Lectures on Nuclear and Hadron Physics

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Outline Lecture I

1. Introduction
2. Dispersion Relations
3. S-matrix
4. Scalar Form Factor of the Pion
5. $\gamma\gamma \to \pi^0\pi^0$

Introduction

In the chiral limit $m_u = m_d = m_s = 0$ the QCD Lagrangian is invariant under $U_L(3) \otimes U_R(3)$ symmetry in the quark flavours or types.

$SU_L(3) \otimes SU_R(3) \rightarrow SU_V(3)$ is Spontaneously Broken.

Goldstone bosons appear π, K, η
Mass gap:
$M_\pi \sim 140$ MeV $\ll M_\rho \sim 770$ MeV
$M_K \sim 496$ MeV
$M_\eta \sim 550$ MeV

Conservation of axial-vector current: $\partial_\mu \left(\bar{q}\gamma^\mu\gamma_5\lambda^a q\right) = 0$,

$$\langle 0|\bar{q}\gamma^\mu\gamma_5\lambda^a q|\pi^b(p)\rangle = i\,\delta^{ab}F_\pi\, p^\mu$$

$$\partial_\mu\langle 0|\bar{q}\gamma^\mu\gamma_5\lambda^a q|\pi^b(p)\rangle = i\,\delta^{ab}F_\pi\, p^2 = 0$$

$$p^2 = 0 \rightarrow M_b^2 = 0$$

$L + R$ corresponds to $\bar{q}\gamma^\mu q$
$U_V(1) \equiv U_{L+R}$ Conserved Baryon Number.

$L - R$ corresponds to $\bar{q}\gamma^\mu \gamma_5 q$
$U_A(1) \equiv U_{L-R}$ Neither Conserved nor Goldstone Boson.

Puzzle: (if $U_A(1)$ were spontaneously broken)
Goldstone mode: There should be an η_0 with a mass $< \sqrt{3} m_\pi$
but η is much heavier Weinberg PRD'75

QCD Anomaly

The ninth axial singlet current has an anomalous divergence Adler PR'69 Fujikawa PRD'80

$$J_5^{\mu\,(0)} = \bar{q}\gamma_\mu\gamma_5 q$$
$$\partial_\mu J_5^{\mu\,(0)} = \frac{g^2}{16\pi^2}\frac{1}{N_c} Tr_c(G_{\mu\nu}\widetilde{G}^{\mu\nu})$$

Large N_c QCD 't Hooft NPB'74, Witten NPB'79 $N_c \to \infty$, $g^2 N_c \to constant$

$U_L(N_F) \otimes U_R(N_F) \to U_{L+R}(N_F)$ Coleman, Witten PRL'80
Entire Nonet of Goldstone bosons results.

Dispersion Relations

Causality conditions → Dispersion Relations

- Analytical properties of transient amplitudes

H. A. Kramers, Atti Cong. Intern. Fisica, Como, **2** (1927) 545;
R. Kronig, Opt. Soc. Amer. **12** (1926) 547.
Scattering of light by a dispersive medium.
The name 'dispersion relation' derives from this first application.
CAUSAL STATEMENT: **Polarization** $P(t)$

$$P(t) = 0 \text{ for } \ t < 0$$

Fourier Transform $E(\nu) = n(\nu) - 1$:

$$E(\nu) = 4\pi \int_{-\infty}^{+\infty} dt P(t) e^{i\nu t} \ .$$

Dispersion Relation:

$$\Re E(\nu) = \frac{1}{\pi} P \int_{-\infty}^{+\infty} d\nu' \frac{\Im E(\nu')}{\nu' - \nu} \ .$$

Proof: Essential point

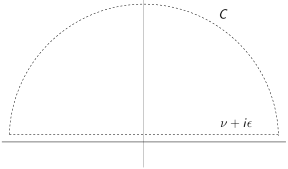
$$\nu = \nu_1 + i\,\nu_2, \;\; \nu_2 > 0 \ ,$$

$$E(\nu) = 4\pi \int_{-\infty}^{+\infty} dt P(t) \exp[i\nu_1 t - \nu_2 t] \ .$$

It converges because $P(-\infty) = 0$.

