following the steps above for $\ell^{\alpha pk}_{12,m}$

$$L^{\alpha pk}_{12,d} = \frac{1}{2}L^{\alpha k}_{11,d}(\cos\beta') - \frac{A + \mathbf{p}^2 + m_\pi^2}{2} L^{\alpha k}_{12,d} \tag{H.114}$$

with $L^{\alpha k}_{11,d}$ evaluated in eq.(G.48) and $L^{\alpha k}_{12,d}$ in eq.(H.63). The resulting equation is then

$$L^{\alpha pk}_{12,d} = L^{T\alpha}_{12,d}\alpha^2\vec{\alpha}\cdot\mathbf{p} + L^{Tp}_{12,d}|\mathbf{p}|^2\vec{\alpha}\cdot\mathbf{p} + L^{Tp'}_{12,d}\vec{\alpha}\cdot\mathbf{p}'\,\mathbf{p}\cdot\mathbf{p}' + L^{Tpp'}_{12,d}\,(\vec{\alpha}\cdot\mathbf{p}\,\mathbf{p}\cdot\mathbf{p}' + \vec{\alpha}\cdot\mathbf{p}'\,\mathbf{p}^2)$$
$$+ L^{T\alpha p}_{12,d}(\alpha^2\mathbf{p}^2 + (\vec{\alpha}\cdot\mathbf{p})^2) + L^{T\alpha p'}_{12,d}\,(\alpha^2\,\mathbf{p}\cdot\mathbf{p}' + \mathbf{p}\cdot\vec{\alpha}\,\vec{\alpha}\cdot\mathbf{p}') \ . \tag{H.115}$$

Contracting with $\alpha^a p'^b$ gives

$$L^{\alpha p' k}_{12,d} = \frac{1}{2}L^{\alpha k}_{11,d}(\cos\beta) - \frac{A + \mathbf{p}^2 + m_\pi^2}{2} L^{\alpha k}_{12,d} \tag{H.116}$$

and the equation

$$L^{\alpha p' k}_{12,d} = L^{T\alpha}_{12,d}\alpha^2\vec{\alpha}\cdot\mathbf{p}' + L^{Tp}_{12,d}\vec{\alpha}\cdot\mathbf{p}\,\mathbf{p}'\cdot\mathbf{p} + L^{Tp'}_{12,d}|\mathbf{p}|^2\vec{\alpha}\cdot\mathbf{p}' + L^{Tpp'}_{12,d}\,(\vec{\alpha}\cdot\mathbf{p}\,\mathbf{p}^2 + \vec{\alpha}\cdot\mathbf{p}'\,\mathbf{p}'\cdot\mathbf{p})$$
$$+ L^{T\alpha p}_{12,d}(\alpha^2\mathbf{p}'\cdot\mathbf{p} + \mathbf{p}'\cdot\vec{\alpha}\,\vec{\alpha}\cdot\mathbf{p}) + L^{T\alpha p'}_{12,d}\,(\alpha^2\,\mathbf{p}^2 + (\mathbf{p}'\cdot\vec{\alpha})^2) \ . \tag{H.117}$$

The contraction with $p^a p^b$ can also be readily worked out from eq.(H.106)

$$L^{2pk}_{12,d} = -\frac{\mathbf{p}^2 + m_\pi^2 + A}{2} L^{pk}_{12,d} + \frac{1}{2}\left\{L^{\alpha}_{11,d}(c\beta')\vec{\alpha}\cdot\mathbf{p} + L^{p}_{11,d}(c\beta')\mathbf{p}\cdot\mathbf{p}'\right\}$$
$$= L^{T\alpha}_{12,d}(\vec{\alpha}\cdot\mathbf{p})^2 + L^{Tp}_{12,d}|\mathbf{p}|^4 + L^{Tp'}_{12,d}(\mathbf{p}\cdot\mathbf{p}')^2 + L^{Tpp'}_{12,d} 2\mathbf{p}^2\,\mathbf{p}\cdot\mathbf{p}' + L^{T\alpha p}_{12,d} 2\mathbf{p}^2\,\vec{\alpha}\cdot\mathbf{p} + L^{T\alpha p'}_{12,d}\,2\vec{\alpha}\cdot\mathbf{p}\,\mathbf{p}\cdot\mathbf{p}' \ . \tag{H.118}$$

And similarly for contracting with $p'^a p'^b$

$$L^{2p'k}_{12,d} = -\frac{\mathbf{p}^2+m_\pi^2+A}{2} L^{p'k}_{12,d} + \frac{1}{2}\left\{ L^{\alpha}_{11,d}(c\beta)\vec{\alpha}\cdot\mathbf{p}' + L^{p}_{11,d}(c\beta)\mathbf{p}\cdot\mathbf{p}' \right\}$$
$$= L^{T\alpha}_{12,d}(\vec{\alpha}\cdot\mathbf{p}')^2 + L^{Tp}_{12,d}(\mathbf{p}\cdot\mathbf{p}')^2 + L^{Tp'}_{12,d}|\mathbf{p}|^4 + L^{Tpp'}_{12,d}2\mathbf{p}^2\,\mathbf{p}\cdot\mathbf{p}' + L^{T\alpha p}_{12,d}2\mathbf{p}\cdot\mathbf{p}'\,\vec{\alpha}\cdot\mathbf{p}' + L^{T\alpha p'}_{12,d}2|\mathbf{p}|^2\,\vec{\alpha}\cdot\mathbf{p}' \; , \tag{H.119}$$

