Scalar radius of the pion and $\gamma\gamma \to \pi\pi$ Improved treatment of the scalar Omnès function

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1. Introducion

The non-strange I=0 pion scalar form factor:

$$F(t) = \langle \pi(q') | m_u \bar{u}u + m_d \bar{d}d | \pi(q) \rangle \quad t = (q' - q)^2$$

Quadratic scalar radius of the pion, $\langle r^2 \rangle_s^\pi$

$$F(t) = F(0) \left\{ 1 + \frac{1}{6} t \langle r^2 \rangle_s^\pi + \mathcal{O}(t^2) \right\}$$

• $\langle r^2 \rangle_s^\pi$ contributes 10% to the a_0^0 and a_0^2 scattering lengths from Roy equations+CHPT to two loops (2% of precision). It is a big contribution.
Colangelo, Gasser and Leutwyler, NPB603, 125(2001). (CGL)

• It gives $\bar{\ell}_4$ which controls the departure of F_π from its value in the chiral limit
$\langle r^2 \rangle_s^\pi = \frac{3}{8\pi^2 f_\pi^2} \left\{ \bar{\ell}_4 - \frac{13}{12} \right\} \quad f_\pi = f \left\{ 1 + \frac{M_\pi^2}{16\pi^2 f_\pi^2} \bar{\ell}_4 \right\}$

- One loop CHPT, Gasser and Leutwyler PLB125,325 (1983)
  $\langle r^2 \rangle_s^\pi = 0.55 \pm 0.15 \text{fm}^2$

- Donoghue, Gasser, Leutwyler NPB343,341(1990) from the solution of the Muskhelishvili-Omnès (MO) equations, updated value in CGL.
  $\langle r^2 \rangle_s^\pi = 0.61 \pm 0.04 \text{ fm}^2$

- Mousallam EPJC14,11(200) allowed for two different $T-$matrices in MO and obtained the same values.

The scalar radius of the pion is noticeably larger than the charged one,
$\langle r_\pi^2 \rangle = 0.432 \pm 0.006 \text{ fm}^2$
This is due to the pionic cloud (strong final state interactions)

MO equations have systematic uncertainties like their dependence on values of strong amplitudes for non-physical values of energy.
Assumptions on which are the channels that matter.
Multipion states are neglcted.
Other approaches are then most welcome.

- Elastic Omnès representation and numerical-CHPT to two loops
  Gasser, Meißner NPB357,90(1991)

- Two-loop CHPT, Bijnens, Colangelo and Talavera JHEP 9805, 014 (1998)

- One loop UCHPT, Meißner, JAO, NPA679, 671 (2001); Meißner, Lahde, PRD74, 034021(2006)

- Ynduráin's approach based on the Omnès representation of $F(t)$
  Ynduráin, PLB578, 99 (2004); (E)-ibid B586, 439 (2004) (Y1)
  Ynduráin, PLB612, 245 (2005) (Y2)
  Ynduráin, arXiv: hep-ph/0510317 (Y3)
  } Yall
  $\langle r^2 \rangle_s^\pi = 0.75 \pm 0.07$ fm^2 "robust" lower bound: $\langle r^2 \rangle_s^\pi = 0.70 \pm 0.06$ fm^2

This value is much larger than 0.61 ± 0.04 fm^2 and both are incompatible (less than 3%)

Consequences in the scattering lengths of CGL

$$a_0^0 = 0.220 \pm 0.005 \text{ fm} \ , \ a_0^2 = -0.0444 \pm 0.0010 \text{ fm} \ , \text{ precision } 2\%$$

$$\delta a_0^0 = +0.027\Delta_{r^2} \quad \delta a_0^2 = -0.004\Delta_{r^2} \quad \langle r^2\rangle_s^\pi = 0.61(1+\Delta_{r^2})\ \mathrm{fm}^2$$

$\Delta_{r^2} = +0.23$ between Yall and CGL

$\delta a_0^0 = +0.006$ and $\delta a_0^2 = -0.001$
(in I=2 S-wave final state interactions are much smaller, exotic channel)
The shift in the central value is one sigma of CGL

$$a_0^0 = 0.220 \pm 0.005\ \mathrm{fm}\ ,\ a_0^2 = -0.0444 \pm 0.0010\ \mathrm{fm}\ ,\ \text{precision } 2\%$$

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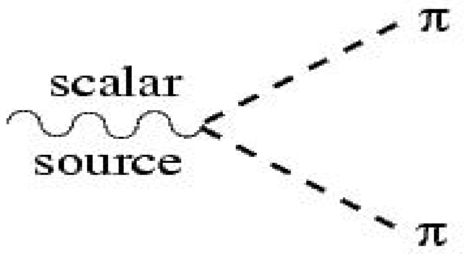
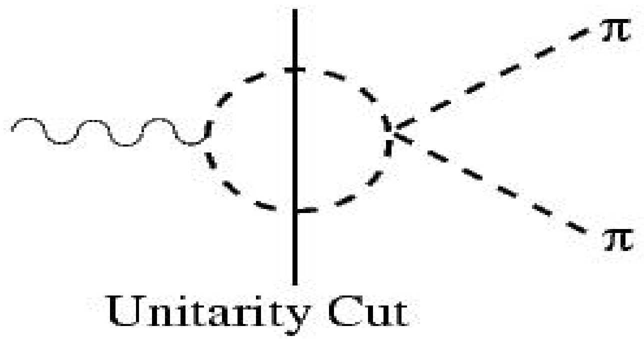
2. Disperson Relations

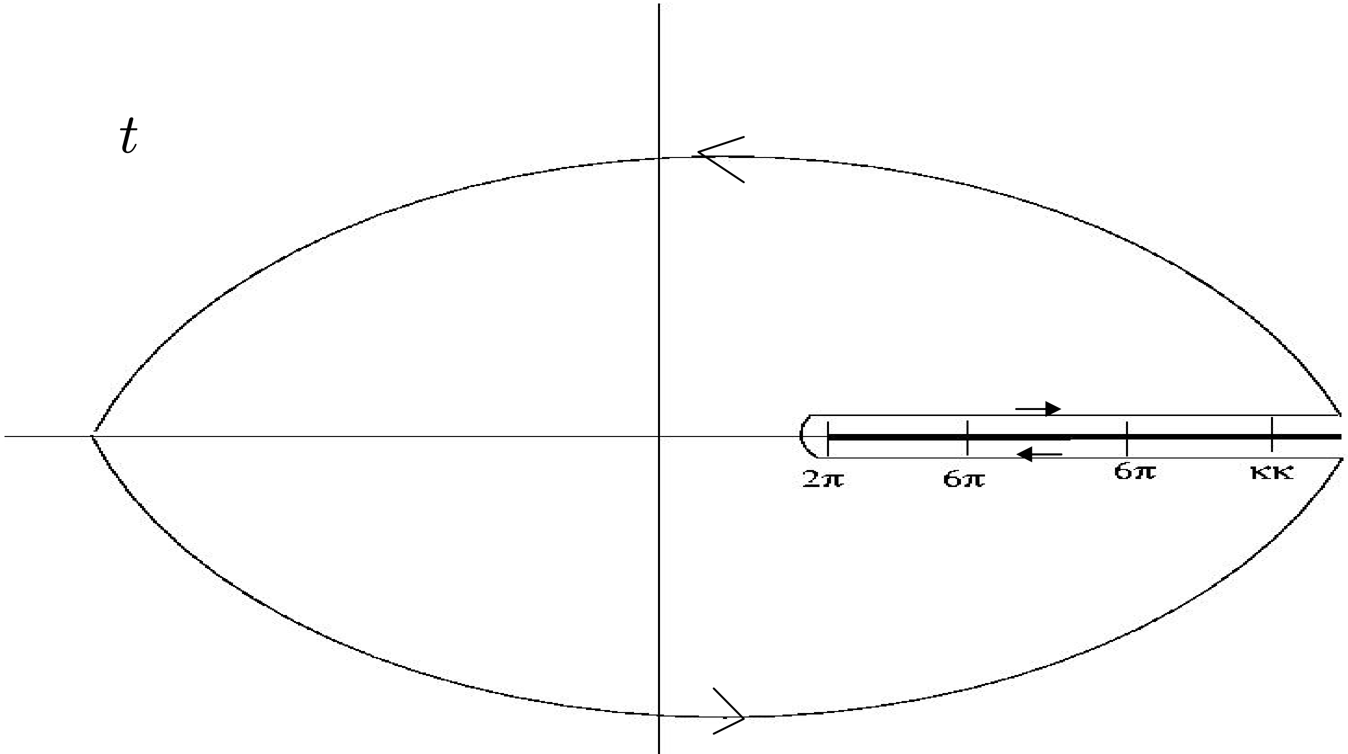
The pion non-strange scalar form factor

$$F(t) = \langle \pi(q')|m_u \bar{u}u + m_d \bar{d}d|\pi(q)\rangle \quad t = (q' - q)^2$$

It is an analytic function having only a right hand cut (or unitarity cut) for $t \geq 4m_\pi^2$ due to the intermediate Isospin 0 J=0 states $\pi\pi$, 4π, ..., $K\bar{K}$, $\eta\eta$, etc.





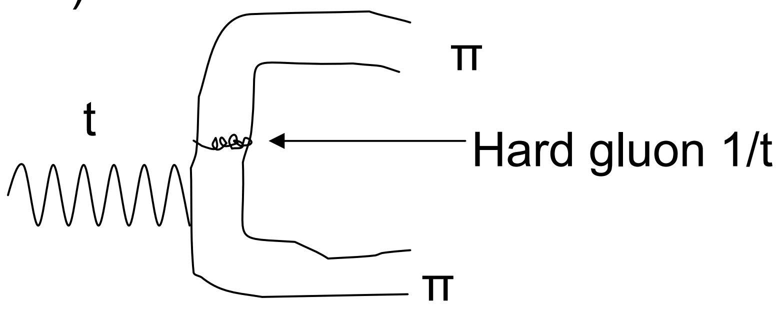


Dispersion Relation

$$F(t) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} \frac{\mathrm{Im}F(s)}{s-t}$$

For "physical" values of t ($t \geq 4m_\pi^2$) one should take $t + i\epsilon$, $\epsilon \to 0^+$

For the scalar form factor F(z) vanishes as 1/z because of QCD.
(Brodsky-Farrar counting rules).



Omnès representation

It is valid for a function with the same analytic properties as $F(t)$ (analytic except for the right hand cut).

One must first remove the zeroes (also the poles for the general case, not in the present one) of F(t) and consider the function

$$g(t) = \frac{F(t)}{P(t)}$$

$s_1,\ s_2,\ \ldots,$ zeroes of $F(s)$

$$P(t) = \frac{F(0)}{s_1 \cdots s_n}(s_1 - t)\cdots(s_n - t)$$

Then one performs a dispersion relation of

$$f(t) = \log\frac{F(t)}{P(t)}$$

$$\log z = \log|z| + i\arg z$$

$$f(s^+) - f(s^-) = \log\frac{F(s+i\eta)}{P(s)} - \log\frac{F(s-i\eta)}{P(s)} = 2i\arg\frac{F(s)}{P(s)}$$

$argF(s)/P(s) = \phi(s)$, and this phase must be continuous ($\log g(s)$ is analytic in D). So it can be larger than 2π, if necessary.
Another requirement: $\phi(4m_\pi^2) = 0$.

$$\log \frac{F(t)}{P(t)} = \gamma_0 + \frac{t}{\pi} \int_{4m_\pi^2}^{+\infty} \frac{\phi(s)}{s-t-i\epsilon}$$

$$F(t) = P(t) \exp \frac{t}{\pi} \int_{4m_\pi^2}^{+\infty} \frac{\phi(s)}{s-t-i\epsilon}$$

Note that $\exp\gamma = F(0)$ and then it is absorbed in $P(t)$

A simple exercise:

$F(t) \to (-1)^n e^{i\phi(+\infty)}\, t^{n-\phi(+\infty)/\pi}$ for $t \to +\infty$.

So $\phi(+\infty) \to (n+1)\pi$ in order that $F(t) \to -1/(t - i0^+)$, as required by QCD

Watson final state theorem

- Elastic case, only $\pi\pi$. Above threshold,

$$\mathrm{Im}F(t) = F(t)\rho(t)T^*_{\pi\pi}(t)$$

Since the left hand side is real then the phases of $F(t)$ ($\delta(s)$) and $T_{\pi\pi}(t)$ ($\varphi(s)$) are equal (modulo π)

For the coupled channel case, this theorem can also be applied if $\eta \simeq 1$

Corolary:

$t_{\pi\pi} = \rho\, T_{\pi\pi} = \sin\delta_\pi\, e^{i\delta_\pi}$, $\delta_\pi(4m_\pi^2) = 0$, δ_π is continuous and at most differs by modulo π from the phase of $T_{\pi\pi}$
This happens when δ_π crosses π ($\sin\delta_\pi < 0$)

$$F(t) = P(t)\exp\frac{t}{\pi}\int_{4m_\pi^2}^{+\infty}\frac{\phi(s)}{s(s-t)}$$

$$\boxed{\phi(s) = \delta_\pi(s) \text{ for } s \leq 4m_K^2.}$$

3. Ynduráin's method

We follow Y1.
MO equations neglect multipion contributions that
for the electromagnetic form factor account a 6% of the result.

I) Let us call by $\delta(t)$ the *continuous* phase of $F(t)$, $\delta(4m_\pi^2) = 0$.