- **Cauchy's theorem:** (Assume $E(\nu) \rightarrow 0$ for $\nu \rightarrow \infty$)

$$E(\nu + i\epsilon) = \frac{1}{2\pi i} \int_C d\nu' \frac{E(\nu')}{\nu' - \nu - i\epsilon} \quad (1)$$



$$0 = \frac{1}{2\pi i}\int_C d\nu' \frac{E(\nu')}{\nu' - \nu + i\epsilon} \quad \overset{\text{Complex Conjugate}}{\longrightarrow} \quad 0 = \frac{1}{2\pi i}\int_C d\nu' \frac{E^*(\nu')}{\nu' - \nu - i\epsilon} \tag{2}$$

- Subtracting Eqs.(1), (2):

$$E(\nu) = \lim_{\epsilon \to 0^+} \frac{1}{\pi}\int_{-\infty}^{+\infty} d\nu' \frac{\Im E(\nu')}{\nu' - \nu - i\epsilon} \tag{3}$$

Adding Eqs.(1), (2):

$$E(\nu) = \lim_{\epsilon \to 0^+} \frac{-i}{\pi}\int_{-\infty}^{+\infty} d\nu' \frac{\Re E(\nu')}{\nu' - \nu - i\epsilon} \tag{4}$$

Taking into account

$$\int_{-\infty}^{+\infty} dx \frac{g(x)}{x - x_0 \pm i\epsilon} = P\int_{-\infty}^{+\infty} dx \frac{g(x)}{x - x_0} \mp i\pi g(x_0)$$

From Eqs. (3) and (4), respectively:

$$\Re E(\nu) = P\frac{1}{\pi}\int_{-\infty}^{+\infty} d\nu' \frac{\Im E(\nu')}{\nu' - \nu} \ , \qquad (5)$$

$$\Im E(\nu) = P\frac{1}{\pi}\int_{-\infty}^{+\infty} d\nu' \frac{\Re E(\nu')}{\nu' - \nu} \ . \qquad (6)$$

Eq. (5) is by far much more useful.
It expresses the Dispersive part ($\Re E(\nu)$) in terms of the Absorptive part ($\Im E(\nu)$), once it is known for all frequencies.

The same results are obtained directly from

$$E(\nu + i\epsilon) = \frac{1}{2\pi i}\int_C d\nu' \frac{E(\nu')}{\nu' - \nu - i\epsilon}$$

Applying

$$\frac{1}{x - x_0 - i\epsilon} \to P\int \frac{dx}{x - x_0} + i\pi\delta(x - x_0) \ .$$

Subtractions

Let's relax the assumption $E(\nu) \to 0$ for $\nu \to \infty$

- Now:

$$\lim_{\nu \to \infty} \frac{E(\nu)}{\nu} = 0 \tag{7}$$

One Subtraction

$$E(\nu + i\epsilon) - E(\nu_0 + i\epsilon) = \frac{1}{2\pi i} \int_C d\nu' \left[\frac{E(\nu')}{\nu' - \nu - i\epsilon} - \frac{E(\nu')}{\nu' - \nu_0 - i\epsilon} \right]$$
$$= \frac{\nu - \nu_0}{2\pi i} \int_C d\nu' \frac{E(\nu')}{(\nu' - \nu - i\epsilon)(\nu' - \nu_0 - i\epsilon)}$$

In the same way as before:

$$\Re E(\nu) - \Re E(\nu_0) = P \frac{1}{\pi} \int_{-\infty}^{+\infty} d\nu' \left[\frac{\Im E(\nu')}{\nu' - \nu} - \frac{\Im E(\nu')}{\nu' - \nu_0} \right]$$
$$= \frac{\nu - \nu_0}{\pi} P \int_{-\infty}^{+\infty} d\nu' \frac{\Im E(\nu')}{(\nu' - \nu)(\nu' - \nu_0)}$$

The addition of further subtractions is iterative. Assume:

$$\lim_{\nu\to\infty} \frac{E(\nu)}{\nu^2} = 0 \ .$$

Two subtractions: One considers

$$E(\nu) - E(\nu_0) - \frac{\nu-\nu_0}{\nu_1-\nu_0}\left\{E(\nu_1)-E(\nu_0)\right\}$$

