Introduction/Comment to the Khuri-Treiman Formalism ($\eta \to 3\pi$)

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Outline

Introduction

I have written a review for PPNP on coupled-channel dynamics

Contents

I am going to review on the Khuri-Treiman formalism **(KT)** for $\eta \to 3\pi$ *A comment is also made*

- Final-, Initial-state interactions
- Two-body unitarity
- Three-body scattering
- Lehmann ellipse
- Overlap between right- and left-hand cuts
- Anomalous threshold

KT, Khuri, Treiman, PR119(1960), S-wave $\pi\pi$ final-state interactions (FSI) in $K \to 3\pi$

Three-particle scattering from two-particle scattering

Further studied and advances in the 60's:

Gribov, V.V.Anisovich,Anselm Sov.Phys.JETP 15(1962)
Bonnevay, Nuovo Cimento 30(1963)
Bronzan,Kacser, PR132(1963)
Kacser, PR132(1963)
Bronzan, PR134(1964)
Aitchison, PR133(1963), PR137(1965)
R.Pasquier,J.Y.Pasquier, PR170(1968), PR177(1969)
Neveu, Scherk, Ann.Phys.57(1970)

The KT approach was "brought to life" in the half 90's:

S- and *P*-waves two-body FSI

η decays + Chiral Perturbation Theory (ChPT) + new experiments:

A.V.Anisovich, Phys.Atom.Nucl.58(1995)
Anisovich, Leutwyler, PLB375(1996)
Kambor,Wiesendanger,Wyler,NPB465(1996)

Very-recent studies:

P.Guo *et al.* [JPAC], PLB771(2017)
Gasser, Rusetsky, EPJC78(2018)
Colangelo *et al.*, EPJC78(2018)

Extension of KT to coupled channels

Albaladejo,Moussallam, EPJC77(2017) (AM)

$\pi\pi$ ($I = 0, 1, 2$), $\eta\pi$ ($I = 1$), $K\bar{K}$ ($I = 0, 1$)

$f_0(500)$, $\rho(770)$, $f_0(980)$: Final-State Interactions (FSI)

$a_0(980)$: Initial-State Interactions (HSI)

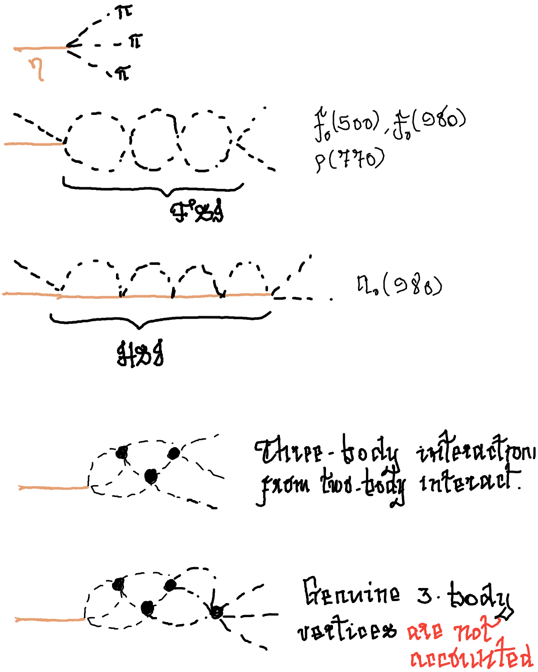
Application of KT to other processes:

Niecknig,Kubis, *Consistent Dalitz plot analysis of Cabibbo-favored $D^+ \to \bar{K}\pi\pi^+$ decays*, PLB780(2018)

Isken,Kubis,Schneider,Stoffer, *Dispersion relations for $\eta' \to \eta\pi\pi$*, EPJC77(2017)

Albaladejo *et al.* [JPAC Collaboration], *Khuri-Treiman equations for $\pi\pi$ scattering*, EPJC78(2018)

ETC


π
π
π
η
f0(500), f0(980)
ρ(770)
a0(980)
Three-body interaction from two-body interact.
Genuine 3-body vertices are not accounted

$\eta \to 3\pi$ is an interesting history

$$\Gamma_c = \Gamma(\eta \to \pi^+\pi^-\pi^0) \ ; \ \Gamma_n = \Gamma(\eta \to 3\pi^0) \ ; \ r = \Gamma_n/\Gamma_c$$

Th. Calculation	Experiment
Current Algebra	Strong disagreement
Weinberg(1975),Raby(1976)	$\Gamma_c = 200 \pm 30$ eV
$\Gamma_c \simeq 65$ eV, $r = 1.5$	$r = 1.34 \pm 0.05$
Unitarization Roiesnel,Truong,NPB(1981)	Better agreement
$\Gamma_c = 200$ eV, $r = 1.6$	
NLO ChPT Gasser,Leutwyler,NPB(1985)	
$\Gamma_c = 160 \pm 50$ eV, $r = 1.43$	Changed
	PDG: $\Gamma_c = 300 \pm 4$ eV
$r = 1.44 \pm 0.04$ Colangelo *et al.*,EPJC(2018)	$r = 1.426 \pm 0.026$(Fit)
$r = 1.45 \pm 0.02$ AM	$r = 1.48 \pm 0.05$(Ave)
$Q = 21.7 \pm 0.2$	

Parameters of the Dalitz plot for $\eta \to 3\pi^0$

	Th. Calculation	PDG
NLO ChPT, Gasser,Leutwyler,NPB(1985)		
	$\alpha = +0.0142$	$\alpha = -0.0318(15)$(Ave)
Colangelo *et al.*(2018)	$\alpha = -0.0307(17)$	
AM(2017)	$\alpha = -0.0337(12)$	

Tension between the two KT calculations

NNLO ChPT Bijnens,Ghorbani,JHEP(2007) is not predictive
too many LECs

Final(Initial)-State Interactions FSI(HSI) are essential

Precise experimental results on Dalitz plot data

KLOE-2 (2016),BESIII(2015),WASA,TAPS,A2 Collaborations

$\eta - \eta'$ decays from unitarized $U(3)$ ChPT

Borasoy,Nissler,NPA716(2003),EPJA26(2005)

2. General Remarks

- $\eta \to 3\pi$ violates isospin: G-parity of η is $+1$, π is -1
- QCD isospin-breaking operator $I = 1,\ t_3 = 0$

$$\frac{m_u - m_d}{2}(u^\dagger d^\dagger)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} u \\ d \end{pmatrix} = \frac{m_u - m_d}{2}(u^\dagger u - d^\dagger d)$$

- $\mathscr{O}(e^2)$ QED contributions, $I = 1,\ t_3 = 0$

$$\hat{q}^2 e^2 = e^2\begin{pmatrix} \frac{4}{9} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} = \frac{5e^2}{18}\mathbb{I} + \frac{e^2}{6}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

QED corrections are further suppressed $\mathscr{O}(e^2 m_{u,d})$

They vanish in the $SU(2)$ chiral limit Sutherland,PL23(1966)

Terms of order $e^2 m_u$ and $e^2 m_d$ are calculated in
Ditsche,Kubis,Meißner, EPJC60(2009)

$\mathcal{O}(e^2(m_u - m_d))$ is only suppression by a factor $\sim (m_u - m_d)/(m_u + m_d) \approx 1/3$