II) At large t, QCD implies (Brodsky-Farrar counting rules), Y3
$F(t) \to C \frac{1}{-t \log^\nu(-t/\Lambda^2)} \quad t < 0 \quad C > 0$

Analytical extrapolation (e.g. taking the circle at infinity)
with $\delta(t) = 0$ for $t < 0$
QCD requires $\delta(+\infty) = \pi$.

III) Omnès representation

$$F(t) = F(0) \exp \frac{t}{\pi} \int_{4m_\pi^2}^{+\infty} \frac{\delta(s)}{s(s-t)} \quad P(t) = F(0) \text{ because } \delta(+\infty) = \pi$$

Weak point of the argument

Not always compatible

IV) Taking the derivative at $t = 0$

$$\langle r^2 \rangle_s^\pi = \frac{6}{\pi} \int_{4m_\pi^2}^{+\infty} \frac{\delta(s)}{s^2}$$

V) Watson's final state theorem

•$\delta(s) = \delta_\pi(s)$ for $s < 4m_K^2$.

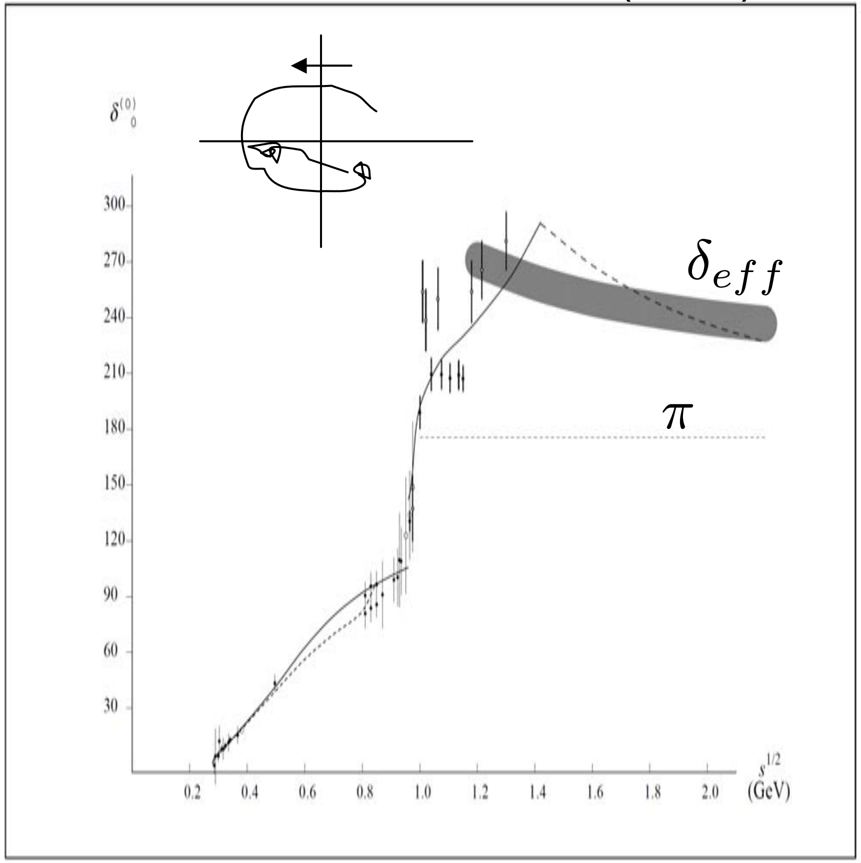
•Again inelasticity is "zero" for $1.1 \leq s^{1/2} \leq 1.5$ GeV as follows from experimental data on $\pi\pi$ scattering
Hyams et al. NPB64, 134(1973); Grayer et al. NPB75, 189 (1974).

$\delta(s) = \delta_\pi(s)$ for $1.1 \leq s^{1/2} \leq 1.42$ GeV.

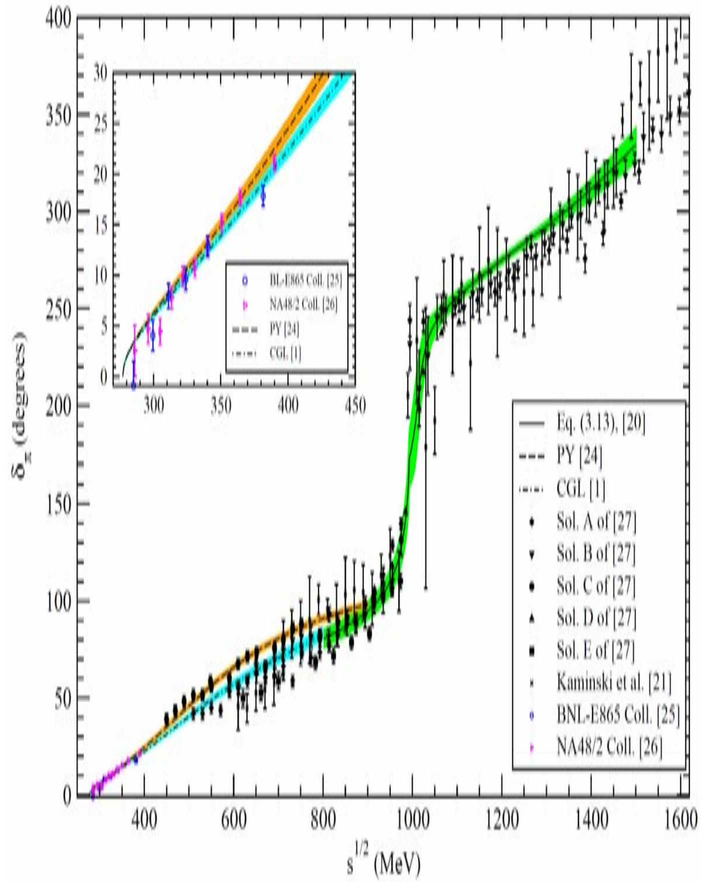
•The region where inelasticity is not zero (for $4m_K < s^{1/2} < 1.1$ GeV) is very narrow and has little numerical impact.
Inelastic effects are estimated at around 10% (although this is not shown, just an statement).

VI) $\delta_{ef}(s) = \pi + [\delta_\pi(s_0) - \pi] \frac{s_0}{s}$ for $s > s_0 = 1.42^2$ GeV^2

From F.J.Ynduráin, PLB578,99(2004)



Rapid motion, not shown in the figure, $\delta_\pi(s) \geq 2\pi$ for s slightly above 1.4 GeV Kaminski, Lesniak, Rybicki, ZPC74, 79(1997)



$$\langle r^2 \rangle_s^\pi = \frac{6}{\pi} \int_{4m_\pi^2}^{+\infty} \frac{\delta(s)}{s^2}$$

$$\langle r^2 \rangle_s^\pi = 0.75 \pm 0.07 \text{ fm}^2$$

Problem: $\delta_\pi(s)$ keeps rising up to around 400^0, then it seems to stabilize and slight decreasing. I do not see a very natural matching to π, somewhat forced.

This approach was critized by
Ananthanarayan, Caprini, Leutwyler, IJMP A21,954 (2006) (ACGL).

The main objection to the previous method is the way that the Watson's final state theorem is applied in the region above the $K\overline{K}$ threshold.

It fixes the phase modulo π, how can one know whether a flip in π has not occured in the region between $2m_K \leq s^{1/2} \leq 1.1$ GeV where inelasticity is not zero?

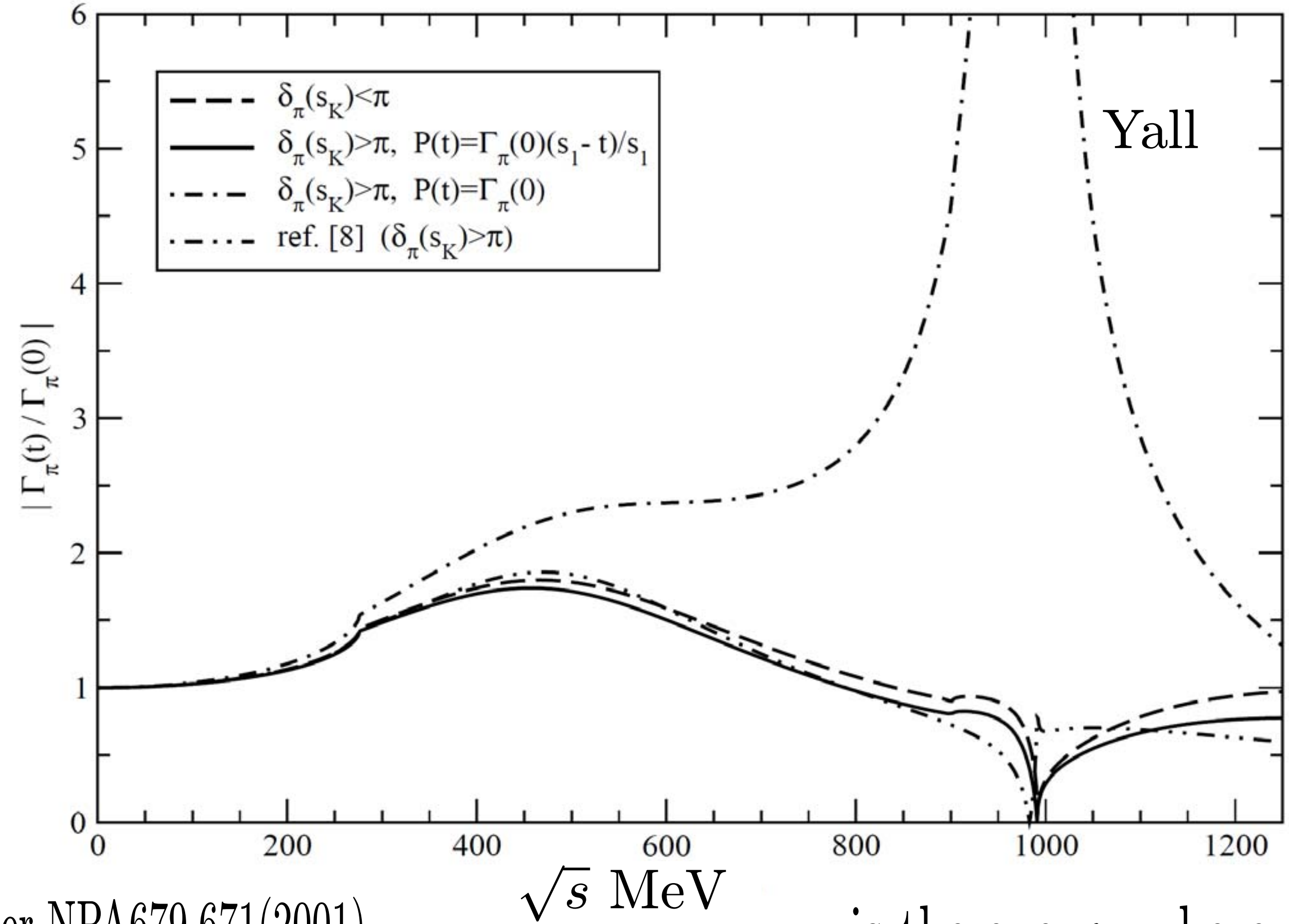
The point is that $\frac{6}{\pi}\int \frac{\delta_\pi(s)-\pi}{s^2} ds$ above the $K\overline{K}$ threshold gives again $\langle r^2\rangle_s^\pi \simeq 0.61\ \text{fm}^2$.

ACGL concludes that the $\pm\pi$ ambiguity in the Watson's theorem can be resolved only by explicit inclusion of inelastic channels in MO equations.

They also mention other studies in which the pion scalar form factor has a minimum just below the $K\bar{K}$ threshold and not the strong maximum that Y1 gives. (More later)

4. Extended Y's method

L. Roca and J.A.O. Phys. Lett. B651,139(2007) arXiv:0704.0039 [hep-ph]



[8] Meißner, Oller NPA679,671(2001)

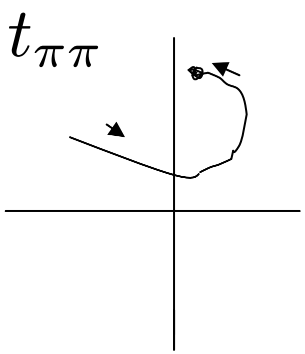
s_1 is the energy where $\delta_\pi(s_1) = \pi$

•We show first that continuity arguments require that $F(t)$ has a zero and its phase also jumps by $-\pi$ at s_1 for $\delta_\pi(s_K) > \pi$. Ynduráin's first step is not always true.