At $\nu=\nu_0 \rightarrow E(\nu_0)$. At $\nu=\nu_1 \rightarrow E(\nu_1)$.

$$\begin{aligned}
&E(\nu) - E(\nu_0) - \frac{\nu-\nu_0}{\nu_1-\nu_0}\left\{E(\nu_1)-E(\nu_0)\right\} \\
&= \frac{1}{2\pi i}\int_C d\nu' E(\nu')\left[\frac{1}{\nu'-\nu}-\frac{1}{\nu'-\nu_0}-\frac{\nu-\nu_0}{\nu_1-\nu_0}\left(\frac{1}{\nu'-\nu_1}-\frac{1}{\nu'-\nu_0}\right)\right] \\
&= \frac{(\nu-\nu_0)(\nu-\nu_1)}{2\pi i}\int_C d\nu' \frac{E(\nu')}{(\nu'-\nu)(\nu'-\nu_0)(\nu'-\nu_1)} \ .
\end{aligned}$$

$$\Re E(\nu) - \Re E(\nu_0) - \frac{\nu - \nu_0}{\nu_1 - \nu_0} \{\Re E(\nu_1) - \Re E(\nu_0)\}$$
$$= \frac{(\nu - \nu_0)(\nu - \nu_1)}{\pi} P \int_C d\nu' \frac{\Im E(\nu')}{(\nu' - \nu)(\nu' - \nu_0)(\nu' - \nu_1)} \ .$$

Dispersion relations were applied to particle physics following a suggestion by R. Kronig, Physica **12** (1946) 543:

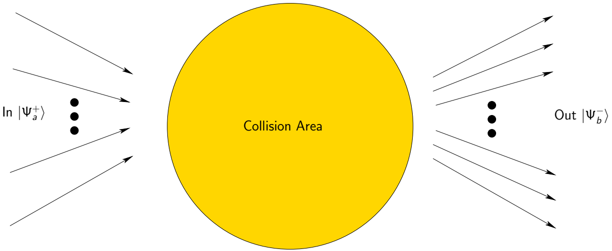
Causality requirements should be added to the usual conditions on the S-matrix, namely, Lorentz invariance and unitarity
M. Gell-Mann, M. L. Goldberger and W. E. Thirring, Phys. Rev. **95** (1954) 1612
The causality requirement: **"the commutator of two Heisenberg operators for the field in question shall vanish if the operators are taken at space-like points."**

$$[\phi(x_1), \phi(x_2)] = 0 \text{ for } (x_1 - x_2) \text{ space-like}$$

They demonstrated that the amplitude for the forward scattering of photons by nucleons satisfies:

$$\Re A(\nu) = \frac{1}{\pi} \int_{-\infty}^{+\infty} d\nu' \frac{\Im A(\nu')}{\nu' - \nu}$$

S-matrix



$|\Psi^\pm\rangle$ are eigenstates of the full Hamiltonian H in Heisenberg picture.

$$|\Psi_a^\mp\rangle = \lim_{t\to\pm\infty} U(0,t)|\phi_a\rangle$$

With the $|\phi_a\rangle$ *free states*. $U(t_2, t_1)$ is the evolution operator in the interaction picture.

$$S_{ba} = \langle \Psi_b^- | \Psi_a^+ \rangle = \lim_{\substack{t_1 \to -\infty \\ t_2 \to +\infty}} \langle \phi_b | U(t_2, t_1) | \phi_a \rangle$$

S-matrix is a Unitary Operator

$$S \cdot S^\dagger = S^\dagger S = I$$

T-matrix: $S = 1 + iT$

Unitarity: $S^\dagger \cdot S = 1$

$$\langle \phi_b | T | \phi_a \rangle - \langle \phi_b | T^\dagger | \phi_a \rangle = i \sum_c \int d\mathcal{Q}_c \langle \phi_b | T^\dagger | \phi_c \rangle \langle \phi_c | T | \phi_a \rangle \; .$$

In the sum only those states which are open contribute!