The QED corrections remain very small ($\lesssim$ %) through the $\eta \to 3\pi$ physical region

Pure QCD: amplitudes are proportional to Q^{-2}

Long standing aim: Accurate determination of

$$Q^2 = \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2} \qquad \hat{m} = \frac{m_u + m_d}{2}$$

Object of desire: $T(\eta \to \pi^+\pi^-\pi^0) = A(s,t,u)$

Mandelstam variables: $\eta(p_0) \to \pi^+(p_1)\pi^-(p_2)\pi^0(p_3)$

$$
\begin{aligned}
s =& (p_0 - p_3)^2 = (p_1 + p_2)^2 \\
t =& (p_0 - p_1)^2 = (p_2 + p_3)^2 \\
u =& (p_0 - p_2)^2 = (p_1 + p_3)^2
\end{aligned}
$$

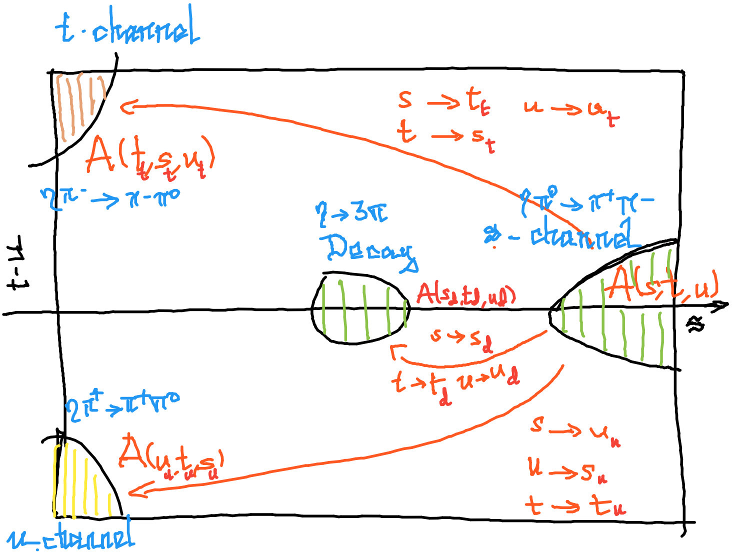
$$s + t + u = 3s_0 \equiv m_\eta + 3m_\pi^2$$

Crossing Symmetry

$$
\begin{aligned}
T(\eta \to \pi^+\pi^-\pi^0) &= A(s,t,u) \\
T(\eta\pi^0 \to \pi^+\pi^-) &= A(s,t,u) \\
T(\eta\pi^- \to \pi^-\pi^0) &= A(t,s,u) \\
T(\eta\pi^+ \to \pi^+\pi^0) &= A(u,t,s)
\end{aligned}
$$

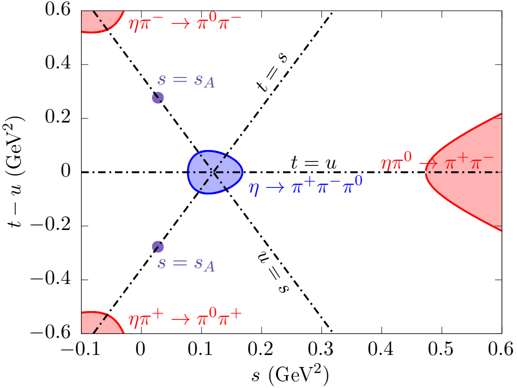
$$
\begin{aligned}
T(\eta \to \pi^0\pi^0\pi^0) &= B(s,t,u) \\
T(\eta\pi^0 \to \pi^0\pi^0) &= B(s,t,u)
\end{aligned}
$$

Bose-Einstein symmetry: $B(s,t,u) = B(t,s,u) = B(u,t,s)$



t-channel
s → t_t u → u_t
t → s_t
A(t_t, s_t, u_t)
ηπ⁻ → π⁻π⁰
η → 3π
Decay
ηπ⁰ → π⁺π⁻
s-channel
A(s_d, t_d, u_d)
A(s, t, u)
s → s_d
t → t_d u → u_d
s
t-u
ηπ⁺ → π⁺π⁰
A(u_u, t_u, s_u)
s → u_u
u → s_u
t → t_u
u-channel

Physical regions: Mandelstam plane $(s, t-u)$



$$t,\ u(s,\cos\theta) = \frac{1}{2}\left(m_\eta^2 + 3m_\pi^2 - s \pm \cos\theta\sqrt{\lambda(s)\sigma(s)}\right)$$

$$t - u = \cos\theta\sqrt{\lambda(s)\sigma(s)}$$

$$\lambda(s) = s^2 + m_\eta^4 + m_\pi^4 - 2s(m_\pi^2 + m_\eta^2) - 2m_\pi^2 m_\eta^2$$

$$\sigma(s) = 1 - \frac{4m_\pi^2}{s}$$

- s-channel: $m_{\eta\pi} \equiv m_\eta + m_\pi$

$$\partial D_{s-\text{channel}} = \left\{ (s, \pm\sqrt{\lambda(s)\sigma(s)}) \ , \ s \geq m_{\eta\pi}^2 \right\}$$

- Decay-channel

$$\partial D_{\text{decay}-\text{channel}} = \left\{ (s, \pm\sqrt{\lambda(s)\sigma(s)}) \ , \ (m_\eta - m_\pi)^2 \geq s \geq 4m_\pi^2 \right\}$$

- u-channel

$$\partial D_{u-\text{channel}} = \left\{ (\frac{1}{2}\left(m_\eta^2 + 3m_\pi^2 - u \mp \sqrt{\lambda(u)\sigma(u)} \right) , \frac{1}{2}\left(m_\eta^2 + 3m_\pi^2 - 3u \pm \sqrt{\lambda(u)\sigma(u)} \right)) \ , \ u \geq m_{\eta\pi}^2 \right\}$$

$$t, \ s(u, \cos\theta_u) = \frac{1}{2}\left(m_\eta^2 + 3m_\pi^2 - u \pm \cos\theta_u \sqrt{\lambda(u)\sigma(u)} \right) \ ,$$

- t-channel

Mirror image across the s axis of the u-channel physical region

3. Isospin decomposition of $\eta\pi \to \pi\pi$

Wigner-Eckart theorem is applied: the isospin-breaking transition operator has $I = 1$, $t_3 = 0$

$$C(11I'|m_1 0 m_1)\langle I'||T_V||1\rangle$$

$$A(\eta\pi^0 \to \pi^+\pi^-) = A(s,t,u) = -\frac{1}{3}M^2(s,t,u) + \frac{1}{3}M^0(s,t,u)$$
$$A(\eta\pi^+ \to \pi^+\pi^0) = A(u,t,s) = +\frac{1}{2}M^2(s,t,u) + \frac{1}{2}M^1(s,t,u)$$
$$A(\eta\pi^- \to \pi^0\pi^-) = A(t,s,u) = +\frac{1}{2}M^2(s,t,u) - \frac{1}{2}M^1(s,t,u)$$

$$M^0(s,t,u) = 3A(s,t,u) + A(u,t,s) + A(t,s,u)$$
$$M^1(s,t,u) = A(u,t,s) - A(t,s,u)$$
$$M^2(s,t,u) = A(u,t,s) + A(t,s,u)$$

