Number-operator interpretation of the compositeness of bound and resonant states

J. A. Oller

Departamento de Física
Universidad de Murcia[1]
Murcia, Spain

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Outline

Composite

"Elementary"

Building Blocks

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"Elementary"

Building Blocks

At a more microscopic level they are all composite of concrete

1. Basic set-up

Bound state near a two-body threshold. Non-Relativistic Dynamics

S. Weinberg, PR130,776(1963); PR131,440(1963); PR137,B672(1964)

$$H = H_0 + V$$

$H_0 = \sum p_\alpha^2/2m_\alpha$: Kinetic energy term

Spectrum:

$$H|\psi_\alpha\rangle = E_\alpha|\psi_\alpha\rangle \;\;, \;\; \text{Continuum spectrum}$$
$$H|\psi_{B_i}\rangle = E_{B_i}|\psi_{B_i}\rangle \;\;, E_{B_i} < 0 \;, \;\; \text{Discrete Spectrum}$$

Bare spectrum:

$$H_0|\varphi_\alpha\rangle = E_\alpha|\varphi_\alpha\rangle \;, \;\; \text{Continuum}$$
$$H_0|\varphi_n\rangle = E_n|\varphi_n\rangle \;, \;\; \text{Discrete}$$

$|\varphi_\alpha\rangle$ is made up of several free particles in the continuum
$|\varphi_n\rangle$ One-particle bare elementary states

Elementariness: Z
Compositeness: X

$$\langle\psi_B|\psi_B\rangle = 1 = \underbrace{\sum_n |\langle\varphi_n|\psi_B\rangle|^2}_{Z} + \underbrace{\int d\alpha |\langle\varphi_\alpha|\psi_B\rangle|^2}_{X}$$

$$1 = Z + X$$

$$X = 1 - Z = \int d\alpha \, \frac{|\langle\varphi_\alpha|V|\psi_B\rangle|^2}{(E_\alpha - E_B)^2}$$

$$H|\psi_B\rangle = E_B|\psi_B\rangle = (H_0 + V)|\psi_B\rangle$$
$$E_B\langle\varphi_\alpha|\psi_B\rangle = E_\alpha\langle\varphi_\alpha|\psi_B\rangle + \langle\varphi_\alpha|V|\psi_B\rangle$$
$$\langle\varphi_\alpha|\psi_B\rangle = \frac{\langle\varphi_\alpha|V|\psi_B\rangle}{E_B - E_\alpha}$$

Wave Function Renormalization: $Z^{1/2}$
If there is only **one** "elementary" bare state around E_B

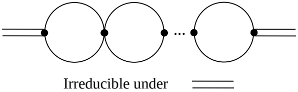
$$\langle\varphi_1|\psi_B\rangle = Z^{1/2}$$

It can be also calculated from the residue of the complete propagator of the bare elementary state:

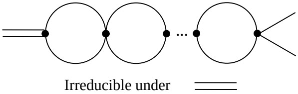
$$\Delta(E) = \frac{1}{E - E_0 - \Pi(E)} \xrightarrow[E \to E_B]{} \frac{Z}{E - E_B}$$

$$\Pi(E) = \langle \varphi_1 | T_1 | \varphi_1 \rangle$$

Weinberg, PR130,776(1963)



$$\begin{aligned} \widetilde{g}_\alpha(k_\alpha) &= \langle k_\alpha, \alpha | T_1(E_B) | \varphi_1 \rangle \\ &= Z^{-1/2} g_\alpha(k_\alpha) \end{aligned}$$



$$\Pi(E) = -\sum_\beta \frac{m_\beta}{\pi^2} \int_0^\infty dk \frac{k^2}{k^2 + \gamma_\beta^2} \widetilde{g}_\beta^2(k_\beta^2) \qquad T_P = \widetilde{g}_\alpha(k_\alpha) \widetilde{g}_\beta(k'_\beta) \Delta(E)$$

$$X_\alpha = \frac{1}{1 + \sum_\beta \frac{2m_\beta^2}{\pi^2} \int_0^\infty dk \frac{k^2}{(k^2+\gamma_\beta^2)^2} \widetilde{g}_\beta^2(k_\beta^2)} \frac{2m_\alpha^2}{\pi^2} \int_0^\infty dk \frac{k^2}{(k^2 + \gamma_\alpha^2)^2} \widetilde{g}_\alpha^2(k^2)$$

$$X = \sum_\alpha X_\alpha = 1 - Z$$

2. Number-operator interpretation

No need of bare-elementary states as an intermediate step

We focus our attention on the continuum spectrum
The continuum spectrum is common to H and H_0

Let there be two **free** particles of types A and B,
$H_0|AB_\gamma\rangle = E_\gamma|AB_\gamma\rangle$
Creation,annihilation operators: $a_\alpha^\dagger a_\alpha,\ b_\beta^\dagger b_\beta$

Number Operators:

$$N_D = \int d\alpha a_\alpha^\dagger a_\alpha + \int d\beta b_\beta^\dagger b_\beta = N_D^A + N_D^B$$
$$= \int d^3x \left[\psi_A^\dagger(x)\psi_A(x) + \psi_B^\dagger(x)\psi_B(x)\right]$$

$$[H_0, N_D] = 0 \longrightarrow N_D(t) = N_D(0)$$

$$H_0 = \int d\alpha E_\alpha a_\alpha^\dagger a_\alpha + \int d\beta E_\beta b_\beta^\dagger b_\beta + \sum_n E_n|\varphi_n\rangle\langle\varphi_n|$$

New definition of X

$$H|\psi_B\rangle = E_B|\psi_B\rangle$$
$$X = \frac{1}{2}\langle\psi_B|N_D|\psi_B\rangle$$

Equivalence to the previous definition

$$|\psi_B\rangle = \int d\gamma C_\gamma |AB_\gamma\rangle + \sum_n C_n|\varphi_n\rangle$$
$$X = \frac{1}{2}\langle\psi_B|N_D^A + N_D^B|\psi_B\rangle = \int d\gamma |C_\gamma|^2 \ .$$

X is an observable because it is the expectation value of N_D, a self-adjoint operator

Specially suitable when using perturbative EFT (e.g. ChPT) with nonperturbative techniques **No bare elementary states**

3. Generation of the σ and ρ

Lowest order χPT Lagrangian implies

$$\mathscr{L}_{4\pi} = \frac{1}{12f^2}\mathrm{Tr}\left[(\partial_\mu\Phi\cdot\Phi - \Phi\partial_\mu\Phi)^2 + M\Phi^4\right]$$

$$\Phi = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} & \pi^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} \end{pmatrix}$$

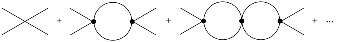
$I=0$, $J=0$ $J^{PC}=0^{++}$

$$V(s) = \frac{s - m_\pi^2/2}{f^2}$$

$I=1$,$J=1$, 1^{--}

$$V(s) = \frac{s-4m_\pi^2}{6f^2}$$

Unitarization employing UχPT



Oller,Oset,NPA620,438(1997);PRD60,074023(1999)

$$T(s) = \frac{V(s)}{1+V(s)G(s)} = \frac{1}{V(s)^{-1}+G(s)}$$

$$G(s) = i\int\frac{d^4p}{(2\pi)^4}\frac{1}{((P-p)^2-m^2+i\varepsilon)(p^2-m^2+i\varepsilon)}$$

$$= \frac{1}{(4\pi)^2}\left[a+\log\frac{m_\pi^2}{\mu^2}+\sigma(s)\log\frac{\sigma(s)+1}{\sigma(s)-1}\right] \quad , \quad \sigma(s) = \sqrt{1-\frac{4m_\pi^2}{s}}$$

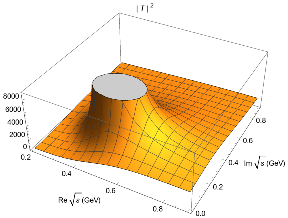
$$a_\sigma \simeq \log \frac{\mu_1^2}{\mu_2^2} = -1 \qquad\qquad a_\rho \simeq -\frac{6(4\pi f)^2}{M_\rho^2}$$

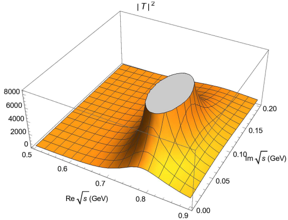
This signals a **very different nature** for the σ and ρ resonances

$\sigma: \quad \mu_2 \simeq \mu_1$

$\rho:$

$$-\frac{6(4\pi f_\pi)^2}{M_\rho^2} = \log \frac{\mu_1^2}{\mu_2^2} \to \mu_2 \simeq 9 \cdot 10^2 \mu_1 \sim 1 \text{ TeV}$$





$$\sqrt{s_\sigma} = 0.46 + i\,0.25 \text{ GeV}\ , \ \sqrt{s_\rho} = 0.77 + i\,0.071 \text{ GeV}$$

There is no bare elementary state in the Lagrangian

No way to calculate Z in a direct manner

Residue of the full propagator of the bare elementary state or by evaluating $|\langle\varphi_n|\psi_B\rangle|^2$.