Phase space

$$d\mathcal{Q}_\alpha = \Pi_{i=1}^{n_\alpha} \int \left[\frac{d^3 p_i}{2 p_i^0 (2\pi)^3} \right] (2\pi)^4 \delta(P_b - P_a) \equiv d\alpha (2\pi)^4 \delta(P_\alpha - P_a)$$

Forward scattering

Very useful case: $a = b$

$$\Im T_{aa} = \frac{1}{2} \sum_c \int dQ_c |\langle \phi_c | T | \phi_a \rangle|^2$$

It provides the imaginary part for positive energy above threshold

Unitarity: $S \cdot S^\dagger = 1$

$$\Im T_{aa} = \frac{1}{2} \sum_c \int dQ_c |\langle \phi_a | T | \phi_c \rangle|^2$$

Important result:

$$\sum_c \int dQ_c |\langle \phi_c | T | \phi_a \rangle|^2 = \sum_c \int dQ_c |\langle \phi_a | T | \phi_c \rangle|^2$$

Rate of all reactions produced by an initial state α

$$\Gamma_\alpha = \frac{1}{c_\alpha} \sum_\beta \int d\mathcal{Q}_\beta |T_{\beta\alpha}|^2$$

From unitarity we then obtain

$$\sum_\beta \int d\beta c_\alpha \frac{d\Gamma(\alpha \to \beta)}{d\beta} = \sum_\beta \int d\beta c_\beta \frac{d\Gamma(\beta \to \alpha)}{d\beta}$$

$P_\alpha d\alpha$ Probability to find the system within volume $d\alpha$

$$\frac{dP_\alpha}{dt} = \sum_\beta \int d\beta P_\beta \frac{d\Gamma(\beta \to \alpha)}{d\alpha} - P_\alpha \sum_\beta \int d\beta \frac{\Gamma(\alpha \to \beta)}{d\beta}$$

Boltzmann H-theorem

Entropy: $-\int d\alpha P_\alpha \ln(P_\alpha / c_\alpha)$
It follows also from forward scattering unitarity
S. Weinberg, The Quantum Field Theory of Fields. Vol. II:

$$-\frac{d}{dt}\int d\alpha P_\alpha \ln(P_\alpha / c_\alpha) \geq 0$$

Partial waves

Unitarity is most simply expressed in terms of Partial Waves
Lacour, Meißner, J.A.O, Ann. Phys. 326 (2011) 241

$$|\mathbf{p},\sigma_1\sigma_2\rangle = \sqrt{4\pi}\sum_{J,\ell,S,m}(\sigma_1\sigma_2 s_3|s_1 s_2 S)(m s_3 \mu|\ell S J)Y_\ell^m(\hat{\mathbf{p}})^*|J\,\mu\,\ell\,S\,s_1\,s_2\rangle$$

Two-particle scattering

$$a(p_1)+b(p_2)\longrightarrow c(p_3)+d(p_4)$$

Spinless case

$$T_\ell(s)=\frac{1}{2}\int_{-1}^{+1}d\cos\theta\,P_\ell(\cos\theta)\,T(s,\cos\theta)$$

$$\Im T_\ell(s)_{fi}=\sum_a T_\ell(s)_{ia}\,T_\ell(s)^*_{fa}\frac{|\mathbf{p}|_a}{8\pi\sqrt{s}}$$

Crossing

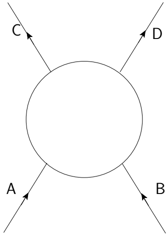
With unitarity alone we do not posses enough information for the standard dispersion relation ($\nu' > 0$)

$$\Re A(\nu) = \frac{1}{\pi} \int_{-\infty}^{+\infty} d\nu' \frac{\Im A(\nu')}{\nu' - \nu}$$

since $\nu' < 0$ is also required.

Relativistic quantum field theory $\rightarrow$ **Crossing Symmetry**.