•We follow Ynduráin's generalized hypothesis: Above the $K\overline{K}$ threshold one can approximately apply Watson's final state theorem

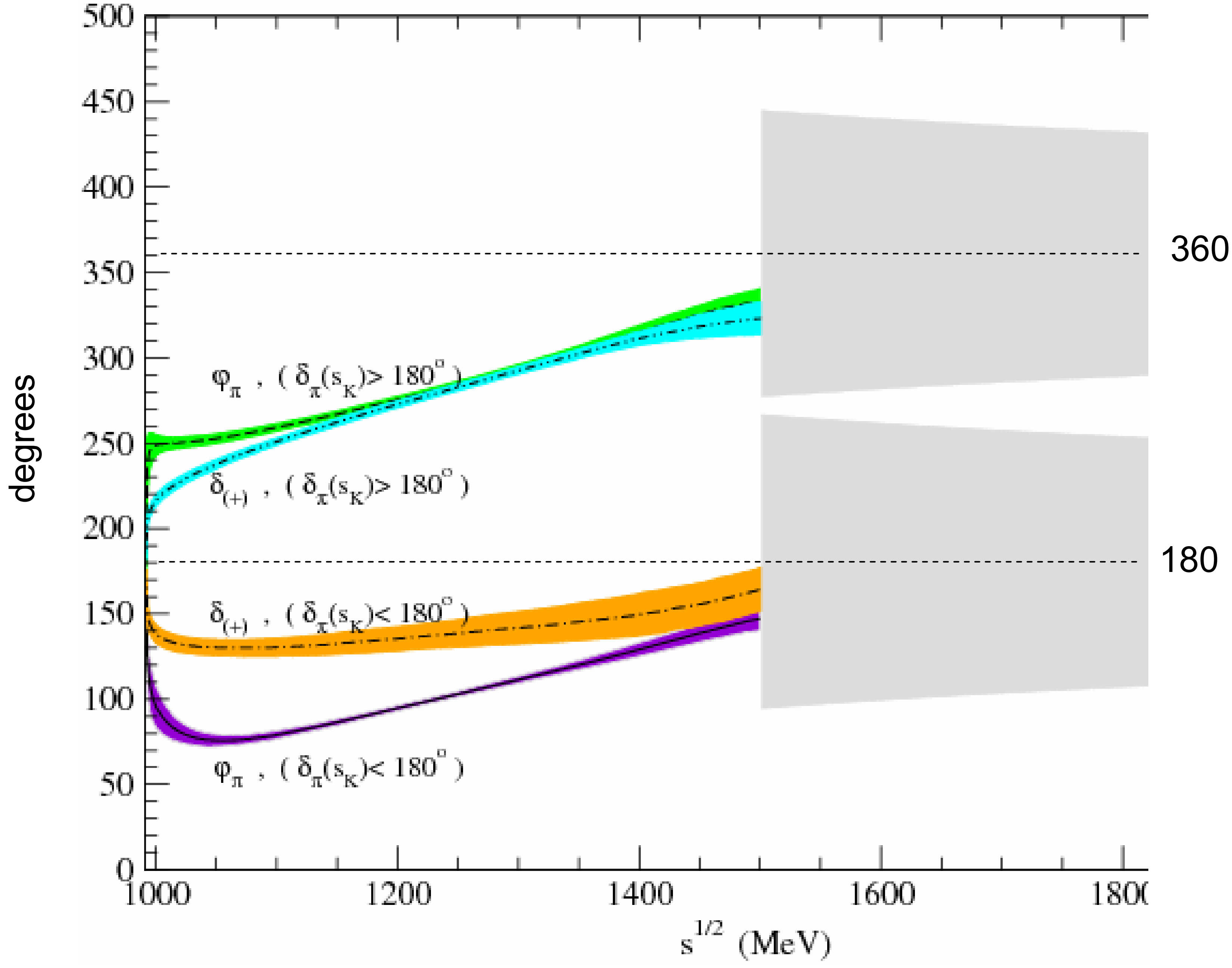
The explicit calculations of Donoghue, Gasser and Leutwyler NPB343,341(1990) as well as those from UχPT, Meißner, Oller NPA679,671(2001) agree on this.

•We start with $\delta_\pi(s_K) < \pi$ (this corresponds to the analysis of CGL)
$t_{\pi\pi} = \frac{1}{2}\eta \sin 2\delta_\pi + \frac{i}{2}(1 - \eta \cos 2\delta_\pi)$
Im$t_{\pi\pi}$ >0 ($\eta < 0$) above the $K\overline{K}$ threshold but the real part changes sign when crossing $\delta_\pi = \pi$. This occurs very quickly.



This rapid motion, from π^- to $\frac{\pi}{2}^-$, produces a strong minimum in $|F(t)|$

$$F(t) = F(0)\exp\left\{\frac{t}{\pi}\int_{4m_\pi^2}^{+\infty}\frac{\phi(s)}{s(s-t)}\right\}$$



500
450
400
350
300
250
200
150
100
50
0
1000
1200
1400
1600
1800
degrees
s^1/2 (MeV)
360
180
φπ , (δπ(sK)> 180°)
δ(+) , (δπ(sK)> 180°)
δ(+) , (δπ(sK)< 180°)
φπ , (δπ(sK)< 180°)

The motion in $\varphi(s)$ is more and more dramatic as $\delta_\pi(s_K) \to \pi^-$
Exactly in this limit it becomes discontinuous with a jump of $-\pi$
This makes the Omnès for $F(t)$ to develop a zero at $s_1 = 4m_K^2$.

Demonstration

$$t_{\pi\pi} = sin\delta_\pi \, e^{i\delta_\pi}, \; s < s_K$$

$\delta_\pi(s_1) = \pi$ with $s_1 > 4m_K^2$. Close to and above s_1, $\varphi(s) \in [0, \pi/2]$.
Now $s_1 \to s_K^+$. In this limit $\varphi(s_K^-) = \pi$ (left) $\quad \varphi(s_K^+) < \pi/2$ (right)
(indeed is 0 because of unitarity, but for simplicity let's go on).

This discontinuity at $s = s_K$ implies a logarithmic singularity in the Omnès as
$\frac{\phi(s_K^-)-\phi(s_K^+)}{\pi} \log \frac{\delta}{s_K}$ with $\delta \to 0^+$.
Exponentiating: $F(t) \to (\delta/s_K)^\nu$, $\nu = (\phi(s_K^-) - \phi(s_K)^+)/\pi = 1 > 0$

$$F(t) = F(0)\,\frac{s_1-t}{s_1}\exp\left[\frac{t}{\pi}\int_{4m_\pi^2}^{+\infty}\frac{\phi(s)}{s(s-t)}ds\right]$$

$$\langle r^2\rangle_s^\pi = -\frac{6}{s_1} + \frac{6}{\pi}\int_{4m_\pi^2}^{+\infty}\frac{\phi(s)}{s^2}ds$$

$\Big\}\ \delta_\pi(s_K) \geq \pi$

$\delta_\pi(s_K) \to \pi^+$ now one obtains continuous results

Determination of s_1

$$F(t) = F(0) + \frac{1}{6}\langle r^2\rangle_s^\pi + \frac{t^2}{\pi}\int_{4m_\pi^2}^{+\infty}\frac{\mathrm{Im}F(s)}{s^2(s-t)}ds$$

Because of Watson's theorem $|\mathrm{Im}F(t)| = |F(t)||\sin\delta_\pi|$ and is zero at $s_1 < s_K$, $\delta_\pi(s_1) = \pi$.
This is the only point where $F(t) = 0$ can be zero for $s < s_K$, otherwise the dispersion relation has an imaginary part that cannot be cancelled since t, $F(0)$, $\langle r^2\rangle_s^\pi$ are all real.

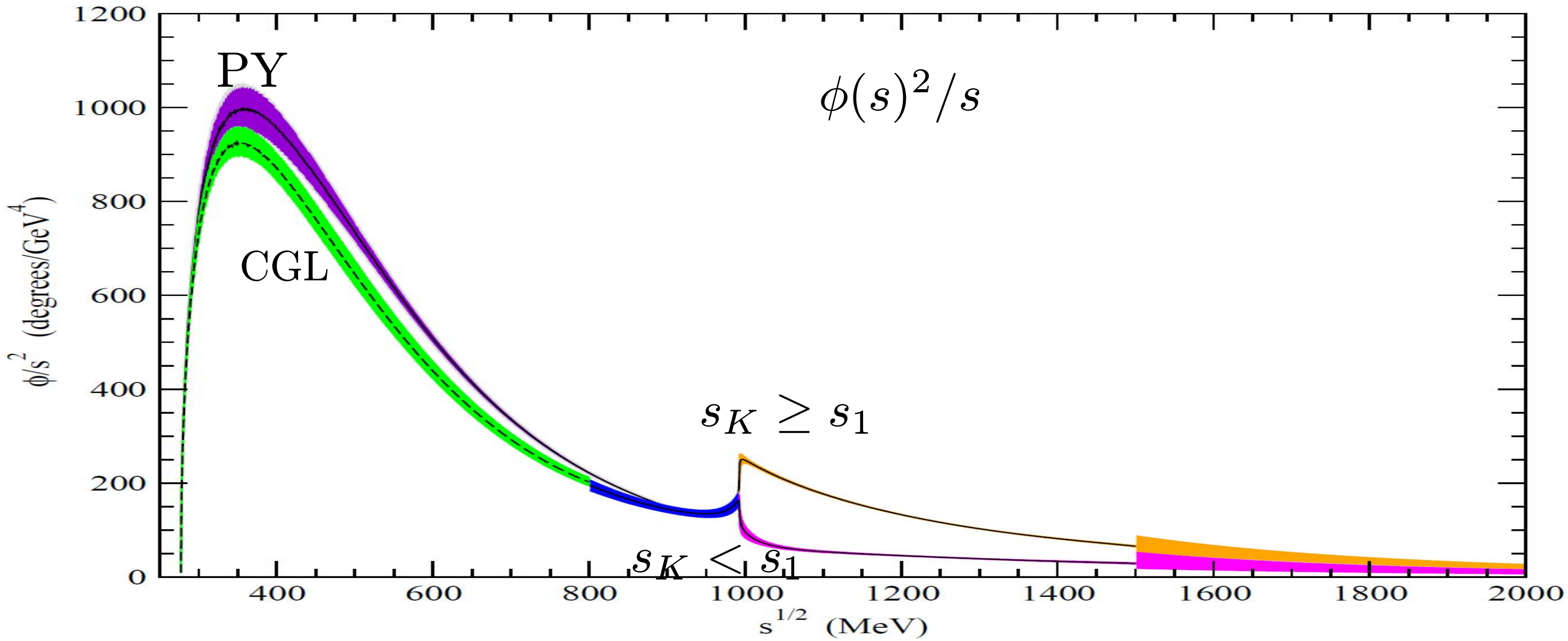
$$F(t) = F(0)\exp\left[\frac{t}{\pi}\int_{4m_\pi^2}^{+\infty}\frac{\phi(s)}{s(s-t)}\right] \quad \delta_\pi(s_K) < \pi \quad \langle r^2\rangle_s^\pi = \frac{6}{\pi}\int_{4m_\pi^2}^{+\infty}\frac{\phi(s)}{s^2}$$

5. Numerical Analysis

$$\langle r^2 \rangle_s^\pi = -\frac{6}{s_1}\theta(s_K - s_1) + \frac{6}{\pi}\int_{4m_\pi^2}^{+\infty} \frac{\phi(s)}{s^2} ds \qquad \langle r^2 \rangle_s^\pi = Q_H + Q_A$$

$$Q_H = -\frac{6}{s_1}\theta(s_K - s_1) + \frac{6}{\pi}\int_{4m_\pi^2}^{s_H} \frac{\phi(s)}{s^2} ds$$

$$Q_A = \frac{6}{\pi}\int_{s_H}^{+\infty} \frac{\phi(s)}{s^2} \qquad s_H = 1.5^2 = 2.25 \text{ GeV}^2$$



We use:
•CGL below 0.8 GeV (upper limit of their analysis)

•Pelez, Ynduráin, PRD68,074005(2003) below 0.9 GeV (PY)
The difference between both parameterization spans well the experimental uncertainties in $\pi\pi$ scattering

• The K-matrix of the energy dependent fit of Hyams et al. NPB64,134(1973)
Simple to pass from $\delta_\pi(s_K) < \pi$(60% events) $\leftrightarrow$ $\delta_\pi(s_K) \geq \pi$(30% events)
above 0.8 GeV when CGL (Parameterization I)
above 0.9 GeV when PY (Parameterization II)

Above the $K\overline{K}$ threshold the application of Watson final state theorem is not straightforward.
Above $s^{1/2} \gtrsim 1.1$ GeV $\eta \simeq 1$ from $\pi\pi$ data up to $s^{1/2} \lesssim$1.5 GeV.

Error estimate of inelastic effects. (Which is not explicitely shown in Y1).

How to get ride of the $\pm\pi$ jump due to the non-elastic zone $2m_K \lesssim s^{1/2} \lesssim 1.1$ GeV ($\eta < 1$) following the generalized Ynduráin's hypothesis.?

- $2m_K \leq s^{1/2} \leq 1.1$ GeV, inelasticity can be substancial.

$\eta = 0.6 - 0.7$ at its minimum value
(more clearly from $\pi\pi \to K\overline{K}$ experiments or from explicit calculations)
Taking typically $\eta \gtrsim 0.6$ then $\epsilon \lesssim 0.5$

	$\delta_\pi(s_K) < \pi$	$\delta_\pi(s_K) \geq \pi$
	$\delta_{(+)} \geq \pi/2$	$\delta_{(+)} \geq \pi$
Correction:	30%	15%

On top of these uncertainties due to inelasticity we also add in quadrature the noise due to the errors in the parameterizations of $t_{\pi\pi}$.

$F(t) \to (-1)^n e^{i\phi(+\infty)}\, t^{n-\phi(+\infty)/\pi}$ for $t \to +\infty$.

So $\phi(+\infty) \to (n+1)\pi$ in order that $F(t) \to -1/(t - i0^+)$, as required by QCD

$\delta_\pi(s_K) < \pi$ we have $n = 0$ then $\phi(+\infty) = \pi$

$\delta_\pi(s_K) \geq \pi$ we have $n = 1$ then $\phi(+\infty) = 2\pi$

$s > s_H = 2.25$ GeV^2 we use the asymptotic $\phi(s)$.