4. QFT Calculation

This new definition is suitable for a QFT treatment

Dirac or Interacting Image

$$V \to Ve^{-\varepsilon|t|} \quad , \quad \varepsilon \to 0^+$$

$$|\psi_B\rangle = |\varphi_B(0)\rangle = U_D(0,-\infty)|\varphi_B\rangle$$

$$|\psi_B\rangle = |\varphi_B(0)\rangle = U_D(0,+\infty)|\varphi_B\rangle$$

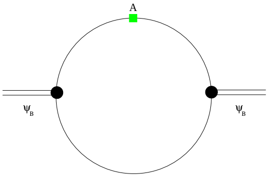
$$X = \frac{1}{n}\langle\psi_B|N_D|\psi_B\rangle$$

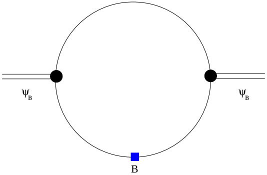
$$= \frac{1}{n}\langle\varphi_B|U_D(+\infty,0)N_D U_D(0,-\infty)|\varphi_B\rangle$$

$$= \frac{1}{n}\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2} dt\langle\varphi_B|U_D(+\infty,t)N_D(t)U_D(t,-\infty)|\varphi_B\rangle$$

$$U_D(t,-\infty)|\varphi_B\rangle = e^{iH_0 t}e^{-iE_B t}U_D(0,-\infty)|\varphi_B\rangle$$

$$X = \frac{1}{n}\lim_{T\to+\infty}\frac{1}{T}\int d^4x\langle\varphi_B|P\left[e^{-i\int_{-\infty}^{+\infty}dt' V_D(t')}\sum_i \psi^\dagger_{A_i}(x)\psi_{A_i}(x)\right]|\varphi_B\rangle$$





$$X_{\ell S} = \int \frac{d^3k}{(2\pi)^3} \frac{g_{\ell S}^2(k^2)}{(k^2/2\mu - E_B)^2}$$

$$X = \sum_{\ell S} X_{\ell S}$$

$\ell = 0$: one has the Weinberg's equation for $1 - Z$

$$g(k) = \frac{1}{2\pi^2} \int_0^\infty k'^2 dk' V(k,k') \frac{1}{k'^2/2\mu - E_B} g(k')$$

$$g_{\ell S}(-k) = (-1)^\ell g_{\ell S}(k) \qquad [T = V + VGT]$$

5. Explicit formulas

If the *finite-range* potential fulfills

$$\int_0^\infty \int_0^\infty dk\, dp |v_{\alpha\beta}(k,p)|^2 < \infty$$

Regular potential

$$\int_0^\infty dp |v_{\alpha\beta}(k,p)|^2 < M$$

If this is not fulfilled (singular potential)
Introduce some cut-off method to regularize V, e.g.
$\omega_{\alpha\beta}(k,p) = v_{\alpha\beta}(k,p)\theta(\Lambda - k)\theta(\Lambda - p)$

Expansion in a complete set of orthonormal linearly independent real functions $\{f_s(k)\}$ in $[0,\infty)$:

$$\omega^{(N)}_{\alpha\beta}(k_\alpha, p_\beta) = \sum_{s,s'=1}^{N} f_s(k_\alpha)\omega_{\alpha\beta;ss'} f_{s'}(p_\beta)$$

$$\omega_{\alpha\beta;ss'} = \int_0^\infty \int_0^\infty dkdp\, f_s(k)\omega_{\alpha\beta}(k,p) f_{s'}(p)$$

$$\omega_{\alpha\beta}(k_\alpha, p_\beta) = \lim_{N\to\infty} \omega^{(N)}_{\alpha\beta}(k_\alpha, p_\beta)$$

$$t^{(N)}_{\alpha\beta}(k,p;E)=\omega^{(N)}_{\alpha\beta}(k,p)+\sum_\gamma \frac{m_\gamma}{\pi^2}\int_0^\infty dq\frac{q^2}{q^2-2m_\gamma E}\omega^{(N)}_{\alpha\gamma}(k,q)t^{(N)}_{\gamma\beta}(q,p;E)\;.$$

$$t_{\alpha\beta;ss'}(E)=\int_0^\infty\int_0^\infty dkdpf_s(k)t_{\alpha\beta}(k,p;E)f_{s'}(p)\;.$$

$$[\omega]=\begin{pmatrix}[\omega_{11}] & [\omega_{12}] & \dots & [\omega_{1n}]\\ [\omega_{21}] & [\omega_{22}] & \dots & [\omega_{2n}]\\ \dots & \dots & \dots & \dots\\ [\omega_{n1}] & [\omega_{n2}] & \dots & [\omega_{nn}]\end{pmatrix}\;.$$

$$[f(k_\alpha)]^T=(\underbrace{0,\dots,0}_{N(\alpha-1)\text{ places}},f_1(k_\alpha),f_2(k_\alpha),\dots,f_N(k_\alpha),0,\dots,0)$$

$$\omega_{\alpha\beta}(k_\alpha,p_\beta)=[f(k_\alpha)]^T\cdot[\omega]\cdot[f(p_\beta)]$$

$$t_{\alpha\beta}(k_\alpha,p_\beta;E)=[f(k_\alpha)]^T\cdot[t(E)]\cdot[f(p_\beta)]$$

$$[G(E)]=\sum_{\alpha=1}^n[G_\alpha(E)]\;,\quad [G_\alpha(E)]=\frac{m_\alpha}{\pi^2}\int_0^\infty dq\frac{q^2}{q^2-2m_\alpha E}[f_\alpha(q)]\cdot[f_\alpha(q)]^T$$

$$[t(E)] = [\omega(E)] - [\omega(E)] \cdot [G(E)] \cdot [t(E)]$$

$$[t(E)] = [D(E)]^{-1}$$
$$[D(E)] = [\omega(E)]^{-1} + [G(E)]$$

$$X_\alpha = \frac{m_\alpha/\pi^2}{[f(p_\alpha)]^T \cdot [d] \cdot \frac{\partial [D]}{\partial E} \cdot [d] \cdot [f(p_\alpha)]} \frac{\partial}{\partial E} \int_0^\infty dk \frac{k^2}{k^2 - 2m_\alpha E}$$
$$\times [f(p_\alpha)]^T \cdot [d] \cdot [f(k_\alpha)] [f(k_\alpha)]^T \cdot [d] \cdot [f(p_\alpha)] \Big|_{E=E_B, p_\alpha = i\gamma_\alpha}$$

- For $\partial[\omega]/\partial E = 0$ then

$$\frac{\partial [D]}{\partial E} = \frac{\partial [G]}{\partial E}$$

$$1 = \sum_{\alpha=1}^{n} X_\alpha$$

Energy-independent contact interactions

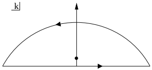
$$v_{\alpha\beta}(k_\alpha, p_\beta) = k_\alpha^{\ell_\alpha} p_\beta^{\ell_\beta} \sum_{i,j}^{N} v_{\alpha\beta;ij} k_\alpha^{2i} p_\beta^{2j}$$

Convergent factor $\mathbf{e^{i\epsilon k}}$, $\epsilon \to 0^+$ (like in many-body)
Analogous to Dimensional Regularization

$$V(k', k) \to V(k', k) e^{i\epsilon(k+k')}$$
$$g(k) \to g(k) e^{i\epsilon k}$$

We rewrite symmetrically the integration in k for X

$$X = \left(\frac{\mu}{\pi}\right)^2 \int_{-\infty}^{+\infty} dk k^2 \frac{g^2(k^2)\mathbf{e^{i\epsilon k}}}{(k^2 - 2\mu E_B)^2}$$