Another condition that together with unitarity and dispersion relations allows dynamical calculations



$$A + B \rightarrow C + D \quad s\text{-channel}$$

$$A + \overline{C} \rightarrow \overline{B} + D \quad t\text{-channel}$$

$$A + \overline{D} \rightarrow \overline{B} + C \quad u\text{-channel}$$

$s = (p_A + p_B)^2$, $t = (p_A - p_C)^2$, $u = (p_A - p_D)^2$

Mandelstam Variables: Only two are independent
$s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2$

$$A(p_A) + B(p_B) \to C(p_C) + D(p_D) \quad s\text{-channel}$$
$$A(p_A) + \overline{C}(-p_C) \to \overline{B}(-p_B) + D(p_D) \quad t\text{-channel}$$
$$A(p_A) + \overline{D}(-p_D) \to \overline{B}(-p_B) + C(p_C) \quad u\text{-channel}$$

Particle→Antiparticle and reverse four-momentum
The amplitudes for the three reactions are given by the same analytic function
Physical processes correspond to certain values of s, t (disconnected physical regions)

$s_c \longrightarrow u$

of these low angular momentum states would be done in the same spirit as the Chew-Low calculations, and the details will not be given in this paper. We thus have a procedure in which the first few angular momentum states are calculated by methods similar to those used in the static theory, while the remaining part of the scattering amplitude, which will be called the "residual part," is calculated by a different procedure which does not make use of a partial-wave expansion. Needless to say, the two parts of the calculation become intermingled by the iteration procedure.

It is only in the calculation of the subtraction terms that use has to be made of the unitarity condition for the pair-annihilation reaction. For the residual part, it is only necessary to use the unitarity condition for pion-nucleon scattering. Had it been possible to use the unitarity condition exactly instead of in the one-meson approximation, the result would also satisfy the unitarity condition for the annihilation reaction in a consistent theory. As it is, we find that the residual part consists of a number of terms which correspond to various intermediate states in the annihilation reaction. In Sec. 5 it is pointed out that the calculation is greatly simplified if we keep only those terms of the residual part corresponding to pair annihilation through states with fewer than a certain number of particles. Such an approximation has already been made in calculating the subtraction terms. The unitarity condition for pion-nucleon scattering is no longer satisfied except for the low angular momentum states. However, the terms neglected are of the order of magnitude of, and probably less than, terms already neglected. The

et al.[1] The momenta of the incoming and outgoing pions will be denoted by q_1 and q_2, those of the incoming and outgoing nucleons by p_1 and p_2. We can then define two invariant scalars

$$\nu = -(p_1+p_2)(q_1+q_2)/4M, \quad (2.1)$$

$$t = -(q_1-q_2)^2. \quad (2.2)$$

The latter is minus the square of the invariant momentum transfer. The laboratory energy will be given by the equation

$$\omega = \nu - (t/4M). \quad (2.3a)$$

It is more convenient to use, instead of the laboratory energy, the square of the center-of-mass energy (including both rest-masses), which is linearly related to it by the equation

$$s = M^2 + \mu^2 + 2M\omega. \quad (2.3b)$$

The Green's function relevant to the process under consideration,

$$\pi_1 + N_1 \rightarrow \pi_2 + N_2, \quad \text{(I)}$$

also gives the processes

$$\pi_2 + N_1 \rightarrow \pi_1 + N_2 \quad \text{(II)}$$

and

$$N_1 + \bar{N}_2 \rightarrow \pi_1 + \pi_2. \quad \text{(III)}$$

The matrix elements for the process II can be obtained from those for the process I by crossing symmetry; the laboratory energy and the square of the center-of-mass energy will now be

reaction. Lines of constant energy for the reaction III are horizontal lines. The reaction will be energetically possible above the line EF, at which $t = 4M^2$, and again the shaded area represents the physical region.