4. Partial-wave expansion

Isospin decomposition of $\eta\pi^0 \to \pi^0\pi^0$

$$
\begin{aligned}
B(s,t,u) &= \frac{2}{3}M^2(s,t,u) + \frac{1}{3}M^0(s,t,u) \\
&= A(s,t,u) + A(t,s,u) + A(u,t,s)
\end{aligned}
$$

Partial-wave amplitudes (PWA) $M^{(IJ)}(s)$

$$
M^{(IJ)}(s) = \frac{1}{2}\int_{-1}^{+1} d\cos\theta P_J(\cos\theta) M^I(s,t,u)
$$

$$
t,\ u(s,\cos\theta)
$$

$$
M^I(s,t,u) = \sum_{J=0}^{\infty}(2J+1)P_J(\cos\theta)M^{(IJ)}(s)
$$

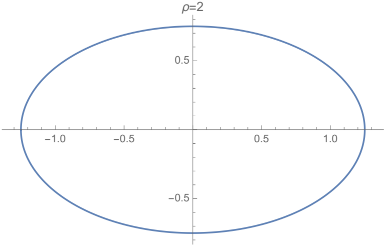
$$
|\mathbf{p},\sigma_1\sigma_2,\alpha_1\alpha_2\rangle = \sqrt{4\pi}\sum_{\ell,m} Y_\ell^m(\hat{\mathbf{p}})^*|\ell m,\sigma_1\sigma_2,\alpha_1\alpha_2\rangle
$$

Convergence of the Legendre expansion, $\cos\theta \in \mathbb{C}$

Lehman ellipse Lehmann, Nuovo CimentoX(1958)

The region of convergence of a Legendre polynomial expansion is the largest ellipse which can be drawn with ± 1 *as foci so that the function represented is analytic within the ellipse*

$$\mathscr{E}_\rho = \left\{ z = \frac{1}{2}(u + u^{-1}) \; , \; u = \rho e^{i\theta} \; , \; \rho > 1 \; , \theta \in [-\pi, \pi] \right\}$$



If $\rho = 1$ the ellipse collapses to a line

Real variable: $\cos\theta \in [-1,1]$

Let $f(z)$ be a finite piecewise continuous function, $z \in [-1,1]$. It can be expanded in series of Legendre polynomials such that

$$S_n(z) = \sum_{k=0}^{n} c_k P_k(z)$$
$$c_k = (\ell + \frac{1}{2}) \int_{-1}^{1} f(z) P_k(z) dz$$

• It converges in the L^2 sense (minimal mean square error).

$$\lim_{n\to\infty} \int_{-1}^{1} dz |f(z) - S_n(z)|^2 = 0$$

Not necessarily everywhere

• Gibbs phenomenon in the discontinuity points Sommerfeld, *Partial Differential Equations in Physics*, AP, 1949

In the singular points it is a poor numerical expansion

Take a t-channel $\pi\pi$ branch-point singularity

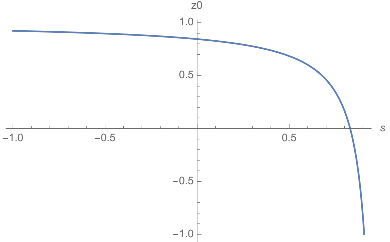
Simpler kinematics: $K\bar{K} \to K\bar{K}$ $\quad t = -2\mathbf{p}^2(1-\cos\theta)$

$$\sqrt{\frac{t(s,z)}{4} - m_\pi^2} = \left\{ \left(\frac{s}{4} - m_K^2\right)(z - z_0) \right\}^{\frac{1}{2}}$$

$$z_0(s) = 1 + \frac{2m_\pi^2}{s/4 - m_K^2}$$

LHC

$|z_0(s)| \leq 1$: $\quad s \in]-\infty, 4(m_K^2 - m_\pi^2)]$



LHC and RHC overlap each other

Left-Hand Cut (LHC)
Right-Hand Cut (RHC) $s > 4m_\pi^2$

$$z_0(-\infty) = 1 \,,\; z_0(4(m_K^2 - 2m_\pi^2)) = 0 \,,\; z_0(4(m_K^2 - m_\pi^2)) = -1$$

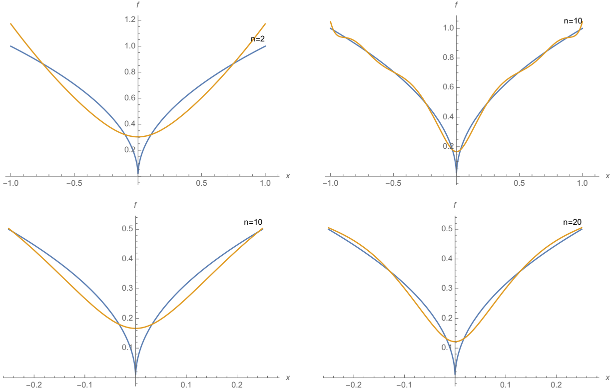
$$f(z) = \sqrt{(z - z_0)}$$

$$f'(z_0) = \infty$$

$$S_n'(z_0) = \sum_{k=0}^{n} c_k P_k'(z_0)$$

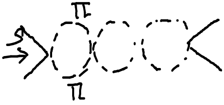
Numerical enhancements affecting error estimates

$z_0 = 0$

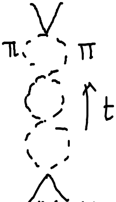


s-channel $K^+K^- \to K^+K^-$ t-channel $K^+K^- \to K^+K^-$

Listillo



I add it
in t-channel
crossing



$\sum_{K}^{n} P_K(s,t)\, M_K(s)$ $\sum P_K(t,s)\, M_K(t)$

But $s \geq 4m_K^2$ implies $\boxed{t \leq 0}$ → No partial-wave amplitude expansion

~~Lehmann-ellipse~~

No analytical extrapolation in $\cos\theta_t$

This can be improved by using the KT formalism

5. Lehmann ellipse in $\pi\pi \to \pi\pi$ Scattering

Violation of the Lehmann ellipse

A comment on Albaladejo *et al.* [JPAC], EPJC78(2018)

Equal mass case $m_K^2 \to m_\pi^2$

From the t-, u-channel dynamics, there is a tension between

- $\cos\theta_t$ value to reproduce s

$$s(t,\cos\theta_t) = -2(t/4 - m_\pi^2)(1-\cos\theta_t)$$

$$z \equiv \cos\theta_t = 1 + \frac{s/2}{t/4 - m_\pi^2}$$

- Lehmann ellipse: limited by the s-channel branch-point at threshold ($s = 4m_\pi^2$)

$$z_0(t) = 1 + \frac{2m_\pi^2}{t/4 - m_\pi^2}$$

Given

$t = -2(s/4 - m_\pi^2)(1-\cos\theta)$, $x = \cos\theta$

$$z = 1 - \frac{s}{(s/4 - m_\pi^2)(1-x) + 2m_\pi^2}$$
$$z_0 = 1 - \frac{4m_\pi^2}{(s/4 - m_\pi^2)(1-x) + 2m_\pi^2}$$