Notation: $\varkappa = i\gamma = \sqrt{2\mu E_B}$

$$
\begin{aligned}
X &= \frac{2i\mu^2}{\pi}\frac{\partial}{\partial k}\left[\frac{k^2 g^2(k^2)}{(k+\varkappa)^2}\right]_{k=\varkappa} \\
&= g^2(\varkappa^2)\frac{\mu^2}{2\pi\gamma} + \frac{\mu^2}{2\pi}\left.\frac{\partial g^2(-\bar{\gamma}^2)}{\partial\bar{\gamma}}\right|_{\bar{\gamma}=\gamma} \\
&= 1
\end{aligned}
$$

The 1st term gives the leading Weinberg contribution ($E_B \to 0$) for S-wave scattering

The 2nd term is the new one.
E.g. it takes into account that $g^2_{\ell S}(k^2) \propto k^{2\ell}$ for $k \to 0$

Aceti,Oset, PRD86,014012(2012)

The 2nd term depends on V

$$
g(k) = \frac{1}{2\pi^2}\int_0^\infty k'^2 dk' V(k,k')\frac{1}{k'^2/2\mu - E_B}g(k')
$$

• 1st Example. S wave.

$$V(k',k) = \left[v_0 + v_2(k^2 + {k'}^2)\right] e^{i\epsilon(k+k')}$$

$$g^2(\varkappa^2)\frac{\mu^2}{2\pi\gamma} = \frac{1-2\gamma^2 v_2/v_0}{1-6\gamma^2 v_2/v_0} \,,$$

$$\frac{\mu^2}{2\pi}\left.\frac{\partial g^2(-\bar{\gamma}^2)}{\partial\bar{\gamma}}\right|_{\bar{\gamma}=\gamma} = -\frac{4\gamma^2 v_2/v_0}{1-6\gamma^2 v_2/v_0}$$

$$X = 1$$

Shallow case $\gamma^2 v_2/v_0 \simeq \gamma r_s/4$ ($\gamma r_s \to 0$)

• 2nd Example. Angular momentum ℓ.

$$V(k',k) = v_\ell {k'}^\ell k^\ell e^{i\epsilon(k+k')}$$

$$g^2(k^2) = \frac{(-k^2)^\ell 2\pi}{\mu^2(2\ell+1)\gamma^{2\ell-1}}$$

$$X = 1 = \frac{1}{2\ell+1} + \frac{2\ell}{2\ell+1}$$

• 3rd example. Energy dependence in V:

$$[v] = \frac{1}{E - E_0} \begin{pmatrix} 0 & v_{12} \\ v_{12} & 0 \end{pmatrix}$$

Cut-off regularization:

$$L_{n+1} = \int_0^\infty dq q^n = \theta_n \Lambda^{n+1}$$

$$E_0 = v_{12}(L_3 + \epsilon\sqrt{L_5(L_1 - \alpha)})$$

$$v_{12} = \frac{2\epsilon\sqrt{L_5(L_1-\alpha)}}{m\left(rL_5 - 2L_3(L_1-\alpha) - 4\epsilon\sqrt{L_5}(L_1-\alpha)^{3/2}\right)}$$

$\Lambda \to \infty$ limit:

$$T(k,k) = \frac{1}{\alpha + \frac{1}{2} r k^2 + i\frac{mk}{2\pi}}$$

$$\frac{T(k,p)}{T(k,k)} = 1 + (k^2 - p^2)\frac{\rho_\Lambda}{\Lambda^2} + \mathscr{O}(\Lambda^{-3})$$

$$X = \frac{1}{\sqrt{1 - 2r_s/a_s}} \leq 1 \;,\; r_s \leq 0 \;,\; a_s > 0$$

Recall: $r_s \geq 0$ cannot be in pure contact-interaction theory

Phillips, Beane, Cohen, *et al.*, AOP263(1998) DR cannot be applied

6. Relativistic case

Traditionally:

1.- The attention is focused on the wave function renormalization Z

2.- There is a lack of a general applicable results

3.- Partial results are available:

$0 \leq Z \leq 1$: Lee model Vaughn, Aaron, Amado PRC124,1258(1961); Yukawa-type interactions Houard,Jouvet,Nuovo Cim.18,466(1960); Salam,Nuovo Cim.25,224(1962); Lurié, Macfarlane, PR136,B816(1964)
$Z = 0$ equivalence between 4-Fermi theories and Yukawa theories

Examples in the recent literature

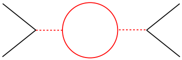
Hyodo,Jido,Hosaka,PRC85,015201(2012)

Agadjanov,Guo,Rios,Rusetsky,JHEP2015,01,118

Yukawa-like model

Scalar fields: $\phi_1(x)$, $\phi_2(x)$
A bare elementary scalar field: $\Phi(x)$, bare mass M_0

$$\mathscr{L}_{\rm int} = g_0 \phi_1(x)\phi_2(x)\Phi(x)$$



$$T = \frac{g_0^2}{s - M_0^2 - g_0^2 G(s)} \qquad T \xrightarrow[s\to M^2]{} \frac{g^2}{s-M^2} = \frac{Z g_0^2}{s-M^2}$$

$$0 \leq Z = \frac{1}{1 - g_0^2 G'(M^2)} \leq 1 \qquad Z = 1 + g^2 G'(M^2)$$

Lurié,Macfarlane,PR136,B816(1964): $Z=0$ composite state
Hyodo *et al* PRC85,015201(2012): $1-Z$ is the compositeness
Agadjanov *et al.*,JHEP2015,01,118: Shallow states, $Z = |\langle\varphi_1|\psi_B\rangle|^2$

The Källen-Lehmann Representation

Weinberg, QFTI, §10.7

$$\Delta_F(p) = \int_0^\infty d\mu^2 \frac{\rho(\mu^2)}{p^2 - \mu^2 - i\varepsilon}$$

Imposing (anti)commutation relations at equal time

$$\left[\frac{\partial \Phi(\mathbf{x}, t)}{\partial t}, \Phi^\dagger(\mathbf{y}, t)\right] = -i\delta(\mathbf{x} - \mathbf{y})$$

Sum rule: $1 = \int_0^\infty d\mu^2 \rho(\mu^2)$

Coupling with a one-particle asymptotic state

$$\rho(\mu^2) = Z\delta(\mu^2 - M^2) + \sigma(\mu^2)$$

$$1 = Z + \int_0^\infty d\mu^2 \underbrace{\sigma(\mu^2)}_{\text{Multi}-\text{Part.States}}$$

$$0 \le Z \le 1$$

Weinberg: "The limit $Z = 0$ has an interesting interpretation as a condition for a particle to be composite rather than elementary"

Not applicable if there is no bare elementary fields in the Lagrangian

The examples of the σ and ρ above.

To keep in mind: In the relativistic case we can also have different number of particles in the continuum states. E.g. $\pi\pi$, 4π, ...