We can then write a similar system of equations as in the case of $\ell^{ab}_{12,m}$ to calculate the coefficients functions in eq.(H.110),

$$\begin{pmatrix} \alpha^2 & |\mathbf{p}|^2 & |\mathbf{p}|^2 & 2\mathbf{p}\cdot\mathbf{p}' & 2\vec{\alpha}\cdot\mathbf{p} & 2\vec{\alpha}\cdot\mathbf{p}' \\ \alpha^4 & (\vec{\alpha}\cdot\mathbf{p})^2 & (\vec{\alpha}\cdot\mathbf{p}')^2 & 2\vec{\alpha}\cdot\mathbf{p}\,\vec{\alpha}\cdot\mathbf{p}' & 2\alpha^2\vec{\alpha}\cdot\mathbf{p} & 2\alpha^2\vec{\alpha}\cdot\mathbf{p}' \\ \alpha^2\vec{\alpha}\cdot\mathbf{p} & \mathbf{p}|^2\vec{\alpha}\cdot\mathbf{p} & \vec{\alpha}\cdot\mathbf{p}'\,\mathbf{p}\cdot\mathbf{p}' & (\vec{\alpha}\cdot\mathbf{p}\,\mathbf{p}\cdot\mathbf{p}'+\vec{\alpha}\cdot\mathbf{p}'\,\mathbf{p}^2) & (\alpha^2\mathbf{p}^2+(\vec{\alpha}\cdot\mathbf{p})^2) & (\alpha^2\,\mathbf{p}\cdot\mathbf{p}'+\mathbf{p}\cdot\vec{\alpha}\,\vec{\alpha}\cdot\mathbf{p}') \\ \alpha^2\vec{\alpha}\cdot\mathbf{p}' & \vec{\alpha}\cdot\mathbf{p}\,\mathbf{p}'\cdot\mathbf{p} & |\mathbf{p}|^2\vec{\alpha}\cdot\mathbf{p}' & (\vec{\alpha}\cdot\mathbf{p}\,\mathbf{p}^2+\vec{\alpha}\cdot\mathbf{p}'\,\mathbf{p}'\cdot\mathbf{p}) & (\alpha^2\mathbf{p}'\cdot\mathbf{p}+\mathbf{p}'\cdot\vec{\alpha}\,\vec{\alpha}\cdot\mathbf{p}) & (\alpha^2\,\mathbf{p}^2+(\mathbf{p}'\cdot\vec{\alpha})^2) \\ (\vec{\alpha}\cdot\mathbf{p})^2 & |\mathbf{p}|^4 & (\mathbf{p}\cdot\mathbf{p}')^2 & 2\mathbf{p}^2\,\mathbf{p}\cdot\mathbf{p}' & 2\mathbf{p}^2\,\vec{\alpha}\cdot\mathbf{p}' & 2\mathbf{p}\cdot\mathbf{p}'\,\vec{\alpha}\cdot\mathbf{p}' \\ (\vec{\alpha}\cdot\mathbf{p}')^2 & (\mathbf{p}\cdot\mathbf{p}')^2 & |\mathbf{p}|^4 & 2\mathbf{p}^2\,\mathbf{p}\cdot\mathbf{p}' & 2\mathbf{p}\cdot\mathbf{p}'\,\vec{\alpha}\cdot\mathbf{p}' & 2|\mathbf{p}|^2\,\vec{\alpha}\cdot\mathbf{p}' \end{pmatrix}$$
$$\cdot \begin{pmatrix} L^{T\alpha}_{12,d} \\ L^{Tp}_{12,d} \\ L^{Tp'}_{12,d} \\ L^{Tpp'}_{12,d} \\ L^{T\alpha p}_{12,d} \\ L^{T\alpha p'}_{12,d} \end{pmatrix} = \begin{pmatrix} AL_{12,d} \\ L^{2\alpha k}_{12,d} \\ L^{\alpha pk}_{12,d} \\ L^{\alpha p'k}_{12,d} \\ L^{2pk}_{12,d} \\ L^{2p'k}_{12,d} \end{pmatrix} \; . \tag{H.120}$$

For the case $\xi_1 \geq \xi_2$ one performs the change of variable $\mathbf{k} \to -\mathbf{k}$ in eq.(H.110) which implies the exchange of the roles of ξ_1 and ξ_2 as well as the changes $\mathbf{p} \to -\mathbf{p}$ and $\mathbf{p}' \to -\mathbf{p}'$. Then one has

$$L^{ab}_{12,d} = L^{T\alpha}_{12,d}(\xi_2,\xi_1,-c\beta,-c\beta')\alpha^a\alpha^b + L^{Tp}_{12,d}(\xi_2,\xi_1,-c\beta,-c\beta')p^a p^b + L^{Tp'}_{12,d}(\xi_2,\xi_1,-c\beta,-c\beta')p'^a p'^b$$
$$+ L^{Tpp'}_{12,d}(\xi_2,\xi_1,-c\beta,-c\beta')(p^a p'^b + p^b p'^a) - L^{T\alpha p}_{12,d}(\xi_2,\xi_1,-c\beta,-c\beta')(\alpha^a p^b + \alpha^b p^a)$$
$$- L^{T\alpha p'}_{12,d}(\xi_2,\xi_1,-c\beta,-c\beta')(\alpha^a p'^b + \alpha^b p'^a) \; . \tag{H.121}$$

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