If $z_0^2 - z^2 < 0 \rightarrow$ No Analytical extrapolation in $\cos\theta_t$ can reach z
It is outside Lehmann ellipse ($|z_0| > 1$) or the latter does not exist ($|z_0| \leq 1$)

$$z_0^2 - z^2 = -8(1+x)\frac{(s-s_1)(s-s_2)}{\left[s(1-x)+4m_\pi^2(1+x)\right]^2}$$

$$s_2 = -4m_\pi^2\frac{1-x}{1+x} < s_1 = 4m_\pi^2$$

IT IS NEGATIVE FOR $s > s_1$

THE PWA EXPANSION IN THE t- AND u-CHANNELS CANNOT BE APPLIED IN THE s-CHANNEL , $s > 4m_\pi^2$

For $x = 1$

$$z = 1 - \frac{s}{2m_\pi^2} \to z < -1 \text{ for } s > 4m_\pi^2$$

$$z_0 = -1$$

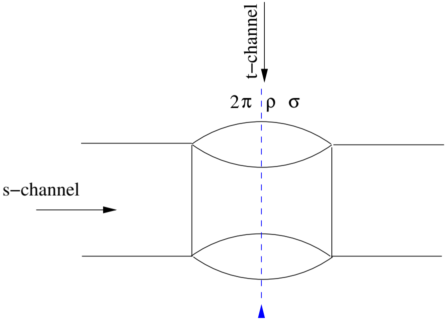
For $s \gg m_\pi^2$ and $x \neq 1$

$$z \to 1 - \frac{4}{1-x} < -3$$
$$z_0 \to 1 - \frac{16 m_\pi^2}{s(1-x)} \to 1 - \epsilon$$

For studying the dynamics along the s-channel Albaladejo *et al* [JPAC] EPJC78(2018) proposes

$$A(s,t,u) = \sum_{\ell=0}^{\ell_{\max}} (2\ell+1) P_\ell(\cos\theta_s) p_s^{2\ell} a_\ell(s) + \sum_{\ell=0}^{\ell_{\max}} (2\ell+1) P_\ell(\cos\theta_t) p_t^{2\ell} a_\ell(t) + \sum_{\ell=0}^{\ell_{\max}} (2\ell+1) P_\ell(\cos\theta_u) p_u^{2\ell} a_\ell(u)$$

Box and more involved contributions



ITS LEGENDRE EXPANSION IS MEANINGLESS $s \geq 16 m_\pi^2$

For $x = 1$

$$z = 1 - \frac{s}{2m_\pi^2}$$

$$z < -7 \quad \text{for } s > 16 m_\pi^2$$

$$z_0 = -7$$

For $s \gg m_\pi^2$ and $x \neq 1$

$$z \to 1 - \frac{4}{1-x} < -3$$

$$z_0 \to 1 - \frac{64 m_\pi^2}{s(1-x)} \to 1 - \epsilon$$

This is also used for large s within DRs

THIS IS **not** THE SAME AS NEGLECTING THE 4π, ETC, INTERMEDIATE CONTRIBUTIONS IN DISPERSION RELATIONS

Here the KT approach drives to an undermined framework for DRs & e.g. Roy Equations

Separation of the RHC from LHC

Complex mass Mandelstam,PRL4(1960)

$$m_K^2 \to m_K^2 + i\epsilon$$

$$t = -2(\frac{s}{4} - m_K^2 - i\epsilon)(1-\cos\theta) \to t + 2i\epsilon(1-\cos\theta)$$

It acquires a positive imaginary part [physical Riemann sheet in the t-channel] $\Im t > 0$ for $t > 4m_\pi^2$ and this is the adequate way to approach a unitarity cut (perpendicularly from above)

$\Im t = 0 \leftrightarrow t = 0$ LHC implies $t \geq 4m_\pi^2$

$$s = 4m_K^2 - \frac{8m_\pi^2}{(1-\cos\theta)} + i\epsilon$$

One can give meaning by analytical continuation to integrals along the LHC

$$\int_{4m_\pi^2}^{\infty} dt' \int_{-1}^{+1} d\cos\theta \frac{1}{t(s,\cos\theta) - t'} \quad , \quad t' \geq 4m_\pi^2$$

6. Elastic unitarity

Unitarity

$$T - T^\dagger = iT^\dagger T$$

Taking this relation between states of well-define I and J (PWA) gives **unitarity in partial waves**

$$2i\Im M^{(IJ)}(s+i\varepsilon) = i\frac{\sigma(s)^{1/2}}{8\pi}M^{(IJ)}(s+i\varepsilon)T^{(IJ)}(s+i\varepsilon)^* \\ = 2ie^{-i\delta^{(IJ)}(s)}\sin\delta^{(IJ)}(s)M^{(IJ)}(s+i\varepsilon)$$

$\delta^{(IJ)}$ are the $\pi\pi$ phase shifts in PWA (IJ)
This is not true for $m_\eta > 3m_\pi$

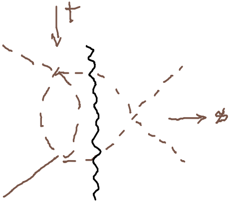
One has to proceed by analytical continuation in the external mass $m_\eta^2 + i\epsilon$ Mandelstam(1959)

To separate the LHC and RHC

For η decays the situation is more involved $4m_\pi^2 < s < m_\eta^2 - 5m_\pi^2$

Overlapping of s-channel and t-,u-channels

Anomalous Threshold



$$\Delta M^{(IJ)}(s) = M^{(IJ)}(s+i\varepsilon) - M^{(IJ)}(s-i\varepsilon) \quad , \quad s \geq 4m_\pi^2$$

$$\Delta M^{(IJ)}(s) = 2ie^{-i\delta^{(IJ)}(s)} \sin\delta^{(IJ)}(s) M^{(IJ)}(s+i\varepsilon) \ .$$

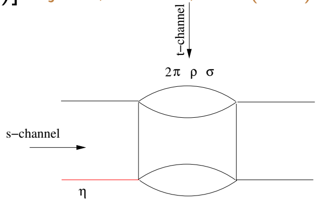
7. Kurhi-Treiman formalism

$\eta \to \pi^+\pi^-\pi^0$, $s \leftrightarrow t$ symmetry (Charge conjugation)

$$A(s,t,u) = M_0(s) - \frac{2}{3}M_2(s) + M_2(t) + M_2(u) + (s-u)M_1(t) + (s-t)M_1(u)$$

- s-channel : $J = 0$, $I = 0, 2$
- t-,u-channels : $J = 0$, $I = 2$; $J = 1$, $I = 1$
- $\cos\theta_t \propto s - u$, $\cos\theta_u \propto s - t$
- This is valid in ChPT up to $\mathcal{O}(p^6)$. [D-wave intermediate states, $\mathcal{O}(p^8)$] Bijnens,Ghorbani,JHEP(2007)

$\mathcal{O}(p^8)$ in ChPT



$$A(\eta\pi^0 \to \pi^+\pi^-) = A(s,t,u) = \frac{1}{3}M^0(s,t,u) - \frac{1}{3}M^2(s,t,u)$$
$$A(\eta\pi^+ \to \pi^+\pi^0) = A(u,t,s) = +\frac{1}{2}M^2(s,t,u) + \frac{1}{2}M^1(s,t,u)$$
$$A(\eta\pi^- \to \pi^0\pi^-) = A(t,s,u) = +\frac{1}{2}M^2(s,t,u) - \frac{1}{2}M^1(s,t,u)$$