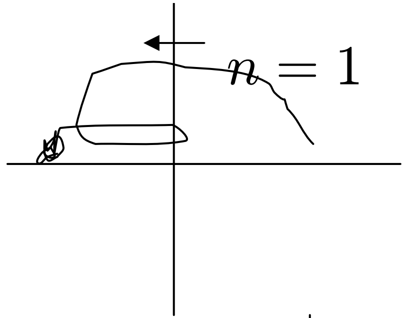
•However, from QCD it is not clear how the phase of $F(t)$ approaches π
There is a controversy between Y3; Espriu, Ynduráin PLB132,187(1983)
and Caprini,Colangelo,Leutwyler IJMA21,954(2006)
wether l.t. or twist3 dominate.

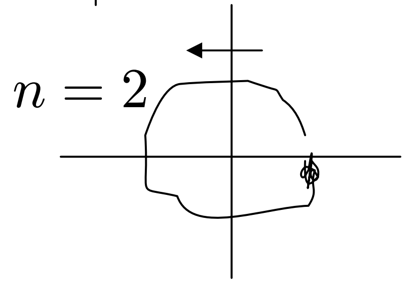
•The phase can approach $n\pi$ from above (l.t), from below (t.3.)
(or maybe can even oscillate?)

•This was relevant before our work since CGL tends to π from below,
while Yall tend to π from above. Ynduráin states (Y3) that
the way to distinguish between the two solutions was to fixed
this asymptotic behaviour. This was also taken seriously by Berna group.

•From our work, one sees now that this is not relevant for $\langle r^2 \rangle_s^\pi$.
The same value within rather small uncertainties results.
Only the error from the asymptotic region could be reduced in a factor 2.

$$\phi_{as}(s) \simeq \pi\left(n \pm \frac{2d_m}{\log(s/\Lambda^2)}\right) \quad \begin{array}{l} n = 2 \text{ for } \delta_\pi(s_K) \geq \pi \\ n = 1 \text{ for } \delta_\pi(s_K) < \pi \end{array}$$





$\pm$ from QCD one cannot fix now how it approaches to π

$0.1 < \Lambda^2 < 0.35$ GeV^2 as suggested in Y1.

We are pretty much conservative for ϕ_{as}
In this way we avoid to enter into hadronic details for $s^{1/2} > 1.5$ GeV
where $\eta < 1$, the $f_0(1500)$ appears.

$$\langle r^2\rangle_s^\pi = -\frac{6}{s_1}\theta(s_K - s_1) + \frac{6}{\pi}\int_{4m_\pi^2}^{+\infty}\frac{\phi(s)}{s^2}ds \qquad \langle r^2\rangle_s^\pi = Q_H + Q_A$$

$$I_1 = \frac{6}{\pi}\int_{4m_\pi^2}^{s_K}\boxed{\frac{\varphi(s)}{s^2}}ds \qquad I_2 = \frac{6}{\pi}\int_{s_K}^{1.1^2}\frac{\varphi(s)}{s^2}ds \qquad I_3 = \frac{6}{\pi}\int_{1.1^2}^{s_H}\frac{\varphi(s)}{s^2}ds$$

$$Q_H = -\frac{6}{s_1}\theta(s_K - s_1) + I_1 + I_2 + I_3$$

$$Q_A = \frac{6}{\pi}\int_{s_H}^{+\infty}\frac{\phi_{as}(s)}{s^2}$$

Note that we use $\varphi(s)$ in the integrals I_2 and I_3.
We could have used $\delta_{(+)}$ instead. But when $\eta \lesssim 1$ then $\delta_{(+)} \simeq \varphi(s)$
The difference is taken into account in the error analysis.
But **consistency** of our approach, so that the same value is obtained for $\delta_\pi(s_K) = \pi - \epsilon$ and $\delta_\pi(s_K) = \pi + \epsilon$, $\epsilon \to 0^+$, requires to use $\varphi(s)$ **because** $\varphi_+(s) = \varphi_-(s) + \pi$

$\phi(s)$	I	I	II	II
$\delta_\pi(s_K)$	$\geq \pi$	$< \pi$	$\geq \pi$	$< \pi$
I_1	0.435 ± 0.013	0.435 ± 0.013	0.483 ± 0.013	0.483 ± 0.013
I_2	0.063 ± 0.010	0.020 ± 0.006	0.063 ± 0.010	0.020 ± 0.006
I_3	0.143 ± 0.017	0.053 ± 0.013	0.143 ± 0.017	0.053 ± 0.013
Q_H	0.403 ± 0.024	0.508 ± 0.019	0.452 ± 0.024	0.554 ± 0.019
Q_A	0.21 ± 0.03	0.10 ± 0.03	0.21 ± 0.03	0.10 ± 0.03
$\langle r^2 \rangle_s^\pi$	0.61 ± 0.04	0.61 ± 0.04	0.66 ± 0.04	0.66 ± 0.04

Our final value: $\langle r^2 \rangle_s^\pi = 0.63 \pm 0.05 \text{ fm}^2$

Our final value: $\langle r^2 \rangle_s^\pi = 0.63 \pm 0.05\ \text{fm}^2$

Y1: $\langle r^2 \rangle_s^\pi = 0.75 \pm 0.07\ \text{fm}^2$

Our value is in good agreement with CGL $\langle r^2 \rangle_s^\pi = 0.61 \pm 0.04\ \text{fm}^2$.

We have also calculated the value of UCHPT paper Meißner,JAO NPA679,671(2001) $\langle r^2 \rangle_s^\pi = 0.64 \pm 0.06\ \text{fm}^2$

Conclusions I

- We reconcile Ynduráín's method and MO equations

- The same is obtained now, with Y extended method, independently of whether $\delta_\pi(s_K) = \pi - \epsilon$ or $\delta_\pi(s_K) = \pi + \epsilon$.

- Independently of whether $\delta(+\infty) = \pi^+$ or $\delta(+\infty) = \pi^-$ (or even if there are oscillations)

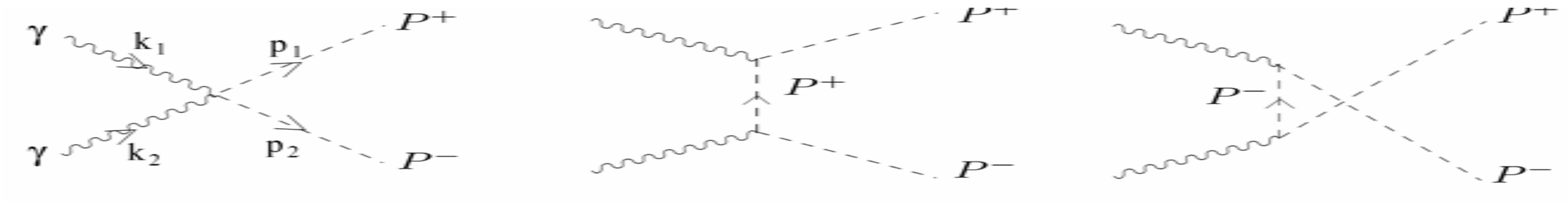
The latest two points were sustained in Y2 and Y3 as the way to disentangle between Y1 and CGL solutions. We see now that this is superfluous.

- More straightforward matching with $\phi_{as}(s)$

6. $\gamma\gamma \to \pi^0\pi^0$

L. Roca, C. Schat and J.A.O., arXiv:hep-ph/0708.1659

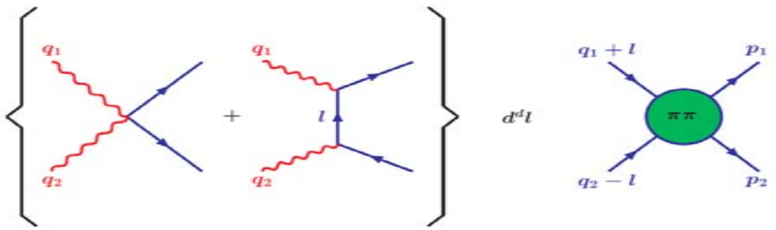
There is no Born term as π^0 is neutral



Final state interactions are enhanced

The final state is a two body hadronic state

Good reaction to study the $I = 0$ $\pi\pi$ S-wave



Silver mode for χPT

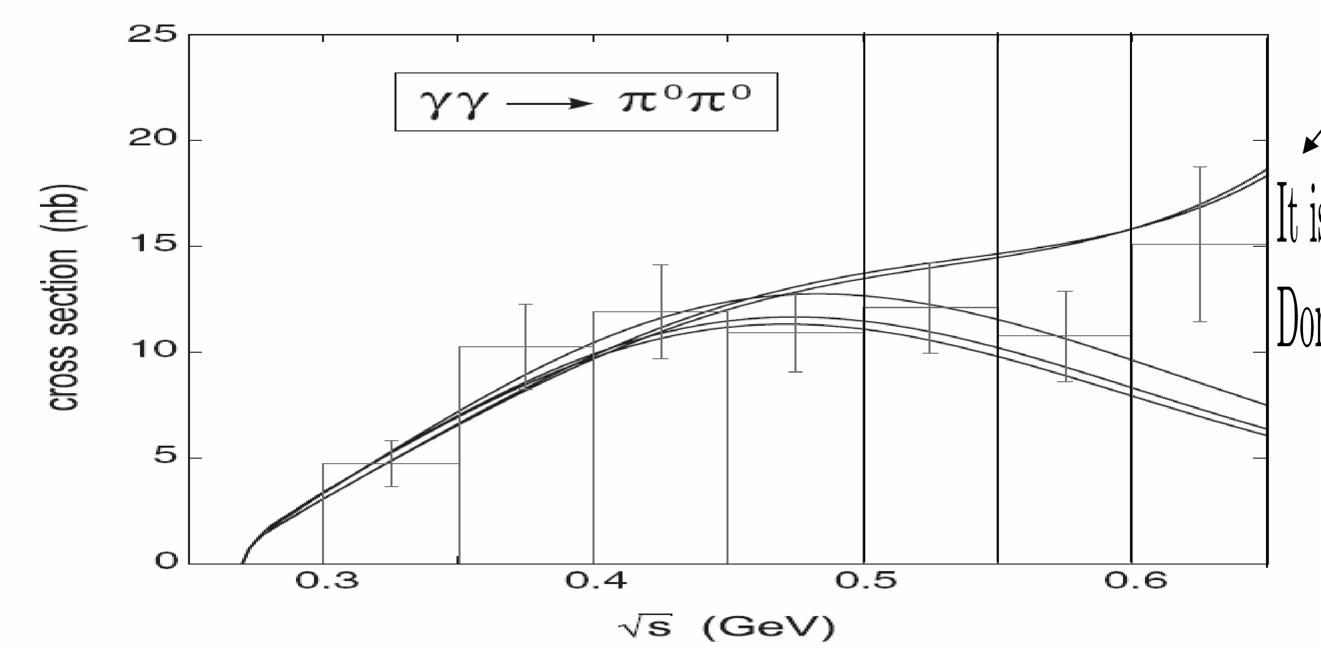
- One loop χPT is the leading contribution: Bijnens, Cornet NPB296,557(1988); PRD37,2423(1988)
- No counterterms, pure χPT quantum prediction (the agreement with data was not satisfactory)
- Two loop calculation in χPT was performed: Bellucci, Gasser, Sainio NPB423,80 (1994), revised in Gasser,Ivanov, Sainio NPB728,31 (2005).
- Better agreement with data. The three counterterms were fixed according to the resonance saturation hypothesis.

$\sigma(\gamma\gamma \to \pi^0\pi^0)$ was measured for $|\cos\theta| < 0.8$ by Crystal Ball Collaboration, H. Marsiske *et al.* PRD41,3324 (1990)

New accurate data on $\gamma\gamma \to \pi^+\pi^-$ $|\cos\theta| < 0.6$
T. Mori *et al.* [Belle Coll.] PRD75,051101 (2007)
Remarkable resolution of the $f_0(980)$ resonance.

Recent activity:

Pennington, PRL97,011601(2006) employs a dispersive method to calculate the S-wave $\gamma\gamma \to \pi\pi$ for low energies.
This approach was settled in Pennington,Morgan PLB272,134(1991), revised in Pennington DAFNE Physics Handbook, Vol.1



Explodes at around 1 GeV

It is also the case for Donoghue and Holstein, PRD48,137(1993)

Large ambiguity because of the phase of the $\gamma\gamma \to \pi\pi$ $I = 0$ S-wave for $s > 4m_K^2$
$\sqrt{s} = 0.5$, 0.55, 0.6, 0.65 GeV one has 20, 45, 92 and 200% error, respectively

$\Gamma(\sigma \to \gamma\gamma) = 4.09 \pm 0.29$ KeV only a 7% error. He applies a narrow resonance formula for the σ without discussion about its large width, ~ 550 MeV

- Is it possible to reduce such large uncertainty for $\sqrt{s} \gtrsim 0.5$ GeV?

- Revise the given error for the $\Gamma(\sigma \to \gamma\gamma)$ width

In a recent paper Mennessier, Minkowski, Narison and Ochs, arXiv:hep-ph/0707.4511 they calculate $\Gamma(\sigma \to \pi^0\pi^0) = 1.4 - 3.2$ KeV. Lower values are then favoured.

- How to extend the dispersive formalism to study more resonances apart from the σ, e.g. the $f_0(980)$?

$F_I(s)$ is the S-wave $\gamma\gamma \to \pi\pi$ with isospin I, $I = 0, 2$

On the complex s-plane it is an analytic function except for two cuts along the real axis:
Unitarity cut: $s \geq 4m_\pi^2$
Left hand cut: $s \leq 0$.