This is why a specific contribution, like the "two-body" contribution in NRQM, is not isolated in the multiparticle contribution

- $[H_0, N_D] = 0$

Non-Relativistic formalism:

$$|\psi_B\rangle = \int d\gamma C_\gamma |AB_\gamma\rangle + \sum_n C_n |\varphi_n\rangle$$

Relativistic formalism:

$$|\psi_B\rangle = \int d\gamma C_\gamma |AB_\gamma\rangle + \int d\eta D_\eta |AAB_\eta\rangle + \int d\mu\, \delta_\mu |ABB_\mu\rangle + \ldots$$
$$+ \int d\eta_\nu F_\nu |CD_\nu\rangle + \ldots + \sum_n C_n |\varphi_n\rangle + \sum_n \int d\alpha C_{n\alpha} |A_\alpha \varphi_n\rangle + \ldots$$

Number operator for each species of particle

$$N_D^A = \int d\alpha a_\alpha^\dagger a_\alpha$$

$$\langle\psi_B|N_D^A|\psi_B\rangle = \int d\alpha |C_\gamma|^2 + 2\int d\eta |D_\eta|^2 + \int d\mu |\delta_\mu|^2 + \ldots$$
$$+ \sum_n \int d\alpha |C_{n\alpha}|^2 + \ldots$$

$$N_D = \sum_A N_D^A + \sum_E N_D^E$$

$$|\psi_B\rangle = \sum_{n,i} C_{ni} |n, i\rangle$$

$$\langle\psi_B|\psi_B\rangle = 1 = \sum_{n,i} |C_{ni}|^2$$

$$\langle\psi_B|N_D|\psi_B\rangle = \sum_{n,i} |C_{ni}|^2 n$$

New criterion for an elementary stable particle

$$\langle\psi_B|N_D|\psi_B\rangle = 1$$

- This **implies** that $C_{ni} = 0$ for $n \geq 2$,

$$\langle\psi_B|N_D^A|\psi_B\rangle = 0 \;, \; \forall A$$

If $\langle\psi_B|N_D|\psi_B\rangle > 1$ we would have multi-particle state contributions

Pure Composite state

- **Necessary Condition** $\langle\psi_B|N_D|\psi_B\rangle \geq 2$

- **Sufficient Condition** $\langle\psi_B|N_D^E|\psi_B\rangle = 0 \ , \ \ \forall E$

  It is not necessary because one can have $C_{1E} = 0$ but $C_{nE} \neq 0$
  $|A\varphi_n\rangle$, $|\varphi_n\varphi_m\rangle$, ...

Other consequences

- If $\langle\psi_B|N_D|\psi_B\rangle \geq 2+m$

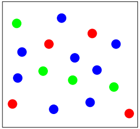
  The multiparticle components with $2+m$ and more particles are important.
  Other similar *exclusive* conditional conclusions can be also established

- **Sum Rule** for the type of particles:

$$N_D = \sum_{i=1}^{n_f} N_D^i \ , \ i \in \{A, E\}$$

$$1 = \sum_{i=1}^{n_f} \frac{\langle\psi_B|N_D^i|\psi_B\rangle}{\langle\psi_B|N_D|\psi_B\rangle}$$

Probability ● ● ●?



$\langle\psi_B|N_D^i|\psi_B\rangle/\langle\psi_B|N_D|\psi_B\rangle$

E.g. for the Deuteron, 50% for p ($i=1$), 50% for n ($i=2$)

We can calculate $\langle\psi_B|N_D^i|\psi_B\rangle$ within QFT too

$$V \to V e^{-\varepsilon|t|}$$

$$|\psi_B\rangle = |\varphi_B(0)\rangle = U_D(0,-\infty)|\varphi_B\rangle$$

$$|\psi_B\rangle = |\varphi_B(0)\rangle = U_D(0,+\infty)|\varphi_B\rangle$$

$$\langle N_D^A\rangle = \langle\psi_B|N_D^A|\psi_B\rangle$$

$$= \langle\varphi_B|U_D(+\infty,0)N_D^A U_D(0,-\infty)|\varphi_B\rangle$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2} dt\langle\varphi_B|U_D(+\infty,t)N_D^A(t)U_D(t,-\infty)|\varphi_B\rangle$$

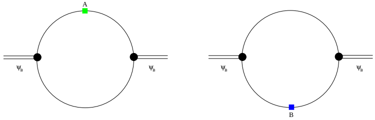
$$U_D(t,-\infty)|\varphi_B\rangle = e^{iH_0 t}e^{-iE_B t}U_D(0,-\infty)|\varphi_B\rangle$$

▷ Technicalities

In general there are many more diagrams now apart from those in the NR case

• If *there are good reasons* for dominance of two-body channels

$$X_{AB} = \frac{1}{2}(\langle N_D^A \rangle + \langle N_D^B \rangle) \approx \int d\gamma |C_\gamma|^2$$



• In the case in which we are close to a two-body threshold (AB) we come back to the NR case for evaluating X_{AB}

7. On-shell methods

General unitarization formula (UχPT) for the on-shell T matrix

$$T(s) = \left[V(s)^{-1} + G(s)\right]^{-1}$$

In general $V(s)$ has dynamical cuts, e.g. left-hand cut for $\pi\pi$ scattering

For simplicity let us consider the uncoupled case

Take the limit $s \to s_B$, $T \to -g^2/(s - s_B)$

$$-g^2 = \frac{1}{-V'(s_B)/V(s_B)^2 + G'(s_B)}$$

$$\begin{aligned} 1 &= -g^2 G'(s_B) + g^2 V'(s_B)/V(s_B)^2 \\ &= \underbrace{-g^2 G'(s_B)}_{X} + \underbrace{g^2 V'(s_B) G(s_B)^2}_{Z} \end{aligned}$$

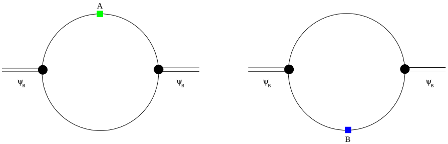
X is given by the same expression as in the Yukawa-like models with constant interaction

Pure energy-dependent (partial-wave projected) interaction: $V(s)$

From Lippmann-Schwinger or Bethe-Salpeter equation:

$$T(s) = V(s) - V(s)G(s)T(s)$$
$$T(s) = \left[V(s)^{-1} + G(s)\right]^{-1}$$

Moving to the pole position $g(s_P)$ is just a constant



$$X_{AB} = \frac{1}{2}(\langle\psi_B|N_D^A|\psi_B\rangle + \langle\psi_B|N_D^B|\psi_B\rangle)$$
$$X = -g^2 G'(s_P)$$

Thus, our formalism with $V(s)$ gives rise to these results too

- Redefine

$$T(s) \to \widetilde{T}(s) = T(s)/N(s)$$

$$\widetilde{T}(s) = \left[N(s)V(s)^{-1} + N(s)G(s)\right]^{-1} = \left[\widetilde{V}(s)^{-1} + \widetilde{G}(s)\right]^{-1}$$

$$\widetilde{X} = -\widetilde{g}^2 \widetilde{G}'(s_P) = X - g^2 G(s_P) N'(s_P)/N(s_P)$$

$$\widetilde{T}(s) \to T(s) = \widetilde{T}(s) N(s)$$

Conclusions have to be taken with a grain of salt

A priori there should be some good reason to change the normalization

For higher partial wave remove the threshold behavior $p^{2\ell} = N(s)$ Aceti,Oset,PRD86,014012(2012)

- For the ρ $\quad X \simeq x_\rho/(1+x_\rho) \ll 1$

$$x_\rho = \frac{1}{6}\left(\frac{M_\rho}{4\pi f_\pi}\right)^2 = -\frac{1}{a_\rho} = 0.073 \ll 1$$

The ρ is a "$q\bar{q}$" resonance. Large N_c running of its pole position Peláez,PRL92,102001(2004): $M_\rho = \mathcal{O}(N_c^0)$ and $\Gamma_\rho = \mathcal{O}(1/N_c)$
Compact object, $\sqrt{\langle r_V^2\rangle} \simeq 1/M_\rho \simeq 0.25 \text{ fm}$
LQCD, simple quark models.

- For the σ, $|X| = 0.38$

There is a strong dependence on m_π of the pole position

	E_P (MeV)
Resonance	$250 \gtrsim m_\pi \geq 0$ MeV
$\|X\|$ decreases	$1 \geq \|X\| \geq 0.20$
Virtual state	$340 \gtrsim m_\pi \geq 250$ MeV
No meaningful $\|X\|$	
Bound state	$m_\pi \geq 340$ MeV
X decreases	$X \leq 1$

Chiral limit ($m_\pi \to 0$)

$$X = \left| \frac{x_\sigma}{1 + x_\sigma} \right| \simeq 0.2$$

$$x_\sigma = s_\sigma/(4\pi f_\pi)^2$$

A strong dependence of s_σ with m_π is obtained in LQCD Hadron Spectrum Coll. PRL118(2017)

Very unusual behavior of s_σ with N_c QCD Oller,Oset,PRD60(1998); Peláez,PRL92(2004); Guo,Oller,Ruiz-Elvira,PRD86(2012)

Indications of the σ as a compact hadronic state for actual m_π:

Calculation of $\langle r^2\rangle_s^\sigma \simeq 0.20$ fm^2 Oller,Albaladejo,PRD86(2012)
We interpret it as a kind of fusion of 2 pions in $qq\bar{q}\bar{q}$