Sum of the s-, t-,u-channel amplitudes

$$\begin{aligned}&\frac{1}{3}\hat{M}_0(s) - \frac{1}{3}\hat{M}_2(s)\\ &+\frac{1}{2}\hat{M}_2(t) + \frac{1}{2}(s-u)\hat{M}_1(t)\\ &+\frac{1}{2}\hat{M}_2(u) - \frac{1}{2}(t-s)\hat{M}_1(u)\end{aligned}$$

$\eta \to 3\pi^0$

$$B(s,t,u) = \frac{1}{3}\left[M_n(s) + M_n(t) + M_n(u)\right]$$
$$M_n(s) = M_0(s) + 4M_2(s)\ .$$

$$M^0(s,t,u) = 3M_0(s) \quad \textbf{RHC}$$
$$+ M_0(t) + M_0(u) + \frac{10}{3}\left[M_2(t) + M_2(u)\right] \quad \textbf{crossed cuts}$$
$$+ 2(s-u)M_1(t) + 2(s-t)M_1(u) \quad \textbf{crossed cuts}$$
$$M^1(s,t,u) = 2(u-t)M_1(s)$$
$$+ (u-s)M_1(t) - (t-s)M_1(u) + M_0(u) - M_0(t) + \frac{5}{3}\left[M_2(t) - M_2(u)\right]$$
$$M^2(s,t,u) = 2M_2(s)$$
$$+ \frac{1}{3}\left[M_2(t) + M_2(u)\right] + M_0(t) + M_0(u) - (s-u)M_1(t) - (s-t)M_1(u)$$

To isolate the s-dependence one projects in PWA

$$\langle z^n M_I \rangle(s) = \frac{1}{2}\int_{-1}^{+1} dz z^n M_I(t(s,z))$$

$$M^{00}(s) \equiv 3\big[M_0(s) + \hat{M}_0(s)\big]$$
$$M^{11}(s) \equiv -\frac{2}{3}\kappa\big[M_1(s) + \hat{M}_1(s)\big]$$
$$M^{20}(s) \equiv 2\big[M_2(s) + \hat{M}_2\big]$$

$$\hat{M}_0(s) = \frac{2}{3}\langle M_0 \rangle + \frac{20}{9}\langle M_2 \rangle + 2(s-s_0)\langle M_1 \rangle \frac{2}{3}\kappa \langle M_1 \rangle$$
$$\kappa(s) = \sqrt{\lambda(s)\sigma(s)}$$

Unitarity relations

$$\Delta M_I(s) = 2ie^{-i\delta^{(IJ)}}\sin\delta^{(IJ)}\big[M_I(s) + \hat{M}_I(s)\big]$$

DRs for the $M_I(s)$

$$M_0(s) = \widetilde{\alpha}_0 + \widetilde{\beta}_0 s + \frac{s^2}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\Delta M_0(s')}{(s')^2(s'-s)} ~,$$

$$M_1(s) = \frac{s}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\Delta M_1(s')}{(s')(s'-s)} ~,$$

$$M_2(s) = \frac{s^2}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\Delta M_2(s')}{(s')^2(s'-s)} ~.$$

$A(s,t,u)$ grows at most linearly in all directions $s,t,u \to \infty$

$M_{0,2}(s) \xrightarrow[s\to\infty]{} s,~ M_1(s) \xrightarrow[s\to\infty]{} \text{const.}$ Anisovich,Leutwyler(1996)

Reshuffling invariance:

$$M_1(s) \to M_1(s) + a_1 ~,$$
$$M_2(s) \to M_2(s) + a_2 + b_2 s$$
$$M_0(s) \to M_0(s) + a_0 + b_0 s$$
$$a_0 = -\frac{4}{3}a_2 + 3s_0(a_1 - b_1)$$
$$b_0 = -3a_1 + \frac{5}{3}b_2$$

$$M_1(0) = 0 ~,$$
$$M_2(0) = 0 ~,$$
$$M_2'(0) = 0 ~,$$

Rewrite the unitarity relation

$$M_I(s+i\varepsilon)=M_I(s-i\varepsilon)e^{2i\delta^{(IJ)}}+2i\hat{M}_I(s)e^{i\delta^{IJ}}\sin\delta^{(IJ)}~,~~s\geq 4m_\pi^2$$

Divide by the Omnés function $\Omega^{(IJ)}(s)$

$$\frac{M_I(s+i\varepsilon)}{\Omega^{(IJ)}(s+i\varepsilon)}=\frac{M_I(s-i\varepsilon)}{\Omega^{(IJ)}(s-i\varepsilon)}+2i\frac{\hat{M}_I(s)\sin\delta^{(IJ)}}{|\Omega^{(IJ)}(s)|}~,~~s\geq 4m_\pi^2$$

$$\Omega^{(JI)}(s)=\exp\left[\frac{s}{\pi}\int_{4m_\pi^2}^{\infty}ds'\frac{\delta^{(IJ)}(s')}{(s'-s)s'}\right]$$

$$M_I(s)=\Omega^{(IJ)}(s)\Big[P_I^{(m)}(s)+\frac{s^n}{\pi}\int_{4m_\pi^2}^{\infty}ds'\frac{\hat{M}_I(s')\sin\delta^{(IJ)}}{|\Omega^{(IJ)}(s')|(s')^n(s'-s)}\Big]$$

$m\geq n-1$

How to fix $P_I^{(m)}(s)$?

Matching with NLO ChPT

$M_{0,2}(s)$ up to $\mathcal{O}(s^2) \to P_{0,2}(s) = \alpha_I + \beta_I s + \gamma_I s^2$, 6 param.

$M_1(s)$ up to $\mathcal{O}(s) \to P_1(s) = \alpha_I + \beta_I s$, 2 param.

$8 - 3 = 5$ **parameters in total**

$\mathcal{O}(p^4)$ $A(s,t,u)$ by expanding $M_I(s)$ up to $\mathcal{O}(s^{2+I(I-2)})$

$u = 3s_0 - s - t$

s, t, s^2, t^2, st and independent term

Only 4 equations

Asymptotic behavior of $\Omega^{(IJ)}(s)$

$$\lim_{s\to\infty} \Omega^{(IJ)}(s) \to s^{-\delta^{(IJ)}(\infty)/\pi}$$

$I = 2$, $J = 0$: Exotic channel

$\delta^{(20)}(\infty) = 0 \to \Omega^{(20)}(s) \to s^0$

$P_2^{(1)}(s) = 0 \to$ **4 parameters**

$I = J = 0, 1$; $\Omega^{(IJ)}(s) \to s^{-1}$

It is enough $\delta^{(IJ)}(\infty) \to \pi$

Omnès solution

$$F(s) = \frac{P^{(p)}(s)}{Q^{(q)}(s)} \Omega^{(IJ)}(s)$$

$P^{(p)}(s)$ and $Q^{(q)}(s)$ are polynomials of degree p and q

Asymptotic behavior $s \to \infty$

$$F(s) \xrightarrow[s\to\infty]{} s^{p-q-\delta^{(IJ)}(\infty)/\pi}$$

Changes in the parameters→changes in $\delta^{(IJ)}(\infty)$

Stability of the results→ $p - q - \varphi(\infty)/\pi = \text{fixed}$

Introduced in Oller,*A Brief Introduction to Dispersion Relations, with modern applications*, SpringerBriefs in Physics, Springer, 2019