Pennington, Morgan approach, PLB272,134 (1991)

$L_I(s)$ is the left hand cut contribution of $F_I(s)$

$F_I(s) - L_I(s)$ has no left hand cut

$$w_I(s) = \exp\left[\frac{s}{\pi}\int_{4m_\pi^2}^{\infty}\frac{\phi_I(s')}{s'(s'-s)}ds'\right]$$

$\phi_I(s)$ is the phase of $F_I(s)$, modulo π.
It must be continuous and $\phi_I(4m_\pi^2) = 0$

$F_I(s)/\omega_I(s)$ has no right hand cut

Twice subtracted dispersion relation for $(F_I(s) - L_I(s))/\omega_I(s)$

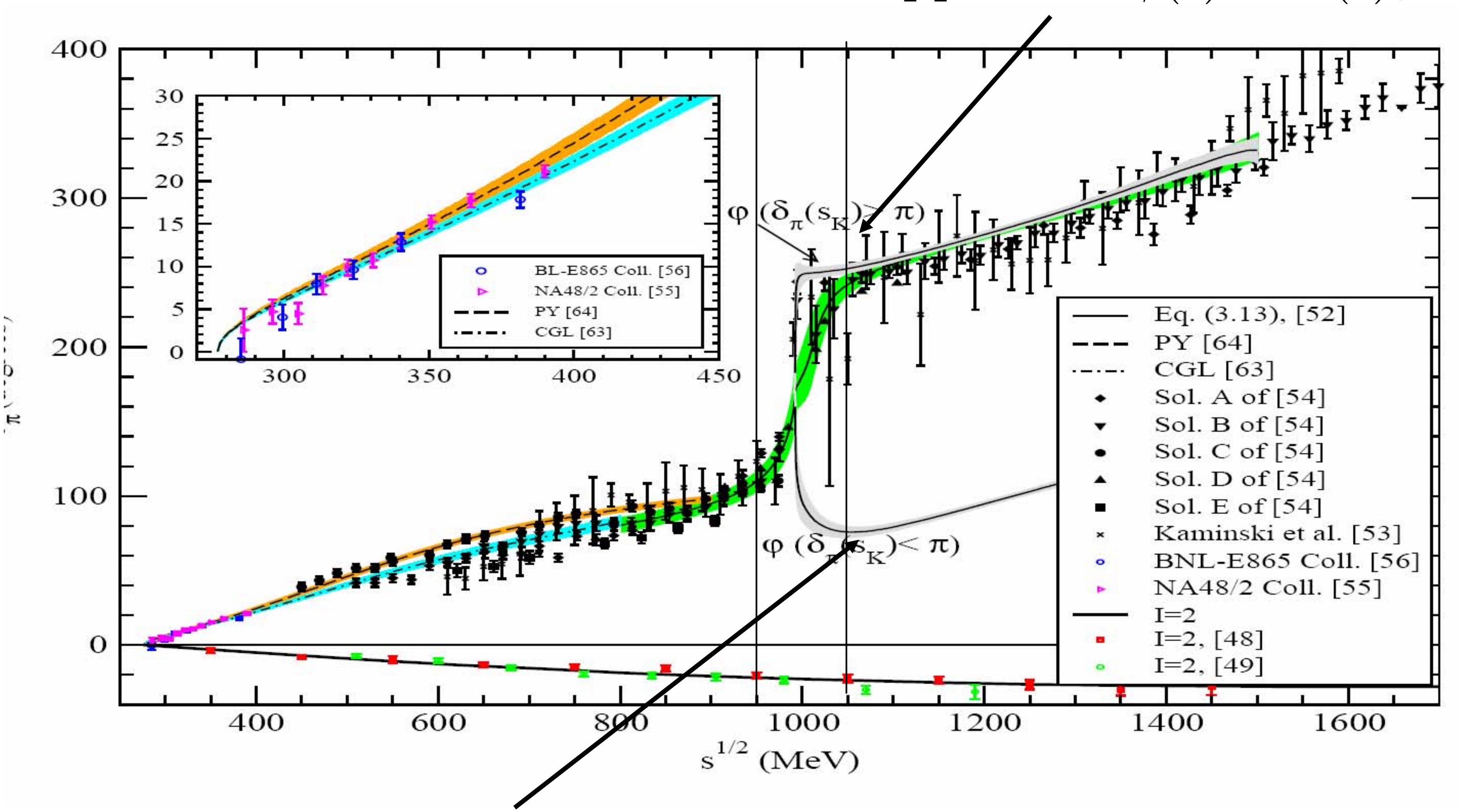
$$F_I(s) = L_I(s) + a_I\,\omega_I(s) + c_I s\,\omega_I(s) + \frac{s^2}{\pi}\omega_I(s)\int_{4m_\pi^2}^{\infty}\frac{L_I(s')\sin\phi_I(s')}{s'^2(s'-s)|\omega_I(s')|}ds'$$

Low's theorem $F_I(s) \to B_I(s) + \mathcal{O}(s)$ for $s \to 0$ then $a_I = 0$.

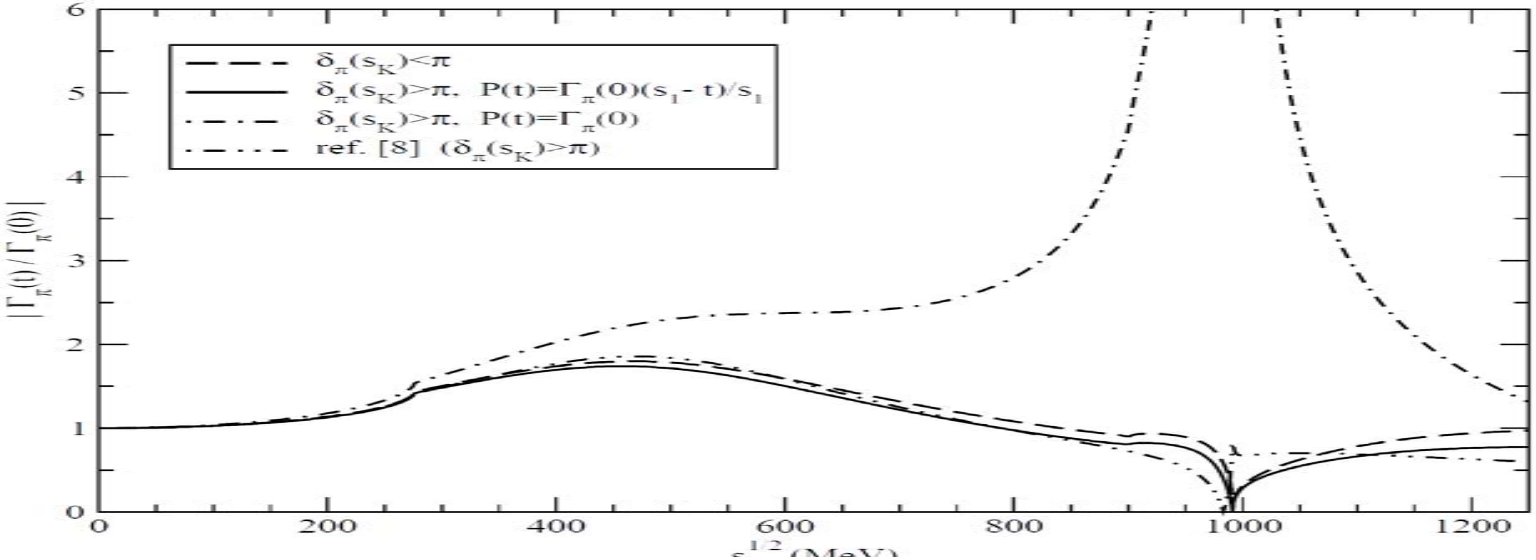
The dramatic increased referred above corresponds to the use of $\omega_0(s)$

Similarly as already commented for the scalar form factor of the pion when $\phi(t) = \delta_\pi(s)_0$, <u>without</u> the zero at s_1

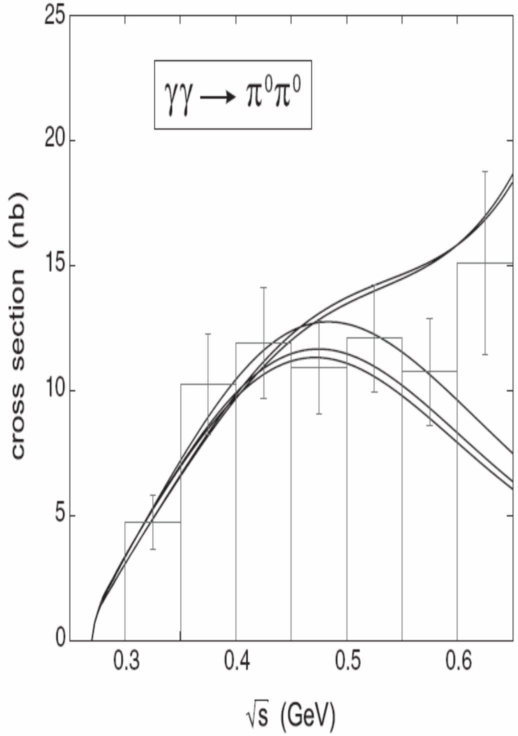
Upper line $\phi(s) \simeq \delta_\pi(s)_0$



Lower line $\phi_0(s) \simeq \delta_\pi(s) - \pi$



$\delta_\pi(s_K)<\pi$
$\delta_\pi(s_K)>\pi,\ P(t)=\Gamma_\pi(0)(s_1-t)/s_1$
$\delta_\pi(s_K)>\pi,\ P(t)=\Gamma_\pi(0)$
ref. [8] $(\delta_\pi(s_K)>\pi)$
$|\Gamma_\pi(t)/\Gamma_\pi(0)|$
0
1
2
3
4
5
6
0
200
400
600
800
1000
1200



$\gamma\gamma \to \pi^0\pi^0$
cross section (nb)
0
5
10
15
20
25
0.3
0.4
0.5
0.6
$\sqrt{s}$ (GeV)

The same problem arises here about the discontinuity of $\omega_0(s)$ by employing an increasing(maximum) $\phi_0(s)$ above $s_K = 4m_K^2$ or a decreasing(minimum) one.

We now how to solve that. One has to use for an increasing $\phi_0(s)$ above s_K:
$\left(1 - \frac{s}{s_1}\right) \exp\left[\frac{s}{\pi}\int_{4m_\pi^2}^{\infty} \frac{\phi_0(s')}{s'(s'-s)} ds'\right]$

For $I = 2$ Watson's final state theorem applies to good approximation in the whole energy range and $w_2(s)$ is smooth and well behaved.
$\phi_2(s) = \delta_\pi(s)_2$

On the phase $\phi_0(s)$

- Watons's theorem: $\phi_0(s) = \delta_\pi(s)$ for $s \leq s_K$

- For $1.1 \lesssim \sqrt{s} \lesssim 1.5$ GeV, $\eta \simeq 1$ and Watson's final state theorem approximately applies.
Then $\phi_0(s) \simeq \delta^{(+)}(s)$ modulo π

• In order to fix the integer factor in front of π one needs to follow the track of $\phi_0(s)$ in the narrow region $1 \lesssim \sqrt{s} \lesssim 1.1$ GeV (so that continuity can be invoked)

- The appearance of the $f_0(980)$ on top of the $K\bar{K}$ threshold.
- The cusp effect of the latter

For $1.05 < \sqrt{s} < 1.1$ GeV there are no further narrow structures.
Observables evolve smoothly with energy.

Two Options:

A.- $\phi_0(s)$ keeps increasing with energy for $s > s_K$.
Then matches smoothly with $\simeq \delta_\pi(s)_0$ for $\sqrt{s} \gtrsim 1.05$ GeV.

$$z = +1$$

E.g. $\varphi(s)$ for $\delta_\pi(s_K)_0 \geq \pi$

B.- The cusp effect makes the deriative of $\phi_0(s)$ discontinuous.
$\phi_0(s)$ decreases rapidly with energy for $s > s_K$.
Then matches smoothly with $\simeq \delta_\pi(s)_0 - \pi$ for $\sqrt{s} \gtrsim 1.05$ GeV

$$z = -1$$

E.g. $\varphi(s)$ for $\delta_\pi(s_K)_0 < \pi$

An ambiguity of π is left for $\phi_0(s)$ and $s > s_K$

We perform the twice subtracted dispersion relation for $(F_0(s) - L_0(s))/\Omega_0(s)$

$$
\begin{aligned}
F_0(s) &= L_0(s) + c_0 s \Omega_0(s) + \frac{s^2}{\pi}\Omega_0(s)\int_{4m_\pi^2}^{\infty}\frac{L_0(s')\sin\overline{\phi}_0(s')}{s'^2(s'-s)|\Omega_0(s')|}ds' \\
&+ \theta(z)\frac{\omega_0(s)}{\omega_0(s_1)}\frac{s^2}{s_1^2}(F_0(s_1) - L_0(s_1)) \ .
\end{aligned}
$$

$$\Omega_0(s) = \left(1 - \theta(z)\tfrac{s}{s_1}\right)\omega_0(s)$$

$\overline{\phi}_0(s)$ is the phase of $\Omega_0(s)$.
For $s > s_1$ and $z = +1$ $\overline{\phi}_0(s) = \phi_0(s) - \pi$

c_0, c_2 and $F_0(s_1) - L_0(s_1)$ to be fixed

Our equation is equivalent to a *three times* subtracted dispersion relation for $(F_0(s) - L_0(s))/\omega_0(s)$