Unusual Regge trajectory of the σ.
As a low-energy resonance produced by a short-range Yukawa potential, $a \simeq 0.4$ fm, between pions Londergan *et al.* PLB729(2014)

8. Resonances. Number-operator-number interpretation

In my developments a resonance follows by analytical continuation from the physical axis

- Let $|\psi_\alpha^+\rangle$ be a two-body in-state

$$
\begin{aligned}
|\psi_\alpha^+\rangle &= U_D(0,-\infty)|\varphi_\alpha\rangle \\
&= |\varphi_\alpha\rangle + \int d\gamma \frac{T_{\gamma\alpha}(E+i\varepsilon)}{E-E_\gamma+i\varepsilon}|\varphi_\gamma\rangle + \sum_n \frac{T_{n\alpha}(E)}{E-E_n}|\varphi_n\rangle
\end{aligned}
$$

S. Weinberg, QFT, Vol.1

$$
\langle\psi_\alpha^+|\underbrace{\int d\gamma a_\gamma^\dagger a_\gamma}_{N_D^A} + \underbrace{\int d\eta b_\eta^\dagger b_\eta}_{N_D^B}|\psi_\alpha^+\rangle = 2\langle\varphi_\alpha|\varphi_\alpha\rangle \quad \text{Fine!}
$$

There are cancellations because of unitarity

Problem: This expectation value cannot be analytically continued to the resonance pole

$$\langle\psi_\alpha^+| = \langle\varphi_\alpha| + \int d\gamma \frac{T_{\gamma\alpha}(E-i\varepsilon)}{E-i\varepsilon-E_\gamma}\langle\varphi_\gamma| + \sum_n \frac{T_{n\alpha}(E-i\varepsilon)}{E-i\varepsilon-E_n}\langle\varphi_n|$$

$$T(E\pm i\varepsilon)^\dagger = T(E\mp i\varepsilon)$$

The analytical continuation to $E = M_R - i\Gamma/2$ remains in the 1st or physical Riemann Sheet (RS)
No resonance pole there

The analytical continuation must be done as in the calculation of the S-matrix:

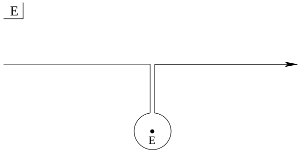
out state $|\psi_\alpha^-\rangle$, $E - i\varepsilon$

$$\langle \psi_\alpha^- | N_D^A + N_D^B | \psi_\alpha^+ \rangle$$

$$\langle \psi_\alpha^- | = \langle \varphi_\alpha | + \int d\gamma \frac{T_{\gamma\alpha}(E+i\varepsilon)}{E+i\varepsilon-E_\gamma} \langle \varphi_\gamma | + \sum_n \frac{T_{n\alpha}(E+i\varepsilon)}{E+i\varepsilon-E_n} \langle \varphi_n |$$

When crossing the real positive energy axis

$$T(E+i\varepsilon) \to T^{II}(E-i\varepsilon)$$



The resonance pole is now reached both for the ket and the bra

9. QFT calculation

Dirac or Interacting Image

$$V \to V e^{-\varepsilon|t|}$$

$$|\psi_R^+\rangle = U_D(0,-\infty)|\varphi_R^+\rangle$$

$$\langle\psi_R^-| = \langle\varphi_R^-|U_D(+\infty,0)$$

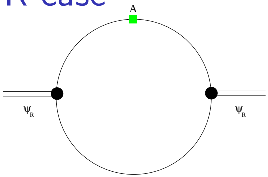
$$\langle\psi_R^-|N_D|\psi_R^+\rangle = \langle\varphi_R^-|U_D(+\infty,0)N_D U_D(0,-\infty)|\varphi_R^+\rangle$$

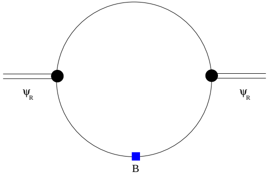
$$= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2} dt\langle\varphi_R^-|U_D(+\infty,t)N_D(t)U_D(t,-\infty)|\varphi_R^+\rangle$$

$$U_D(t,-\infty)|\varphi_R^+\rangle = e^{iH_0 t}e^{-iHt}|\psi_R^+\rangle = e^{-(iM_R+\frac{\Gamma}{2})t}e^{iH_0 t}U_D(0,-\infty)|\varphi_R^+\rangle$$

$$\langle\varphi_R^-|U_D(+\infty,t) = \langle\psi_R^-|e^{iHt}e^{-iH_0 t} = \langle\varphi_R^-|U_D(+\infty,0)e^{-iH_0 t}e^{(iM_R+\frac{\Gamma}{2})t}$$

10. NR case

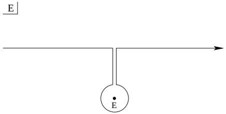




2nd Riemann Sheet: $E_R = \varkappa^2/2\mu$

$$X_{\ell S} = \int \frac{d^3k}{(2\pi)^3} \frac{g_{\ell S}^2(k^2)}{(k^2/2\mu - E_R)^2} + \frac{i\mu^2}{\pi\varkappa} \frac{\partial}{\partial k} \left[k\, g_{\ell S}^2(k^2)\right]_{k=\varkappa}$$

$$X = \sum_{\ell S} X_{\ell S}$$



$$g(k) = \frac{\mu}{\pi^2} \int_0^\infty dk' k'^2 \frac{V(k,k')g(k')}{k'^2 - \varkappa^2} + \frac{i\mu\varkappa V(k,\varkappa)/\pi}{1 - i\mu\varkappa V(\varkappa,\varkappa)/\pi} \frac{\mu}{\pi^2} \int_0^\infty dk' k'^2 \frac{V(\varkappa,k')g(k')}{k'^2 - \varkappa^2}$$

$$g_{\ell S}(-k) = (-1)^\ell g_{\ell S}(k)$$

Explicit formulas. Resonance case.

Expansion in a complete set of orthonormal linearly independent real functions $\{f_s(k)\}$ in $[0,\infty)$:

$$\omega_{\alpha\beta;ss'} = \int_0^\infty \int_0^\infty dk dp f_s(k) \omega_{\alpha\beta}(k,p) f_{s'}(p)$$

$$\omega_{\alpha\beta}(k_\alpha, p_\beta; E) = [f_\alpha(k_\alpha)]^T \cdot [\omega] \cdot [f_\beta(p_\beta)]$$

$$t^{II}_{\alpha\beta}(k_\alpha, p_\beta; E) = [f_\alpha(k_\alpha)]^T \cdot [t^{II}(E)] \cdot [f_\beta(p_\beta)]$$

$$[G^{II}_\alpha(E)] = \frac{m_\alpha}{\pi^2} \int_0^\infty dq \frac{q^2}{q^2 - 2m_\alpha E} [f_\alpha(q_\alpha)] \cdot [f_\alpha(q_\alpha)]^T + \frac{i m_\alpha}{\pi} \sqrt[II]{2m_\alpha E} \left[f_\alpha(\sqrt[II]{2m_\alpha E}) \right] \cdot \left[f_\alpha(\sqrt[II]{2m_\alpha E}) \right]^T$$

$$[t^{II}(E)] = [D^{II}(E)]^{-1}$$

$$[D^{II}(E)] = [\omega(E)]^{-1} + [G^{II}(E)]$$

$$X_\alpha = \left([p_\alpha]^T \cdot [d^{II}] \cdot \frac{\partial [D^{II}]}{\partial E} \cdot [d^{II}] \cdot [p_\alpha] \right)^{-1}$$

$$\times \frac{\partial}{\partial E} \left(\frac{m_\alpha}{\pi^2} \int_0^\infty dk \frac{k^2}{k^2 - 2m_\alpha E} [k_\alpha]^T \cdot [d^{II}] \cdot [p_\alpha] \, [k_\alpha]^T \cdot [d^{II}] \cdot [p_\alpha] \right.$$

$$\left. + i \frac{m_\alpha}{\pi} \sqrt[II]{2m_\alpha E} \, [\sqrt[II]{2m_\alpha E}]^T \cdot [d^{II}] \cdot [p_\alpha] \, [\sqrt[II]{2m_\alpha E}]^T \cdot [d^{II}] \cdot [p_\alpha] \right) \Big|_{E=E_R, p_\alpha = \varkappa_\alpha}$$