Controversy: The scalar radius, scalar form factor of the π

Oller,Roca, PLB651(2007); Yndduráin, PLB612,578(2004,5); Leutwyler *et al*, NPB343(1990),603(2001),PLB125(1983)

KT Integral Equations

$$M_0(s) = \Omega^{(00)}(s)\Big[\alpha_0 + \beta_0 s + \gamma_0 s^2 + \frac{s^2}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\hat{M}_0(s')\sin\delta^{(00)}(s')}{|\Omega^{(00)}(s')|(s')^2(s'-s)}\Big]$$

$$M_1(s) = \Omega^{(11)}(s)\Big[\beta_1 s + \frac{s}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\hat{M}_1(s')\sin\delta^{(11)}(s')}{|\Omega^{(11)}(s')|(s')(s'-s)}\Big]$$

$$M_2(s) = \Omega^{(20)}(s)\Big[\frac{s^2}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\hat{M}_2(s')\sin\delta^{(20)}(s')}{|\Omega^{(20)}(s')|(s')^2(s'-s)}\Big]$$

$\bar{M}_I(s)$ are calculated at NLO ChPT

$$\alpha_0 = \bar{M}_0(0) + \frac{4}{3}\bar{M}_2(0) + 3s_0\Big[\bar{M}_2'(0) - \bar{M}_1(0)\Big] + 9s_0^2\bar{M}_2^r$$

$$\beta_0 = \bar{M}_0'(0) + 3\bar{M}_1(0) - \frac{5}{3}\bar{M}_2'(0) - 9s_0\bar{M}_2^r - {\Omega^{(00)}}'(0)\alpha_0$$

$$\beta_1 = \bar{M}_1'(0) - \mathscr{I}_1(0) + \bar{M}_2^r$$

$$\gamma_0 = \frac{1}{2}\bar{M}_0''(0) - \mathscr{I}_0(0) + \frac{4}{3}\bar{M}_2^r - \frac{{\Omega^{(00)}}''(0)}{2}\alpha_0 - {\Omega^{(00)}}'(0)\beta_0$$

$$\mathscr{I}_I(s) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\hat{M}_I(s')\sin\delta^{(IJ)}(s')}{|\Omega^{(IJ)}(s')|(s')^{2-2J}(s'-s)}$$

Iterative Method for solving the IEs

Input:

- $\delta^{(IJ)}(s) \to \Omega^{(IJ)}(s)$
- NLO ChPT $\to$ subtraction constants $c_0 = \{\alpha_0, \beta_0, \gamma_0, \beta_1\}$

0_{th} $\hat{M}^{(IJ)}(s)_0 = 0 \to c_0 \cup M^{(IJ)}(s)_0 \to \hat{M}^{(IJ)}(s)_1$

1_{st} $\hat{M}^{(IJ)}(s)_1 \to c_1 \cup M^{(IJ)}(s)_1 \to \hat{M}^{(IJ)}(s)_2$

2_{nd} $\hat{M}^{(IJ)}(s)_2 \to c_2 \cup M^{(IJ)}(s)_2 \to \hat{M}^{(IJ)}(s)_2$

etc

AM $\lesssim 10$ iterations; numerical uncertainty $\sim 10^{-5}$

Another strategy: Fit $\eta \to \pi^+\pi^-\pi^0$

AM(2017), Colangelo *et al.*(2018)

Take advantage of the accurate experimental determination by KLOE-2(2016), BESIII(2015), of

$$\left|\frac{A(s,t,u)}{A(s_r,t_r,t_r)}\right|^2$$

all over the Dalitz plot

$s_r = t_r = u_r = s_0$

This allows to fit 3 of the 4 free parameters

α_0 is fixed by the matching with NLO ChPT

The others are fitted

8. Coupled-channel case

Developed by Albaladejo,Moussallam, EPJC77(2017) (AM)

$I=1$ $J=0$, $\eta\pi$-$K\bar{K}$ initial-state interactions (HSI)

$I=J=0,\ 1,\ \pi\pi,\ K\bar{K}$ FSI

$I=2, J=0$ $\pi\pi$ FSI

At the end AM employes elastic $I=J=1$ FSI $(\pi\pi)$

I give here a much more compact notation

- Explicit account of the S-wave $a_0(980)$ $(I=1)$ and $f_0(980)$ $(I=0)$ resonances in HSI,FSI
- These resonances couple strongly with the $K\bar{K}$ channels
- The threshold of $\eta\pi$ is not so far $(m_\eta+m_\pi)^2$

(I,J)	HSI	FSI
(1,0)	$\eta\pi(1)$, $K\bar{K}(2)$	
(0,0)		$\pi\pi(1)$, $K\bar{K}(2)$
(2,0)		$\pi\pi(1)$
(1,1)	$\pi\eta(1)$	$\pi\pi(1)$, $K\bar{K}(2)$

Unitarity

$$\Im T_{fi}^{(J)} = \sum_h \frac{|\mathbf{q}_h|}{8\pi\sqrt{s}} T_{fh}^{(J)}(s)^* T_{hi}^{(J)}(s) \ ,$$

- Keeping first-order isospin-breaking effects
- T_S (Strong) and T_W (Isospin-Breaking)
- Mixing $a_0(980) - f_0(980)$: Different masses of $K^\pm$ and K^0

$$\Im T_{W;fi}^{(IJ)} = \sum_h \frac{|\mathbf{q}_h|}{8\pi\sqrt{s}} \left[T_{S;fh}^{(IJ)}(s)^* T_{W;hi}^{(IJ)}(s) + T_{W;fh}^{(IJ)}(s)^* T_{S;hi}^{(1J)}(s) \right] + \delta_{I0} T_{S;f2}^{(00)}(s)^* \Sigma_K|_{22} T_{S;2i}^{(10)}$$

$$\Sigma_K = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\sigma_{K^+}^{1/2}(s)-\sigma_{K^0}^{1/2}(s)}{16\pi\sqrt{s}} \end{pmatrix} ~,~ \sigma_P(s) = s/4 - m_P^2 ~.$$

Separation between RHC–LHC

$$m_\eta^2 + i\epsilon \qquad m_K^2 + i\epsilon$$

$$T^{IJ}_{W;fi}(s) = M^{(IJ)}_{fi}(s) + \hat{M}^{(IJ)}_{fi}(s)$$

$$\Delta M^{(IJ)}_{fi} = \sum_h \frac{|\mathbf{q}_h|}{8\pi\sqrt{s}}\Big(T^{(IJ)}_{S;fh}(s)^* \left[M^{(IJ)}_{hi}(s+i\varepsilon) + \hat{M}^{(IJ)}_{hi}(s)\right]$$
$$+ \left[M^{(IJ)}_{fh}(s-i\varepsilon) + \hat{M}^{(IJ)}_{fh}(s)\right] T^{(1J)}_{S;hi}(s)\Big)$$
$$+ \delta_{I0}\, T^{(00)}_{S;f2}(s)^* \Sigma_K|_{22}\, T^{(10)}_{S;2i}$$