We have taken two subtractions at $s = 0$ and one at s_1

We could have taken them also at $s = 0$

$$F_0(s) = L_0(s) + c_0 s w_0(s) + d_0 s^2 w_0(s) + \frac{s^3 w_0(s)}{\pi} \int_{4m_\pi^2}^{\infty} \frac{L_0(s') \sin\phi_0(s')}{s'^3 (s' - s)|\omega_0(s')|} ds' \,,$$

In the other form the more physical (continuous) Omnès function $\Omega_0(s)$ is used

In this form the role of the $f_0(980)$ is not so easy to indentify

$$F_N(s) = -\frac{1}{\sqrt{3}}F_0 + \sqrt{\frac{2}{3}}F_2 \qquad \gamma\gamma \longrightarrow \pi^0\pi^0$$

$$F_C(s) = -\frac{1}{\sqrt{3}}F_0 - \sqrt{\frac{1}{6}}F_2 \qquad \gamma\gamma \longrightarrow \pi^+\pi^-$$

c_0 and c_2 are fixed by Low's theorem and χPT:

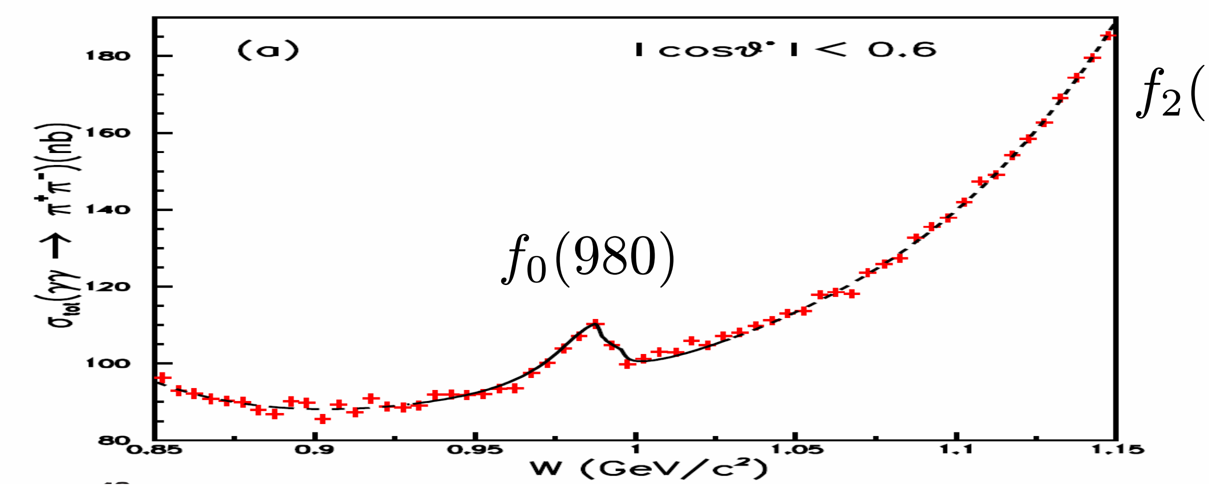
1. $F_C(s) - B_C(s)$ vanishes linearly in s for $s \to 0$
2. $F_N(s)$ vanishes linearly in s for $s \to 0$

The coefficients are calculated from one loop χPT
We use either f_π^2 or f^2 in the expressions $\propto 1/f^2$
Estimate for higher orders, $\sim 12\%$ of uncertainty
(taken into account in the error analysis)

B_C is the $\gamma\gamma \to \pi^+\pi^-$ Born term

Fixing $F_0(s_1) - L_0(s_1)$ for $z = +1$

3. This constant controls the size of the $f_0(980)$ peak.
It has been clearly seen in $\gamma\gamma \to \pi^+\pi^-$
T. Mori *et al.* [Belle Coll.] PRD75, 051101 (2007)



$f_2(1270)$ peak

$M = 985.6^{+1.2}_{-1.6}(stat)^{+1.1}_{-1.6}(sys)$ MeV , $\Gamma = 34.2^{+13.9}_{-11.8}(stat)^{+8.8}_{-2.5}(sys)$ MeV
$\Gamma(f_0(980) \to \gamma\gamma) = 205^{+95}_{-83}(stat)^{+147}_{-117}(sys)$ eV

From Unitary χPT:
J.A.O., NPA727,353(2003): M=988 MeV, Γ=28 MeV,
E.Oset,J.A.O, NPA629,739(1998): $\Gamma(f_0(980) \to \gamma\gamma)$=0.2 KeV

$$
\begin{aligned}
F_0(s) &= L_0(s) + c_0 s \Omega_0(s) + \frac{s^2}{\pi}\Omega_0(s)\int_{4m_\pi^2}^{\infty}\frac{L_0(s')\sin\overline{\phi}_0(s')}{s'^2(s'-s)|\Omega_0(s')|}ds' \\
&+ \theta(z)\frac{\omega_0(s)}{\omega_0(s_1)}\frac{s^2}{s_1^2}\left(F_0(s_1) - L_0(s_1)\right) .
\end{aligned}
$$

$\Omega(s_1) = 0$, only the last term gives contribution at s_1
It is proportional to $F_0(s_1) - L_0(s_1)$ which is small

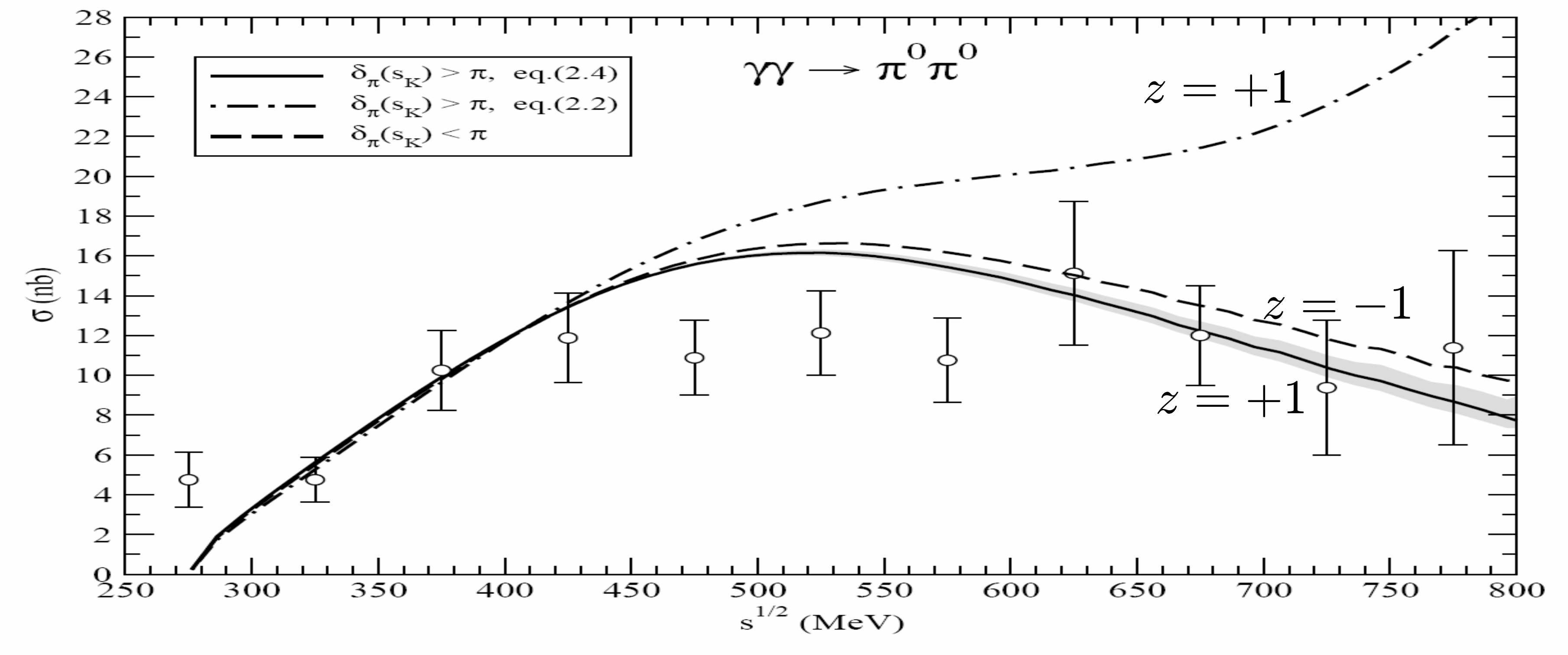
- We require $\sigma(\gamma\gamma \to \pi^0\pi^0) < 400$ nb at s_1, experiment is $\simeq 40$ nb

---

$\boxed{L_I(s)}$ is due to the $\gamma\pi \to \gamma\pi$ dynamics
At $s = 0$ it is given by the Born Term by Low's theorem
This exchange of pions gives rise to the left hand cut for $s < 0$
The main contribution for low energies.

Vector $J^{PC} = 1^{--}$ and Axial-Vector 1^{++}, 1^{+-} exchanges
$s < M_R^2 - m_\pi^2/2$
The axial-vector 1^{++} exchanges are the most important
They give rise to $L_9 + L_{10}$ the $\mathcal{O}(p^4)$ counterterm in $\gamma\gamma \to \pi^+\pi^-$

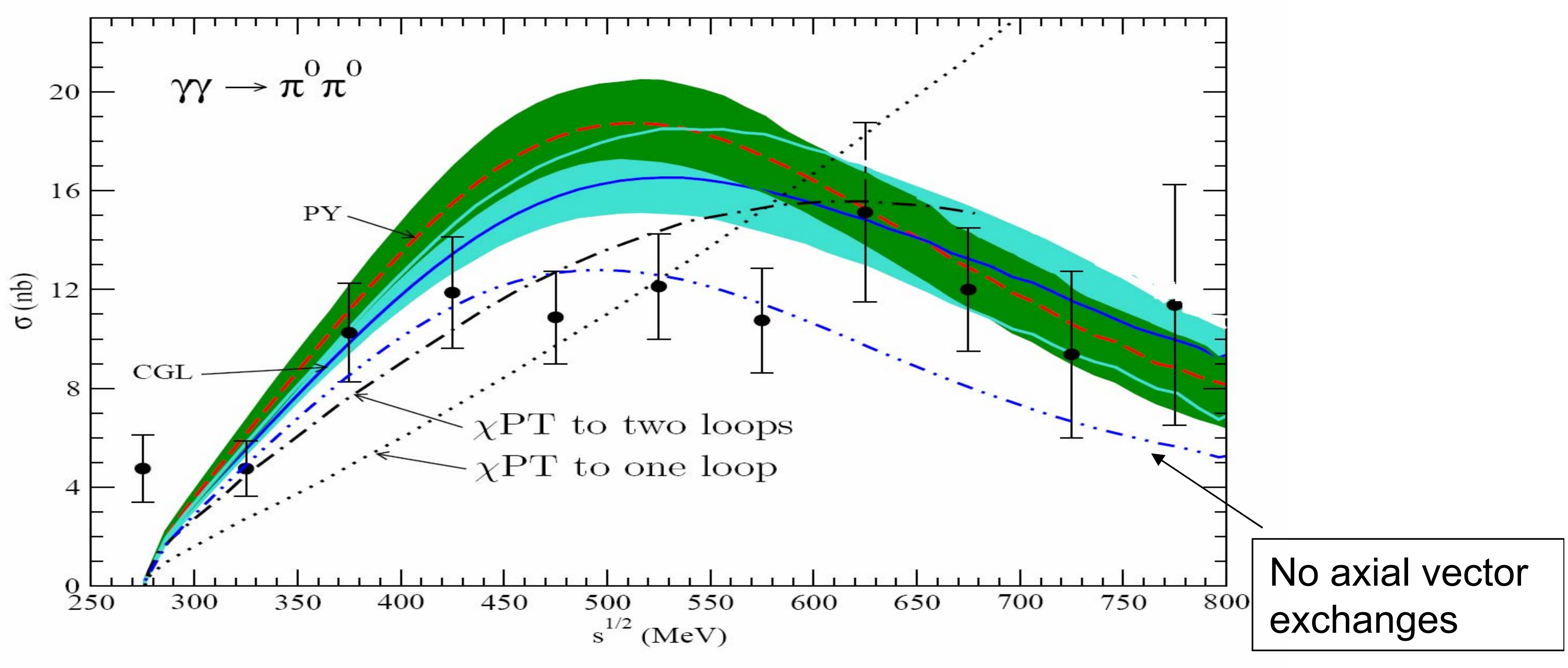
Influence of the uncertainty in the phases above 1 GeV and in the bound $\sigma(\gamma\gamma \to \pi^0\pi^0) < 400$ nb at s_1



We pass from the difference between the dot-dasshed and dashed lines

To that between the solid and dashed line

The gray band is the effect of the bound



Parameterizations for $\delta_\pi(s)_0$

Light Blue: Colangelo, Gasser, Leutwyler, NPB603,125 (2001)

$\sqrt{s} < 0.8$ GeV (Parameterization I)

Dark Green: Yndurain, Pelaez, PRD68,074005 (2003)

Similar phase shifts to Unitary χPT results

$\sqrt{s} < 0.9$ GeV (Parameterization II)

Above those energies up to $\sqrt{s} = 1.5$ GeV

Energy dependent analysis (K-matrix) of data Hyams *et al.* NPB64,134 (1973).

Error bands include:
Uncertainties in the parameterizations CGL, PY and Hyams et al.
The large uncertainties in $\phi_0(s)$ above s_K
The bound $\sigma(\gamma\gamma \to \pi^0\pi^0) < 400$ nb at s_1
c_0 and c_2 calculation employing either f_π^2 or f^2.