- For $\partial[\omega]/\partial E = 0$ then

$$1 = X = \sum_{\alpha=1}^{n} X_\alpha$$

Energy-independent contact potential

We include the convergent factor for the 2nd RS calculation

$$v_{\alpha\beta}(k_\alpha, p_\beta) = k_\alpha^{\ell_\alpha} p_\beta^{\ell_\beta} \sum_{i,j}^{N} v_{\alpha\beta;ij} k_\alpha^{2i} p_\beta^{2j}$$

$$V(k', k) \to V(k', k) e^{-i\epsilon(k+k')}$$

$$X_{\ell S} = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dk k^2 \frac{g_{\ell S}^2(k^2) e^{-i\epsilon k}}{(k^2/2\mu - E_R)^2} + \frac{i\mu^2}{\pi\varkappa} \left. \frac{\partial k g_{\ell S}^2(k^2)}{\partial k} \right|_{k=\varkappa}$$

$$= g^2(\varkappa^2) \frac{i\mu^2}{2\pi\varkappa} + \frac{i\mu^2\varkappa}{\pi} \left. \frac{\partial g^2(k^2)}{\partial k^2} \right|_{k=\varkappa}$$

$$X_{\ell S} = \frac{2\mu^2}{\pi^2} \int_0^\infty dk^2 \sqrt[II]{k^2 + i\varepsilon} \frac{g_{\ell S}^2(k^2)}{(k^2 - \varkappa^2)^2}$$

$$X = 1 = \sum_{\ell S} X_{\ell S}$$

Resonances are composite

For a *regular* energy-independent potential $X = 1$

Hernández,Mondragón, PRC29,722(1984)

X is in general complex for a resonance

$$V(k,k') = f(k^2)f(k'^2)\,\mathrm{V(E)}$$

$$g(k^2) = \mathrm{V}^{\frac{1}{2}} f(k^2)\left[\frac{\partial(\mathrm{V}\,\widetilde{G}^{\mathrm{II}})}{\partial E_R}\right]^{-1}$$

$$X = \left[\frac{\partial(\mathrm{V}\,\widetilde{G}^{\mathrm{II}})}{\partial E_R}\right]^{-1}\frac{-1}{(2\pi)^2}\int_0^\infty dk k^2 \frac{\mathrm{V} f(k^2)^2}{(E_R - k^2/2\mu)^2}$$

$$= \left[\frac{\partial(\mathrm{V}\,\widetilde{G}^{\mathrm{II}})}{\partial E_R}\right]^{-1}\left\{\mathrm{V}\frac{\partial\widetilde{G}^{\mathrm{II}}}{\partial E_R} + \widetilde{G}^{\mathrm{II}}(E_R)\frac{\partial \mathrm{V}(E_R)}{\partial E_R}\right\}$$

11. Redefinition of "phases"

in-,out-states:
$\eta(E)$ is a complex function with RHC: $\eta(E^*) = \eta(E)^*$

$$|\psi_\alpha^+\rangle \longrightarrow e^{\eta(E_\alpha + i\varepsilon)} |\psi_\alpha^+\rangle$$

$$\begin{aligned}\langle\psi_\alpha^-| &\longrightarrow \langle\psi_\alpha^-| e^{\eta(E_\alpha - i\varepsilon)^*} \\ &= \langle\psi_\alpha^-| e^{\eta(E_\alpha + i\varepsilon)}\end{aligned}$$

Analytical continuation $E_\alpha \to E_R = M_R - i\Gamma/2$

$$\eta(E_\alpha + i\varepsilon) \to \eta^{\mathrm{II}}(E_\alpha - i\varepsilon) \to \eta^{\mathrm{II}}(M_R - i\Gamma/2)$$

An specific fact of resonances; no analogue for bound states.

These phase factors make X_{AB} be positive definite

There could be dependence on the channel, $\eta_{AB}(E)$

$$g_{AB}^2(k^2) \to g_{AB}^2(k^2)\, e^{2\eta_{AB}^{\rm II}(E_R)}$$

$$\langle \psi_R^- | N_D^{AB} | \psi_R^+ \rangle\, e^{2\eta_{AB}^{\rm II}(E_R)} \in \mathbb{R}^+$$

Plausible dispersion relation for $\eta(E)$

Narrow-Resonance Case:

$$\eta(E) = \frac{1}{\pi}\int_0^\infty dE' \frac{{\rm Im}\eta(E')}{E' - M_R - i\varepsilon}$$
$$= \frac{1}{\pi}\int_0^\infty dE' \frac{{\rm Im}\eta(E')}{E' - M_R} + i{\rm Im}\eta(E')$$

${\rm Im}\eta(E')$ is smooth and $\eta(M_R) \approx i{\rm Im}\eta(M_R)$

Pure phase factor $e^{\eta(M_R)} \approx e^{i\,{\rm Im}\eta(M_R)}$

12. S-matrix transformations

Introduced in Z.H.Guo, Oller, PRD93,096001(2016)

Example: Narrow resonance case

Laurent series around the resonance pole:
$s_P = (M_R - i\Gamma/2)^2$

$$S(s) = \frac{R}{s - s_P} + S_0(s)$$
$$S(s)S(s)^\dagger = I$$

$S_0(s) \to S_0$, constant

$$(s - s_P)(s - s_P^*)S_0 S_0^\dagger + (s - s_P)S_0 R^\dagger + (s - s_P^*)RS_0^\dagger + RR^\dagger = (s - s_P)(s - s_P^*)\ .$$

$$S_0 S_0^\dagger = I$$
$$S_0 R^\dagger + RS_0^\dagger = 0$$
$$-s_P S_0 R^\dagger - s_P^* RS_0^\dagger + RR^\dagger = 0$$

Solution:

$$S_0 = \mathscr{O}\mathscr{O}^T$$
$$\mathscr{O}\mathscr{O}^\dagger = I$$

Rank 1 Symmetric Projection Operator $\mathscr{A}$:

$$\begin{aligned} R =& i\lambda \mathscr{O}\mathscr{A}\mathscr{O}^T ~,~ \lambda\in\mathbb{R} \\ \mathscr{A}^\dagger =& \mathscr{A} \\ \mathscr{A}^2 =& \mathscr{A} \\ \lambda =& 2\mathrm{Im}\, s_P = -2M_R\Gamma_R \end{aligned}$$

Resonant S-matrix $S_R(s)$:

$$S(s) = \mathscr{O}\underbrace{\left(I + \frac{i\lambda\mathscr{A}}{s-s_R}\right)}_{S_R(s)}\mathscr{O}^T$$

Origin of phases: Smooth non-resonant terms, $\mathscr{O}$
E.g. Coulomb phases in nuclear physics

In general, do not take the real part in $\langle\psi_R^-|N_D^A|\psi_R^+\rangle$ to make it real!!

The right procedure is doing the phase or S-matrix transformations

$$S_{\mathscr{O}}(s) \equiv \mathscr{O} S(s) \mathscr{O}^T$$
$$\mathscr{O} = \mathrm{diag}(e^{i\phi_1}, \ldots, e^{i\phi_n})$$
$$g_i^2 \to g_i^2 e^{2i\phi_i}$$

There is an associated $T_{\mathscr{O}}(s)$

E.g. for the case of only one channel:

$$g^2 \to g^2 e^{2i\phi} \qquad \langle\psi_R^-|N_D^A|\psi_R^+\rangle \to |\langle\psi_R^-|N_D^A|\psi_R^+\rangle|$$

NR: Criterion for elementariness of a narrow resonance with respect to the open channels

$$\langle N_D^A \rangle = \langle \psi_R^- | N_D^A | \psi_R^+ \rangle e^{2i\mathrm{Im}\eta_{\mathrm{A}}^{\mathrm{II}}(E_R)} = \left| \langle \psi_R^- | N_D^A | \psi_R^+ \rangle \right| = 0$$

Relativistic case

Necessary condition for a resonance to be qualified as elementary

$$\langle N_D^A \rangle = 0 \ \ , \ \ \forall A$$

Finite width resonances

Necessary Condition for still interpreting $|\langle\psi_R^-|N_D^A|\psi_R^+\rangle|$ as an average number of particles Z.H.Guo, Oller, PRD93,096001(2016)