Isolating the RHC and LHC effects

$\kappa^J X^{(IJ)}(s) = \mathscr{D}^{(IJ)}(s) M^{(IJ)}(s)\, \mathscr{D}_0^{(1J)}(s)^T$

$\mathscr{D}_{(0)}^{(IJ)}(s)^{-1}$ ARE HOLOMORHPIC SOLUTIONS OF THE MUSKHELISHVILI-OMNÈS PROBLEM

Coupled-channel generalization of the single-channel $\Omega^{(IJ)}(s)$

$$S^{(IJ)}(s) = \mathbb{I} + 2iT_S^{(IJ)}(s)\rho(s)$$

$$S^{(IJ)}(s) = e^{2i\delta^{(IJ)}(s)}$$

$$\mathscr{D}^{(IJ)}(s)^{-1} = S^{(IJ)}(s)\mathscr{D}^{(IJ)}(s^*)^{-1}$$

$$\Omega^{(IJ)}(s) = e^{2i\delta^{(IJ)}(s)}\Omega^{(IJ)}(s)^*$$

$$\mathscr{D}^{(IJ)}(s)^* = \mathscr{D}^{(IJ)}(s)S^{(IJ)}(s)$$

$$\frac{1}{\Omega^{(IJ)}(s)^*} = \frac{1}{\Omega^{(IJ)}(s)}e^{2i\delta^{(IJ)}(s)}$$

$T(s) = D^{-1}(s)N \to N = D(s)T(s)$

Integral Equations

$$\kappa^J \Delta X^{(IJ)}(s) = -\Delta\left[\mathscr{D}^{(IJ)}(s)\hat{M}^{(IJ)}(s)\mathscr{D}_0^{(1J)}(s)^T\right] + \delta_{I0}\mathscr{D}^{(00)}(s)^* T_S^{(00)}(s)^* \Sigma_K T_S^{(10)}(s)\mathscr{D}_0^{(10)}(s)^T \ .$$

$$M^{(IJ)}(s) = \kappa^J\,\mathscr{D}^{(IJ)}(s)^{-1}\left[P^{(IJ)}(s) + s^{2-J}\mathscr{I}^{(IJ)}(s)\right]\mathscr{D}_0^{(1J)}(s)^{-1\,T}$$

$$\mathscr{I}^{(IJ)}(s) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\Delta X^{(IJ)}(s')}{(s')^{2-J}(s'-s)}$$

Polynomial	Matrix	Degree
$P^{(00)}(s)$	$(2,2)$	2nd
$P^{(11)}(s)$	$(1,1)$	1st, $\beta_1 s$
$P^{(20)}(s)$	$(1,2)$	$P_{11}^{(20)} = 0$ $P_{12}^{(20)}$ 2nd

16 subtraction constants

- $\mathscr{D}^{(11)}(s)^{-1} = \Omega^{(11)}(s)$ and $\mathscr{D}^{(20)}(s)^{-1} = \Omega^{(20)}(s)$
- $\mathscr{D}^{(00)}(s)^{-1}$, $\mathscr{D}^{(10)}(s)^{-1}$ vanish as $1/s$. AM uses
  Moussallam *et al*,EPJC14(2000);EPJC70(2010);EPJC75(2015)
- Chiral Expansion of $M^{(IJ)}/\kappa(s)^J$ in powers of s
- $A(s,t,u)$ is matched at NLO ChPT
- AM approximates $\hat{M}_{i2}^{(IJ)}(s) = \hat{M}_{2i}^{(IJ)}(s) = 0$
  $M_{i2}^{(IJ)}$, $M_{2i}^{(IJ)}$ are calculated without LHC
- $M_{i2}^{(IJ)}(s)$, $M_{2i}^{(IJ)}(s)$ are matched with LO ChPT (1st degree polynomials in s)
- $M_{11}^{(11)}(0) = M_{11}^{(20)}(0) = 0$
- $M_{11}^{(00)}(s) \xrightarrow[s\to\infty]{} \text{const.}$

The numerical procedure is analogous to the elastic KT case

- The numerical solution is done by iteration
- The subtraction constants fixed by matching with ChPT
- One can also fit the Dalitz plot data on $\eta \to \pi^+\pi^-\pi^0$

$\eta \to \pi^+\pi^-\pi^0$	$O(p^4)$	Single-ch.	Coupled-ch.	KLOE	BESIII
a	−1.328	−1.156	−1.142	−1.095(4)	−1.128(15)
b	0.429	0.200	0.172	0.145(6)	0.153(17)
d	0.090	0.095	0.097	0.081(7)	0.085(16)
f	0.017	0.109	0.122	0.141(10)	0.173(28)
g	−0.081	−0.088	−0.089	−0.044(16)	–
$\eta \to \pi^0\pi^0\pi^0$				PDG	
α	+0.0142	−0.0268	−0.0319	−0.0318(15)	
β	−0.0007	−0.0046	−0.0056	–	

Dalitz plot parameters. Table from AM(2017)

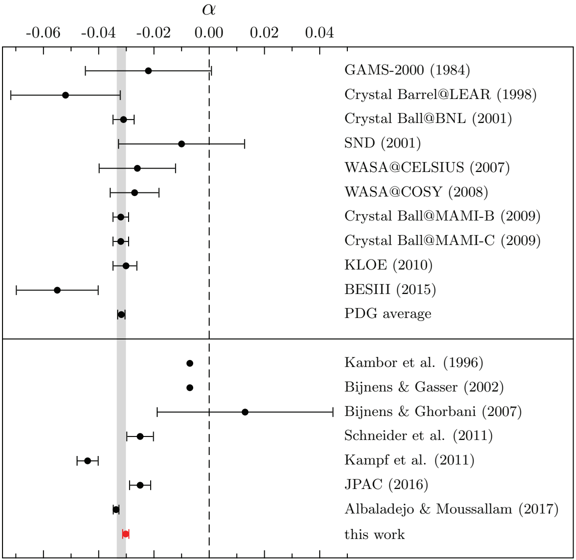


Figure: α parameter for $\eta \to 3\pi^0$. Table from Colangelo *et al.* EPJC78(2018)

9. Lehmann ellipse and separation of RHC–LHC in $\eta \to 3\pi$

Notation

$\Sigma_0^2 = 4m_\pi^2$, $\Sigma_\pm^2 = (m_\eta \pm m_\pi)^2$

$s_{1,2} = \frac{1}{2}(m_\eta^2 - m_\pi^2)$, $m_\pi(m_\eta + m_\pi)$

The branch-point singularities: $t_0 = \Sigma_0^2, \Sigma_-^2$

$$z = \frac{2t_0 - 3s_0 + s}{\kappa(s)}$$

$|z| > 1$ for $s \in [\Sigma_0^2, \Sigma_-^2]$ and $s \neq s_{1,2}$

BUT FOR $s = s_{1,2}$ THEN $z = \pm 1$

No Lehmann ellipse at $s_{1,2}$

Many Legendre polynomials would be needed around $s_{1,2}$

Given the symmetry in the Mandelstam variables $\to$ the same for t, $u = s_{1,2}$