$\Gamma(\sigma \to \gamma\gamma)$

Calculation of the coupling:

Analytical extrapolation to the second Riemann sheet where the σ pole locates, s_σ

Unitarity $4m_\pi^2 \leq s \leq 4m_K^2$

$$F_0(s+i\epsilon) - F_0(s-i\epsilon) = -2iF_0(s+i\epsilon)\rho(s+i\epsilon)T_{II}^0(s-i\epsilon)$$

Continuity in the change of sheets: $F_0(s-i\epsilon) = \widetilde{F}_0(s+i\epsilon)$, $T_I(s-i\epsilon) = T_{II}(s+i\epsilon)$

$\widetilde{F}_0(s) = F_0(s)\left(1 + 2i\rho(s)T_{II}^{I=0}(s)\right)$. On the second sheet: $\widetilde{F}_0(s)$ and $T_{II}^{I=0}(s)$

Around s_σ

$$T_{II}^{I=0} = -\frac{g_{\sigma\pi\pi}^2}{s_\sigma - s} \quad \widetilde{F}_0(s) = \sqrt{2}\frac{g_{\sigma\gamma\gamma}g_{\sigma\pi\pi}}{s_\sigma - s}$$

$$g_{\sigma\gamma\gamma}^2 = -\frac{1}{2}F_0(s_\sigma)^2 g_{\sigma\pi\pi}^2 \left(\frac{\beta(s_\sigma)^2}{8\pi}\right)^2$$

$$\beta = \sqrt{1 - 4m_\pi^2/s}$$

$$\Gamma(\sigma \to \gamma\gamma) = \frac{|g_{\sigma\gamma\gamma}|^2}{16\pi M_\sigma}$$

Finite Width: We use for M_σ in the formula for the width either:
$\text{Real}(\sqrt{s_\sigma})$ or $\sqrt{\text{Real}(s_\sigma)} = \sqrt{M_\sigma^2 - \Gamma_\sigma^2/4}$ → $\begin{cases} \text{CCL } 10\% \\ U\chi PT \ 5\% \end{cases}$

UχPT: $s_\sigma \simeq (469 - i\,203)^2$ MeV^2, $g_{\sigma\pi\pi} \simeq 3$ GeV

Caprini,Colangelo,Leutwyler PRL96,132001 (2006) (CCL)
$s_\sigma = (441^{+16}_{-8} - i\,272^{+9}_{-13})^2$ MeV^2
$|g_{\sigma\pi\pi}^{CCL}| = |g_{\sigma\pi\pi}^{U\chi PT}|\left(\frac{\Gamma^{CCL}(\sigma\to\pi\pi)}{\Gamma^{U\chi PT}(\sigma\to\pi\pi)}\right)^{1/2} = 1.18|g_{\sigma\pi\pi}^{U\chi PT}|$

$\Gamma(\sigma \to \gamma\gamma) = 1.24 \pm 0.06$ KeV with $s_\sigma, g_{\sigma\pi\pi}$ from $U\chi PT$
$\Gamma(\sigma \to \gamma\gamma) = 1.7 \pm 0.2$ KeV with $s_\sigma, g_{\sigma\pi\pi}$ from CCL

Average:
$\boldsymbol{\Gamma(\sigma \to \gamma\gamma) = 1.5 \pm 0.3}$ **KeV**

Pennington PRL97, 011601 (2006) $\Gamma(\sigma \to \pi\pi) = 4.09 \pm 0.29$ KeV
He uses CCL σ
Removing the axial exchanges gives an increase of 10% in our width
Our conclusion: There is a difference by a factor 2 with respect to Pennington's value
We reproduce his calculated $\sigma(\gamma\gamma \to \pi^0\pi^0)$ (once the axial exchanges are removed)

Pennington in his recent paper does not include:
Axial vector exchange contributions 1^{++}, 1^{+-}: 10%
Finite width effects: 10%

His error should be multiplied by a factor
~ 2 if added in quadrature, ~ 3 if added linearly

$$\Gamma(\sigma \to \gamma\gamma) = 1.5 \pm 0.3 \text{ KeV}$$

Our value is smaller than for a $q\bar{q}$

$$\frac{\Gamma(0^{++} \longrightarrow \gamma\gamma)}{\Gamma(2^{++} \longrightarrow \gamma\gamma)} \simeq \frac{15}{4}, 2$$

$\Gamma(f_2(1270) \to \gamma\gamma) = 2.6 \pm 0.2$ KeV ⟶ $5 < \Gamma(0^{++} \to \gamma\gamma) < 10$ KeV

It is more appropriate for a meson-meson resonance, glueball, 4 quark state

Recently Mennessier,Minkowski,Narison,Ochs arXiv:0707.4511 [hep-ph]
They favour a glueball naure.

$$\Gamma(\sigma \to \gamma\gamma) \simeq (1.4 - 3.2) \text{ KeV}$$

In J.A. Oller NPA727, 353 (2003) it is established that the σ is 0.92% a $SU(3)$ *singlet*
Natural explanation if it were a strong dressed ("meson-meson resonance") glueball.
Singlet: Implies both large σ couplings to $\pi\pi$ and $K\bar{K}$.
This is also seen phenomenologically (D. Bugg).

7. Conclusions II

• Drastic reduction in the uncertainty of $\sigma(\gamma\gamma \to \pi^0\pi^0)$ for $\sqrt{s} \gtrsim 0.5$ GeV due to the uncertainty in $\phi_0(s)$ above s_K.

• One can discern among different $I = 0$ S-wave $\pi\pi$ parameterizations when new and more precise data become available.

• We have handled with *three* subtractions constants (more precisision) instead of the *two* used previously in the literature.

• The method is also adequate to study the $f_0(980)$ resonance.
L. Roca, C. Schat and J.A.O. , to appear soon.

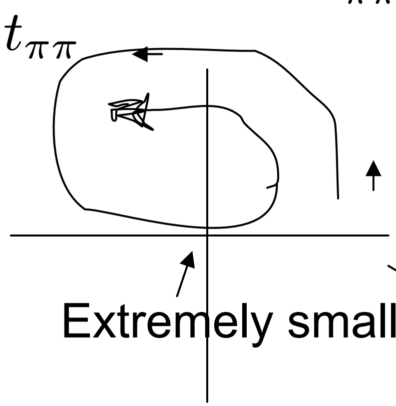
• $\Gamma(\sigma \to \pi\pi) = 1.5 \pm 0.3$ KeV
Non $q\bar{q}$ resonance.

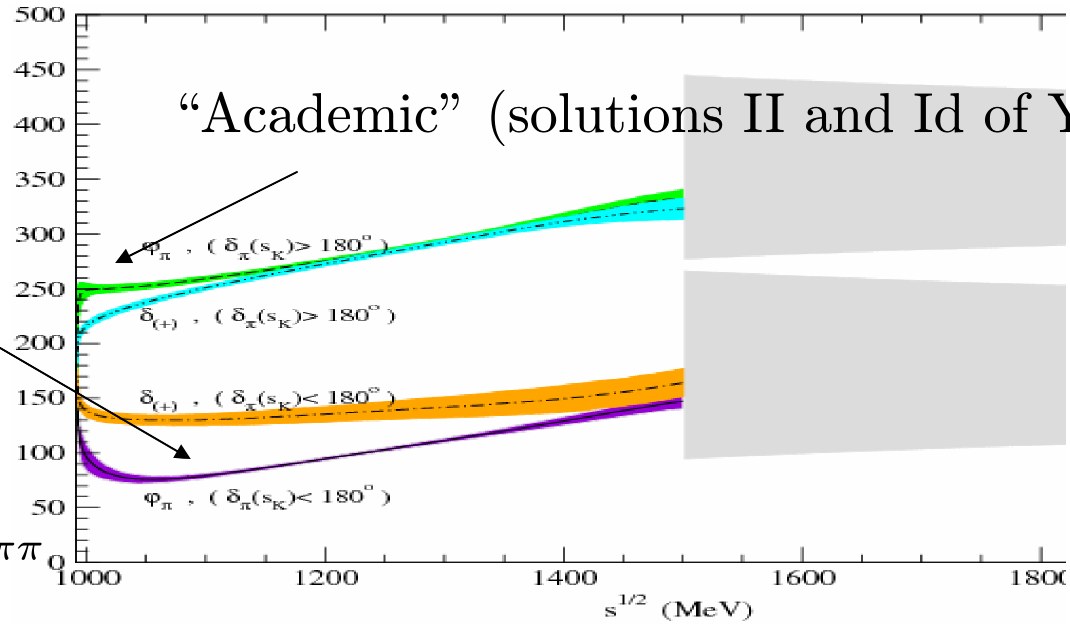
6. Multipion states

$$t_{\pi\pi} = \frac{1}{2}\eta \sin 2\delta_\pi + \frac{i}{2}(1 - 2\eta \cos 2\delta_\pi)$$

As we approach the $f_0(980)$ (below the $K\overline{K}$ threshold)
$\eta < 1$ as it strengthens the transitions among channels 2π, 4π, 6π.

Then $\mathrm{Im}t_{\pi\pi} > 0$ and the phase does not cross π



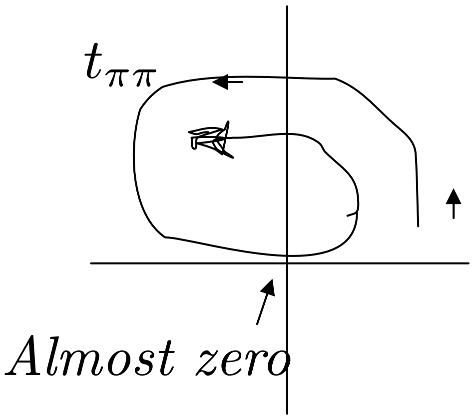


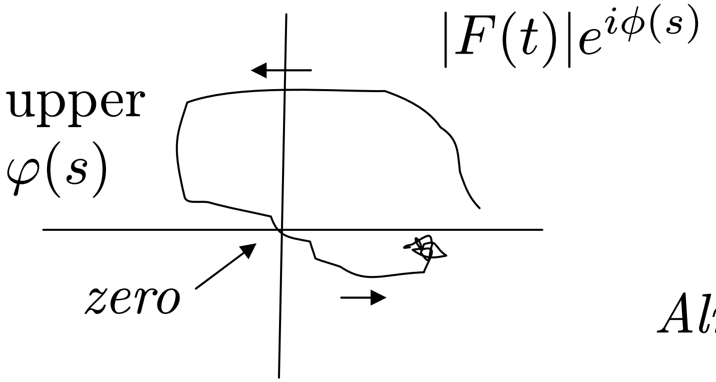
"Academic" (solutions II and Id of Y2)

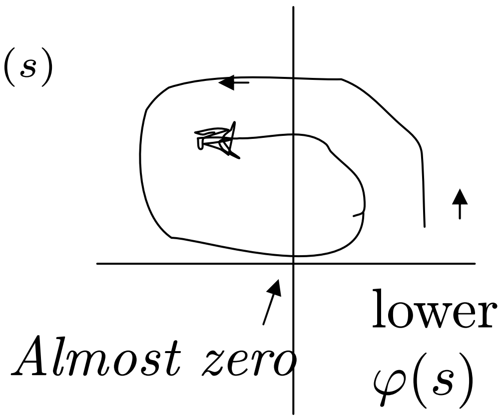
The "true" phase of $t_{\pi\pi}$

Is there $\phi(s)$ for $F(t)$ following the upper $\varphi(s)$?

Maybe, Yes



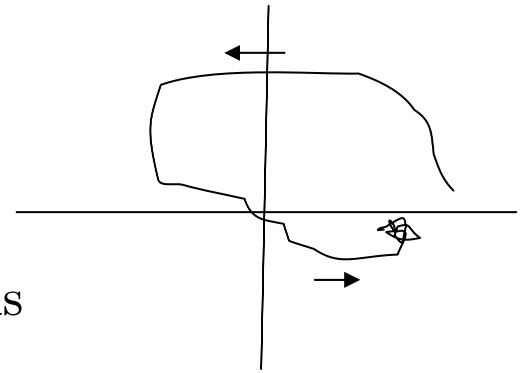




What we learned when considering only the two channels $\pi\pi$ and $K\bar{K}$

$$F(t) \simeq G(t) t_{\pi\pi}$$

Multipion states could i)change the sign of Im$F(t)$, as it is so small, and ii)displace the point where the phase of $F(t)$ crosses π. There would be no zero. In principle,this would resemble to Ynduráin's solutions



Note that $\mathrm{Im}F(t)$ can change sign
This is not the case for $t_{\pi\pi}$ because of unitarity
$\mathrm{Im}t_{\pi\pi} \geq 0$

However, $F(t)$ would then develop almost a pole (strong maximum) and then we pass to an unacceptable situation from the point of view of the starting hypothesis of the perturbative effect of multipion states.

This is remedied if one introduces a zero at s_1 and then, one comes back again to the OR solution with a zero.