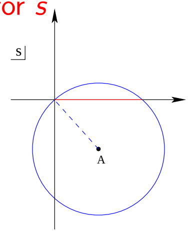
The transformations

$$S_{\mathscr{O}}(s) \equiv \mathscr{O} S(s) \mathscr{O}^T$$
$$\mathscr{O}\mathscr{O}^\dagger = I$$
$$g_i^2 \to g_i^2 \mathscr{O}_{ii}^2$$

make sense only if:

▷The Laurent expansion around s_P is valid in some interval of physical (real values above threshold) for s

$S(s)S(s)^\dagger = I$ is meaningful



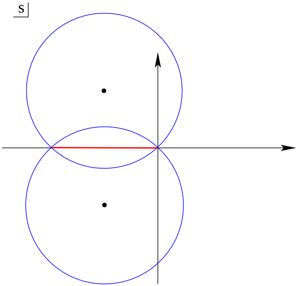
Condition A: $s_n < \mathrm{Re}\, s_P < s_{n+1}$

s_n is the threshold of channel n

Physical idea

- If this condition is fulfilled one can think of a physical process with a clear resonance contribution. E.g. the σ and E791 data on D^+ and D_s^+ decays
- The resonance phenomenon is physically manifest in the open channels
- We preserve $|g_i|$ to the open channels

A resonance is then very different



$$\frac{g^2}{s - s_R} + \frac{g^{2*}}{s - s_R^*} = 2\mathrm{Re}\frac{g^2}{s - s_R}$$

Double-pole like virtual state

This could well be the case for the $X(3872)$, at least as a double-like pole. It could also be triple-like, etc.

X.-W.Kang,Oller,EPJC77,399(2017)

$\bar{D}^0 D^{*0}$ threshold. Tiny width

13. S-wave Effective Range Expansion

X.W.Kang,Z.H.Guo,Oller,PRD94,014012(2016)

$$T(k) = \frac{1}{-\frac{1}{a} + \frac{1}{2} r k^2 - ik}$$
$$G(k) = -ik$$

$$E_R = M_R - i\Gamma/2$$
$$a = -\frac{2k_i}{|k_R|^2}$$
$$r = -\frac{1}{k_i} \quad , \quad a/2 > r$$

$$X = -\gamma^2 \frac{dG}{ds} = -\gamma_k^2 \frac{dG}{dk} = i\frac{k_i}{k_r} = i \tan\frac{\phi}{2}$$

$|X| \leq 1 \leftrightarrow k_r \geq k_i \leftrightarrow M_R \geq 0$
($|X| = 1$ for $M_R = 0$ and $\Gamma > 0$)

If the real part is taken then ALWAYS $X = 0$!

$$\tan\phi = \frac{\Gamma}{2M_R} \longrightarrow \phi \in [0, \frac{\pi}{2}] \text{ for } M_R \geq 0$$

$$k_R = k_r - i\,k_i = \sqrt{2\mu(M_R - i\Gamma/2)} = |k_R|(\cos\phi/2 - i\sin\phi/2)$$

$$X = \left(\frac{2r}{a} - 1\right)^{-1}$$

$Z_b(10610)$ and $Z_b(10650)$ or Z_b and Z_b'

$B^{(*)}\bar{B}^*$ system with $I^G(J^P) = 1^+(1^+)$

$$E_{Z_b} = 10607.2 \pm 2.0 - i(9.2 \pm 1.2)\ \text{MeV}$$

$$E_{Z_b'} = 10652.2 \pm 1.5 - i(5.5 \pm 1.1)\ \text{MeV}$$

M_R is around 3 MeV below $B^{(*)}\bar{B}^*$ threshold

	$Z_b(10610)$	$Z_b(10650)$
$a\,(\text{fm})$	-1.03 ± 0.17	-1.18 ± 0.26
$r\,(\text{fm})$	-1.49 ± 0.20	-2.03 ± 0.38
$X = \gamma_k^2$	0.75 ± 0.15	0.67 ± 0.16

Determining X by making use of the width of the resonance

Meißner,Oller,PLB751(2015)

$$
\begin{aligned}
\Gamma_i^{(1)} =& \frac{2X_i}{\mu} k(M_R)|k_R| \\
\Gamma_i^{(2)} =& \frac{X_i|k_R|M_R^2}{\pi\mu}\int_{M_{\rm th}}^{+\infty} dW \frac{k(W)}{W^2}\frac{\Gamma}{(M_R-W)^2+\Gamma^2/4}
\end{aligned}
$$

For the $Z_b^{(\prime)}$ it gave consistent results with the ERE-based method

14. Spectral density function. NR Resonance

Bogdanova, Hale, Markushin, PRC44,1289(1991);

Baru,Haidenbauer,Hanhart,Kalashnikova,Kudryavtsev,PLB586,53(2004)

Spectral density of the bare state $|\psi_0\rangle$: $\omega(E)$

$$|\psi_0\rangle = \int d\mathbf{k} c_0(k)|\mathbf{k}\rangle$$

$$\omega(E) = 4\pi\mu k|c_0(E)|^2\theta(E)$$

$$\int_0^\infty dE\,\omega(E) = \begin{cases} 1 & \textbf{No bound states} \\ 1-Z & \text{With bound states} \end{cases}$$

How to implement it? Select the *resonant* region around threshold

$$W = \int_{E_-}^{E_+} dE\,\omega(E)$$

Conceptually, it is not *fully* settled as a quantitative estimate of compositeness for resonances

It provides a nice smooth transition from the clear bound states and narrow resonances

It has a clear connection with the pole-counting rule of Morgan NPA543,632(1992), with the presence of nearby CDD poles Kang,Oller, EPJC77,399(2017)

The spectral density method compares well with the on-shell method to get X

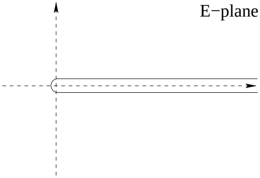
15. Scattering Amplitude $t(E)$

Dispersion Relation for the inverse of $t(E)$

$$\text{Im}\, t(E)^{-1} = -ik$$

One subtraction is needed

$$\oint dz \frac{t(z)^{-1}}{(z-E)(z-C)}$$



The only other structure apart from the threshold that can give rise to a strong distortion in $t(E)^{-1}$ is a pole at M_Z

$$t(E) = \frac{1}{\frac{\lambda}{E-M_Z} + \beta - ik}$$

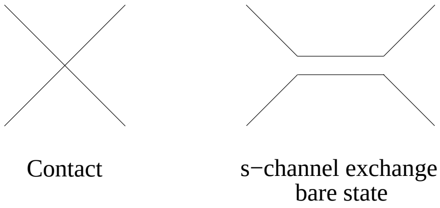
CDD pole Castillejo,Dalitz,Dyson, PR,101,453(1956)

The ERE or a Flatté parametrization break down for $|k| \gtrsim \sqrt{2\mu|M_Z|}$

The general formula for a partial-wave without crossed-channel dynamics was deduced in: Oller, Oset PRD60,074023 (1999)

Contact interaction plus *s*-channel exchange of bare resonances

Kang,Oller, EPJC77,399(2017) study of the $X(3872)$



[BHKKN] Baru,Hanhart,Kalashnikova,Kudryavtsev, EPJA,44,93(2010) *Interplay of quark and meson degrees of freedom in a near-threshold resonance*

[ABK] Artoisenet,Braaten,Kang, PRD,82,014013(2010) *Using line shapes to discriminate between binding mechanisms for the $X(3872)$*

[BHKKN]

$$D_F(E) = E - E_f - \frac{(E-E_f)^2}{(E-M_Z)^2} + \frac{i}{2} g_f^2 k$$

$$t(E) = \frac{g_f^2}{8\pi^2 \mu D_F(E)}$$

$$t(E) = \frac{1}{4\pi^2\mu} \frac{E - E_f + \frac{1}{2} g_f^2 \gamma_V}{(E-E_f)(\gamma_V + ik) + \frac{i}{2} g_f^2 \gamma_V k}$$

$$g_f^2 = \frac{2\lambda}{\beta^2}$$

$$E_f = M_Z - \frac{\lambda}{\beta}$$

$$\gamma_V = -\beta$$

$\gamma_V = 1/a_V$, a_V scattering length in pure contact-interaction theory.