The pathology extends to $\forall s \in [\Sigma_0^2, \Sigma_-^2]$

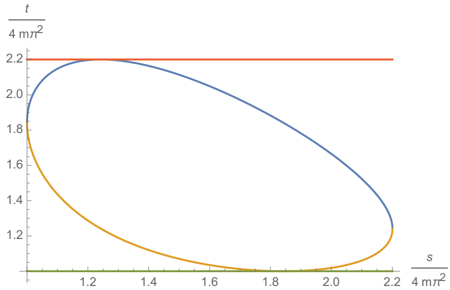


Figure: Physical region of $\eta \to 3\pi$ in the s, t plane

By taking $m_\eta^2 + i\epsilon$ the Lehmann ellipse is restored

We come back to the poor convergence of the Legendre series: E.g. $\sqrt{1-x} \to \sqrt{1+\epsilon-x}$

Separation RHC–LHC, complex mass squared

$m_\pi = 0$

$$t,\ u = (m_\eta^2 + i\epsilon - s)\frac{1 \pm z}{2}$$

$$s = \nu + i\epsilon\ ,\ \ \nu \in [0, m_\eta^2]$$

Physical m_π

$$t,\ u = \frac{1}{2}\left(m_\eta^2 + i\epsilon + 3m_\pi^2 - s \pm \cos\theta\sqrt{\lambda(s)\sigma(s)}\right)$$

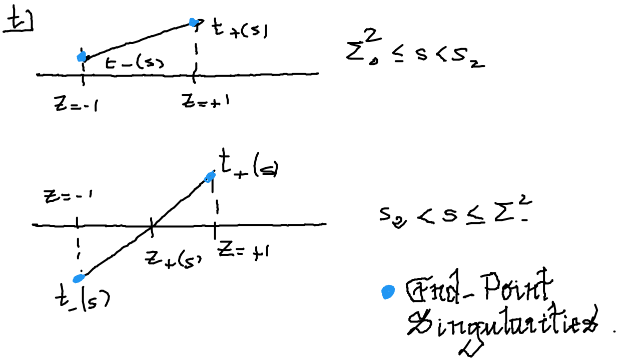
$$\lambda(s) = \lambda(s, m_\eta^2 + i\epsilon, m_\pi^2)$$

$$t,\ u = \frac{1}{2}\Bigg(m_\eta^2 + 3m_\pi^2 - s \pm z\Bigg[\sqrt{\sigma(s)(s-\Sigma_-^2)(s-\Sigma_+^2)}$$

$$+\epsilon^2\sigma(s)^{\frac{1}{2}}\frac{(s-\Sigma_-\Sigma_+)^2 + 4sm_\pi^2}{4\sqrt{(s-\Sigma_-^2)(s-\Sigma_+^2)}}\Bigg]\Bigg)$$

$$+\frac{i\epsilon}{2}\left(1 + z\frac{(m_\eta^2 - m_\pi^2 - s)\,\sigma(s)^{\frac{1}{2}}}{\sqrt{(s-\Sigma_-^2)(s-\Sigma_+^2)}}\right)\ \ ,\ \ \Sigma_0^2 < s < \Sigma_-^2$$

Imaginary part of t, u



- $s_2 = (m_\eta^2 - m_\pi^2)/2$
- Analytical extrapolation is possible, e.g $\sqrt{(s-\Sigma_-^2)(s-\Sigma_+^2)}$ (2nd RS of three-momentum)
- Avoid branch-point singularity at $s = s_2$
  Anomalous Threshold

Imaginary part of t, u

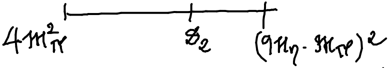
- $\Im t > 0$ for $z \geq 0$ because $m_\eta^2 - m_\pi^2 > \Sigma_-^2$
- $\Im t$ is minimum for $z = -1$
- It only vanishes for $s_2 \leq s \leq \Sigma_-^2$
- Vanishing of the imaginary part at $z_+(s)$:

$$z_+(s) = \mp \frac{\sqrt{(s-\Sigma_-^2)(s-\Sigma_-^2)}}{(m_\eta^2 - m_\pi^2 - s)\sqrt{1-4m_\pi^2/s}} \ ,$$

- $\Im t \leq 0$, $z \in [-1, z_+(s)]$ for $s_2 \leq s \leq \Sigma_-^2$ $\rightarrow$ Crossing of RHC with LHC
- Analytical extrapolation is possible, e.g $\sqrt{(s-\Sigma_-^2)(s-\Sigma_+^2)}$ (2nd RS of three-momentum)
- Avoid branch-point singularity at $s = s_2$

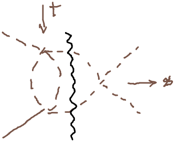
Se produce en $s = s_2 = \frac{1}{2}(m_\eta^2 - m_\pi^2)$

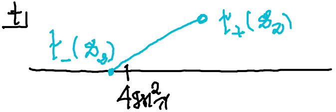
Anomalus threshold



"Aquí no hay threshold"

$t(s_2,-1) = 4m_\pi^2 - \epsilon^2 \nu \quad , \quad \nu > 0$





Calculation of PWAs

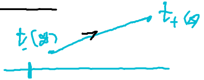
$$M_I(t) = -\frac{1}{\pi}\int_{4m_\pi^2}^{\infty} dt'\, \frac{\Delta M_I(t')}{t(s,z)-t'} + \text{subtractions}$$

$$L(t',s) = \frac{1}{2}\int_{-1}^{+1} \frac{dz}{t(s,z)-t'}~,~~t' \geq 4m_\pi^2~,$$

$t(s,z)$ is calculated with $m_\eta^2 + i\epsilon$

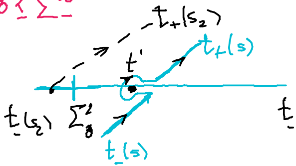
Integration Contours

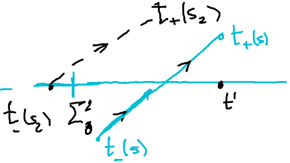
i) $\Sigma_0^2 \le s \le s_2$

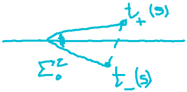


$$L(t',s) = \frac{1}{2}\int_{t_-}^{t_+} dt \frac{1}{t-t'} = \frac{1}{t_+-t_-}\left(\ln[t_+-t'] - \ln[t_--t']\right)$$

ii) $s_2 < s \le \Sigma_-^2$

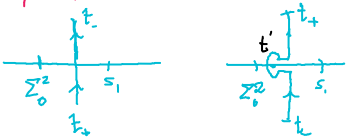






Horseshoe

iii) $\Sigma_-^2 < s \le m_\eta^2 - 5m_\pi^2$



iv) $m_\eta^2 - 5m_\pi^2 < s$ No more crossing $t' \ge \Sigma_0^2$

$$L(t',s) = \frac{1}{t_+(s) - t_-(s)}\left[\ln(t_+ - t') - \ln(t_- - t')\right]$$

In all cases

$\log z$, $\arg z \in [0, 2\pi[$

10. Summary

We have reviewed some aspects of the KT formalism

We have motivated it from considerations from the Lehmann ellipse

We have made a criticism on it

We have developed a compact notation for the coupled-channel case

- Final-, Initial-state interactions
- Two-body unitarity
- Three-body scattering
- Lehmann ellipse
- Overlap between right- and left-hand cuts
- Anomalous threshold