Differences with the strange scalar form factor

$$\mathrm{Im}F_i = \sum_{j=1}^{2} F_j \rho_j \theta(t - s'_j) t^*_{ji} \;, \qquad \text{Unitarity}$$

A general solution to the previous equations is given by

$$F = T\,G \;,\; F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \;,\quad G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$$

G_1 corresponds to “Pion production” and G_2 to “Kaon production”

Gs are free of right hand cut   If $|G_1| >> |G_2|$ then $F_1(t) = G_\pi t_{\pi\pi}$

If $|G_2| >> |G_1|$ (OZI rule for $\bar{s}s$ and after diagnolazing

$F_\pi(t) = -\cos\theta\,\sin\theta\,\rho_2^{-1/2}\rho_1^{-1/2}\,G_2(\widetilde{t}_{11} - \widetilde{t}_{22})$

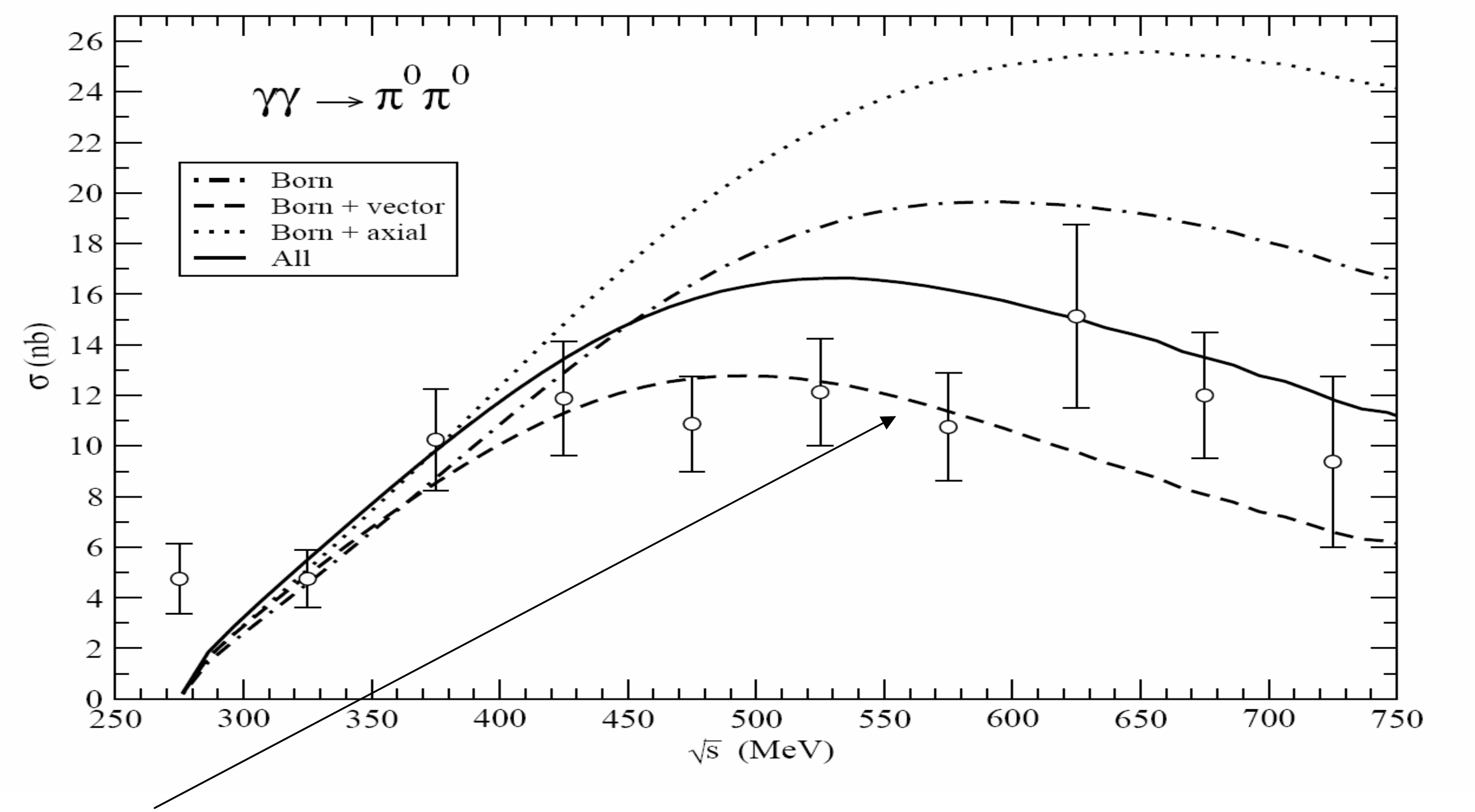
This invalidates our arguments above since for $s_1 \to s_K^+$

boht $\widetilde{t}_{11}$ and $\widetilde{t}_{22}$ tend to zero and $\phi(s)$

is then not given then by $\delta_{(+)}$.

Also for $|G_2| >> |G_1|$ then $|\Gamma'_2/\Gamma'_1| \simeq |\widetilde{t}_{11} \tan\theta / \widetilde{t}_{22}|$
For typical values $|\widetilde{t}_{11}/\widetilde{t}_{22}| \simeq 1$ then
$|\Gamma'_2/\Gamma'_1| \simeq |\tan\theta| < 1$

Different $L_I(s)$ contributions



Pennington overlooked the 1^{++} and 1^{+-} axial vector exchanges altogether

Regarding the $f_0(980)$

With the original approach of Pennington and Morgan it is a metter of fine tuning

$$F_0(s) = L_0(s) + c_0 s \omega_0(s) + \frac{s^2}{\pi}\int_{4m_\pi^2}^{\infty} \frac{L_0(s')\sin\phi_0(s')}{s'^2(s'-s)|\omega_0(s')|} ds'$$

c_0 is fixed from the position of the Adler zero in $F_N(s)$ at m_π^2, $m_\pi^2/2$ or $2M_\pi^2$

$$c_0 + \frac{s_1}{\pi}\int_{4m_\pi^2}^{\infty} \frac{L_0(s')\sin\phi_0(s')}{s'^2(s'-s_1)|\omega_0(s')|} \simeq 0 \; .$$

Then $\phi_0(s)$ must be precisely given such that this cancellation occurs
But $\phi_0(s)$ is not precisely known for $s > s_K$

In our approach one does not need to impose such specific knowledge of $\phi_0(s)$ for $s > s_K$
The $f_0(980)$ is isolated in the last term and its size controlled by $F_0(s_1) - L_0(s_1)$

Y1 $\langle r^2\rangle_s^{K\pi} = 0.31 \pm 0.06\ \mathrm{fm}^2$

- CHPT to one loop, $\langle r^2\rangle_s^{K\pi} = 0.20 \pm 0.05\ \mathrm{fm}^2$

- Y1 ignored the recent theoretical advances in the $K\pi$ scalar form factor
Jamin,JAO,Pich NPB622,279(2002); JHEP02,047(2006);
($K\pi$, $K\eta$, $K\eta'$ MO+CHPT) $\langle r^2\rangle_s^{K\pi} = 0.192 \pm 0.012\ \mathrm{fm}^2$

- Recent experiments in $K_{\ell 3}$ corroborate our value:
Charged kaons, Yushchenko et al, PLB581,31(2004).
$\langle r^2\rangle_s^{K^\pm\pi} = 0.235 \pm 0.014 \pm 0.007\ \mathrm{fm}^2$
Neutral kaons, Alexopoulos et al [KTeV Coll.] PRD70, 092007 (2004)
$\langle r^2\rangle_s^{K^\pm\pi} = 0.165 \pm 0.016\ \mathrm{fm}^2$
Our value (isospin limit) lies in the middle

The last remarks were pointed out in Ananthanarayan, Caprini, Colangelo, Gasser and Leutwyler, PLB602,218(2004).

The controversy about $\langle r^2\rangle_s^{K\pi}$ is over.

We follow here Y2 and diagonalize the 2×2 S-matrix.
We also apply it to calculate inelasticity errors.
We give the expressions directly in terms of observables.

$$T = \begin{pmatrix} \frac{1}{2i}(\eta e^{2i\delta_\pi} - 1) & \frac{1}{2}\sqrt{1-\eta^2} e^{i(\delta_\pi + \delta_K)} \\ \frac{1}{2}\sqrt{1-\eta^2} e^{i(\delta_\pi + \delta_K)} & \frac{1}{2i}(\eta e^{2i\delta_K} - 1) \end{pmatrix} \quad \text{Diagonalization}$$

Orthogonal Matrix C $\qquad C = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$$\cos\theta = \frac{\left[(1-\eta^2)/2\right]^{1/2}}{\left[1 - \eta^2\cos^2\Delta - \eta|\sin\Delta|\sqrt{1-\eta^2\cos^2\Delta}\right]^{1/2}},$$

$$\sin\theta = -\frac{\sin\Delta}{\sqrt{2}} \frac{\eta - \sqrt{1+(1-\eta^2)\cot^2\Delta}}{\left[1 - \eta^2\cos^2\Delta - \eta|\sin\Delta|\sqrt{1-\eta^2\cos^2\Delta}\right]^{1/2}},$$

$\sin\theta \to 0$ as $\sqrt{(1-\eta)/2}$ for $\eta \to 1$

Eigenvalues

$\delta_{(+)}$ follows rather closely $\varphi(s)$

$$e^{2i\delta_{(+)}} = S_{11}\frac{1+e^{2i\Delta}}{2}\left[1-\frac{i}{\eta}\tan\Delta\sqrt{1+(1-\eta^2)\cot^2\Delta}\right]$$

$$e^{2i\delta_{(-)}} = S_{22}\frac{1+e^{-2i\Delta}}{2}\left[1+\frac{i}{\eta}\tan\Delta\sqrt{1+(1-\eta^2)\cot^2\Delta}\right]$$

One has then two channels diagonalized that are ***elastic***

$$\Gamma' \equiv \begin{pmatrix} \Gamma'_1 \\ \Gamma'_2 \end{pmatrix} = C^T Q^{1/2} F = C^T Q^{1/2} \begin{pmatrix} F_\pi \\ F_K \end{pmatrix}$$

$$F_\pi = q_\pi^{-1/2}\left(\lambda\cos\theta\,|\Gamma'_1|e^{i\delta_{(+)}} \pm \sin\theta\,|\Gamma'_2|e^{i\delta_{(-)}}\right)$$

$$F_K = q_K^{-1/2}\left(\pm\cos\theta\,|\Gamma'_2|e^{i\delta_{(-)}} - \lambda\sin\theta\,|\Gamma'_1|e^{i\delta_{(+)}}\right)$$

If $\delta_\pi(s_K) \geq \pi$ one has the zero at $s_1^{1/2} < 2m_K$, this introduces a minus sign due to the prefactor $s_1 - t$. $\lambda = (-1)^{\theta(\delta_\pi(s_K)-\pi)}$

Notice that Γ'_2 is 0 at s_K, this is why we cannot fix the $\pm$ in front of $|\Gamma'_2|$

Shift in $\delta_{(+)}$ because of inelasticity

$$F_\pi = \lambda \cos\theta \, |\Gamma'_1| e^{i\delta_{(+)}} (1+\epsilon \cos\theta) \left(1 + i \frac{\epsilon \sin\rho}{1+\epsilon \cos\rho}\right) \quad \rho = \delta_- - \delta_+$$

With $\epsilon = \pm \tan\theta \left|\frac{\Gamma'_2}{\Gamma'_1}\right|$ $\qquad |\Gamma'_2/\Gamma'_1| \lesssim |\widetilde{t}_{11} \tan\theta/\widetilde{t}_{22}| \simeq |\tan\theta| < 1$

$\tan\theta \to 0$ when $\eta \to 1$. First order correction to $\delta_{(+)}$

$$1 + i \frac{\epsilon \sin\rho}{1+\epsilon \cos\rho} = \exp\left(i \frac{\epsilon \sin\rho}{1+\epsilon \cos\rho}\right) + \mathcal{O}(\epsilon^2) \qquad \delta_{(+)} \to \delta_{(+)} + \frac{\epsilon \sin\rho}{1+\epsilon\cos\rho}$$

- $1.1 \le s^{1/2} \le 1.5$ GeV, $\eta \simeq 1$ experimentally (Hyams, Grayer). Typically $\eta \gtrsim 0.8$ Then $\epsilon \simeq 0.3$.

$\delta_\pi(s_K) < \pi$	$\delta_\pi(s_K) \ge \pi$
$\delta_{(+)} \ge 3\pi/4$	$\delta_{(+)} \ge 3\pi/2$
Correction: $6\% \times 2 \to 12\%$	$12\% \times 2 \to 25\%$

Coupling $\bar{\ell}_4$

$$\langle r^2 \rangle_s^\pi = \frac{3}{8\pi^2 f_\pi^2}\left(\bar{\ell}_4 - \frac{13}{12} + \underbrace{\Delta_r \frac{M_\pi^2}{(4\pi f_\pi)^2}}_{\text{two loops}}\right)$$

With our value for $\langle r^2 \rangle_s^\pi = 0.63 \pm 0.05$

$\bar{\ell}_4 = 4.7 \pm 0.3$ one loop

$\bar{\ell}_4 = 4.5 \pm 0.3$ two loops (taking for $\bar{\ell}_1$, $\bar{\ell}_2$ and $\bar{\ell}_3$ values of CGL) and solving for $\bar{\ell}_4$

CGL: $\bar{\ell}_4 = 4.4 \pm 0.2$

Y1: $\bar{\ell}_4 = 5.4 \pm 0.5$ (One loop $\langle r^2 \rangle_s^\pi(\bar{\ell}_4)$)

(Y3 took $\bar{\ell}_4 = 5 \pm 1$ because of the spurious reasons given above)

We have employed the two loop relation above with $\bar{\ell}_1$,$\bar{\ell}_2$ and $\bar{\ell}_3$ from CGL. Then Y1 values reduces to $\bar{\ell}_4 = 5.0 \pm 0.4$

There is agreement at the level of one sigma