For $|M_Z| \gg |E_f|$ one recovers the standard Flatté approximation

Limitation of [BHKKN] and [ABK]

- They predict only $\lambda \geq 0$

[BHKKN]	[ABK]
$\lambda = \frac{\gamma_V^2}{2} g_f^2$	$\lambda = \frac{2g^2\gamma_0^2(\gamma_1-\kappa_2)^2}{(\gamma_0+\gamma_1-2\kappa_2)^2}$

- Positive effective range r, v_3, v_5, etc, cannot be reproduced with $\lambda \geq 0$:

$$r = -\frac{\lambda}{\mu M_Z^2} < 0$$

$$v_3 = -\frac{\lambda}{8\mu^3 M_Z^4} < 0$$

- $\omega(E) \geq 0 \rightarrow \lambda \geq 0$:

$$\omega(E) = \theta(E)\frac{\lambda k/\pi}{|\lambda + (\beta - ik)(E - M_Z)|^2}$$

Constant contact term plus one s-channel bare-pole exchange picture collapses for $\lambda < 0$

$X(3872)$ Kang,Oller, EPJC77,399(2017)

PDG: $M_X = 3871.69 \pm 0.17$ MeV , $\Gamma_X < 1.2$ MeV
$D^0 \bar{D}^{*0}$ threshold: 3871.81 ± 0.07 MeV
$D^+ D^{*-}$ threshold is $\Delta = +8.1$ MeV higher

We have a scenario with only one-free parameter and having:

- A virtual-state pole $(-i\kappa\ ,\ \kappa > 0)$
- A bound-state pole $(+i\varkappa\ ,\ \varkappa > 0)$

3.I	3.II
$E_V = -0.68 \pm 0.05$ MeV	$E_V - 1.06 \pm 0.05$ MeV
$E_B = -0.50 \pm 0.04$ MeV	$E_B = -0.51 \pm 0.03$ MeV
$X = 0.061$	$X = 0.158$
$W = 0.06$	$W = 0.16$
$M_Z = 0.25 \pm 0.04$ MeV	$M_Z = 3.21 \pm 0.05$ MeV

16.- Conclusions

- A new perspective on compositeness based on the number operators
- Amenable to calculations employing QFT
- The formalism can be extended to relativistic systems
- Universal criterion for a relativistic or non-relativistic bound state to be qualified as elementary
- Generalization to resonances
- Phase-factor transformations, S-matrix transformations
- Necessary condition for a resonance to be elementary
- On-shell methods
- CDD & including bare state explicitly
- Full calculations!: LQCD, EFT+NP methods,...

Other methods to study the nature of resonances

- Study of form factors and determination of the corresponding quadratic radius Sekihara,Hyodo,Jido,PRC83,055202(2011); Albaladejo,Oller,PRD86,034003(2012)
- Pole counting rule Morgan NPA543,632(1992). Presence/absence of nearby CDD poles Kang,Oller, EPJC77,399(2017)
- Evolution of the pole positions with the increase in the number of color of QCD. E.g. for a $q\bar{q}$ $M = \mathcal{O}(N_C^0)$ and $\Gamma = \mathcal{O}(N_c^{-1})$. Pioneer works Oset,Oller,PRD60,074023(1999); Peláez,PRL92,102001(2004); Hyodo,Jido,Hosaka, PRL97,192002(2006)
- Dependence on the mass under quark mass variations. Lattice QCD. Ruiz de Elvira,Meißner,Rusetsky,Schierholz,arXiv:1706.09015
- Regge trajectories Londergan,Nebreda,Peláez,Szczepaniak,PLB729,9(2014)
- Compare predictions within specific models with experiment,e.g. spectrum, decay properties, etc

Name	$\sqrt{s_P}$ [MeV]	$X^R_{\pi\pi}$	$X^R_{\bar{K}K}$	$X^R_{\eta\eta}$	$X^R_{\eta\eta'}$	X^R
$f_0(500)$	$442^{+4}_{-4} - i246^{+7}_{-5}$	$0.40^{+0.01}_{-0.01}$	...	...	...	$0.40^{+0.01}_{-0.01}$
$f_0(980)$	$978^{+17}_{-11} - i29^{+9}_{-11}$	$0.02^{+0.01}_{-0.01}$	$0.65^{+0.10}_{-0.16}$	...	...	$0.67^{+0.11}_{-0.17}$
$f_0(1710)$	$1690^{+20}_{-20} - i110^{+20}_{-20}$	$0.00^{+0.00}_{-0.00}$	$0.03^{+0.02}_{-0.02}$	$0.02^{+0.03}_{-0.03}$	$0.25^{+0.16}_{-0.16}$	$0.30^{+0.17}_{-0.17}$
$\rho(770)$	$760^{+7}_{-5} - i71^{+4}_{-5}$	$0.08^{+0.01}_{-0.01}$	...	...	...	$0.08^{+0.01}_{-0.01}$
		$X^R_{K\pi}$	...	...	...	...
$K^*_0(800)$	$643^{+75}_{-30} - i303^{+25}_{-75}$	$0.94^{+0.19}_{-0.39}$	...	...	...	$0.94^{+0.19}_{-0.39}$
$K^*(892)$	$892^{+5}_{-7} - i25^{+2}_{-2}$	$0.05^{+0.01}_{-0.01}$	...	...	...	$0.05^{+0.01}_{-0.01}$
		$X^R_{\pi\eta}$	$X^R_{\bar{K}K}$	$X^R_{\pi\eta'}$	...	
$a_0(1450)$	$1459^{+70}_{-95} - i174^{+110}_{-100}$	$0.09^{+0.02}_{-0.07}$	$0.02^{+0.12}_{-0.02}$	$0.12^{+0.21}_{-0.08}$	...	$0.23^{+0.35}_{-0.17}$
		$X^R_{\rho\pi}$				
$a_1(1260)$	$1260 - i250$	0.45				0.45
Hyperon with $I=0$		$X^R_{\pi\Sigma}$	$X^R_{\bar{K}N}$	...	...	...
$\Lambda(1405)$ broad	$1388^{+9}_{-9} - i114^{+24}_{-25}$	$0.73^{+0.16}_{-0.07}$	...	...	...	$0.73^{+0.16}_{-0.07}$
$\Lambda(1405)$ narrow	$1421^{+3}_{-2} - i\,19^{+8}_{-5}$	$0.18^{+0.15}_{-0.06}$	$0.81^{+0.18}_{-0.08}$	...	...	$0.99^{+0.33}_{-0.14}$
Hyperon with $I=1$		$X^R_{\pi\Lambda}$	$X^R_{\pi\Sigma}$	$X^R_{\bar{K}N}$	...	...
	$1376^{+3}_{-3} - i\,33^{+5}_{-5}$	$0.04^{+0.01}_{-0.00}$	$0.0^{+0.0}_{-0.0}$	...	...	$0.04^{+0.01}_{-0.00}$
	$1414^{+2}_{-3} - i\,12^{+1}_{-2}$	$0.03^{+0.00}_{-0.00}$	$0.01^{+0.00}_{-0.00}$	$0.13^{+0.03}_{-0.03}$	...	$0.17^{+0.03}_{-0.03}$
		X^R_{DK}	$X^R_{D_s\eta}$	$X^R_{D_s\eta'}$		
$D^*_{s0}(2317)$	2321^{+6}_{-3}	$0.57^{+0.01}_{-0.01}$	$0.12^{+0.01}_{-0.01}$	$0.02^{+0.01}_{-0.01}$	...	$0.71^{+0.03}_{-0.03}$
		$X^R_{J/\psi f_0(500)}$	$X^R_{J/\psi f_0(980)}$	$X^R_{Z_c(3900)\pi}$	$X^R_{\omega\chi_{c0}}$	...
$Y(4260)$	$4232.8 - i36.3$	0.00	0.02	0.02	0.17	0.21
		$X^R_{\Sigma_c^+\pi^0}$	$X^R_{\Sigma_c^{++}\pi^-}$	$X^R_{\Sigma_c^0\pi^+}$	...	...
$\Lambda_c(2595)$	$2592.25 - i1.3$	$0.11^{+0.02}_{-0.02}$	...	...	...	$0.11^{+0.02}_{-0.02}$

Table: Z.H.Guo, Oller, PRD93,096001(2016)