We now examine the analytic properties of the scattering amplitude. To simplify the writing, we shall first neglect spin and isotopic spin; the transition amplitude will then be a scalar function $A(\nu,t)$ of the two invariants ν and t. Its analytic properties as a function of ν, with t constant, are exhibited by the usual dispersion relations

$$A(\nu,t) = \frac{g^2}{2M}\left(\frac{1}{\nu_B-\nu} + \frac{1}{\nu_B+\nu}\right) + \frac{1}{\pi}\int_{\mu+(t/4M)}^{\infty} d\nu' \frac{A_1(\nu',t)}{\nu'-\nu}$$
$$- \frac{1}{\pi}\int_{-\infty}^{-\mu-(t/4M)} d\nu' \frac{A_2(\nu',t)}{\nu'-\nu}, \quad (2.5)$$

where $\nu_B = -(\mu^2/2M) + (t/4M)$. In this and all subsequent such equations, the energy denominators are taken to have a small imaginary part. A_1 and A_2 are the "absorptive parts" associated with the reactions I and II, respectively, and are given by the equations

$$(2\pi)^4 A_1(\nu,t)\delta(p_1+q_1-p_2-q_2) = (2\pi)^6\left(\frac{4p_{01}p_{02}q_{01}q_{02}}{M^2}\right)^{\frac{1}{2}}$$
$$\times \sum_n \langle N(p_1)\pi(q_1)|n\rangle\langle n|N(p_2)\pi(q_2)\rangle, \quad (2.6)$$

$$(2\pi)^4 A_2(\nu,t)\delta(p_1+q_1-p_2-q_2) = (2\pi)^6\left(\frac{4p_{01}p_{02}q_{01}q_{02}}{\ldots}\right)^{\frac{1}{2}}$$

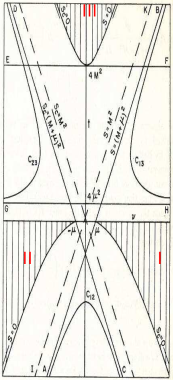


FIG. 1. Kinematics of the reactions I, II, and III.

the possibility of having to perform subtractions is implied.

E.g.

$$N\pi^0 \to N\pi^0$$

s- and u-channel processes are the same
For forward scattering:

$$\Re A(-\nu) = \Re A(\nu)$$
$$\Im A(-\nu) = -\Im A(\nu)$$

$$\Re A(\nu) = \int_{-\infty}^{+\infty} d\nu' \frac{\Im A(\nu')}{\nu' - \nu}$$

We have the needed information to apply the dispersion relation
$\nu' \in]-\infty, -m_\pi]$ and $[m_\pi, +\infty]$

Additionally one also has the one-nucleon pole

$$\frac{g^2}{2m_N} \frac{1}{\nu - m_\pi^2/(2m_N)}$$

The application of crossing is not always so straightforward for obtaining the input in the dispersion relation.

Typical example is fixed t dispersion relations

$$t = -2q^2(1-\cos\theta)$$
$$q^2 = \frac{(s'-(m_N+m_\pi)^2)(s'-(m_N-m_\pi)^2)}{4s'}$$

As s' moves as integration variable $\cos\theta$ will be less than -1 for $q^2 \longrightarrow 0$ (near threshold)

The extension to unphysical values of $\cos\theta$ can be done making use of the partial wave expansion.

However, its convergence its restricted to the Lehmann ellipse
H. Lehmann, Nuovo Cim. Suppl. **14** (1959) 153

πN scattering

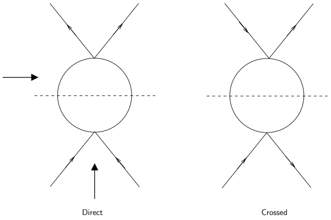
$$0 < -t < \frac{32}{3}\frac{2m_N + m_\pi}{2m_N - m_\pi} m_\pi^2 \sim \frac{32}{3} m_\pi^2$$

The principle of Maximum Analyticity

S. Mandelstam, Rep. Prog. Phys. **25** (1962) 99

"The scattering amplitude is analytic in all its variables except at those points where singularities arise as a consequence of the unitarity condition." (including the crossed channels.)

This is fulfilled to all orders in perturbation theory.



Scalar Form Factor of the Pion

L. Roca and J.A.O., Phys. Lett. B **651** (2007) 139

$$\Gamma_\pi(t) = \int d^4x\, e^{-i(q'-q)x} \langle\, \pi(q')|\left(m_u \bar{u}(x)u(x) + m_d \bar{d}(x)d(x)\right)|\pi(q)\,\rangle \;,$$

$$t = (q'-q)^2$$

Quadratic scalar radius of the pion $\langle\, r^2 \,\rangle_s^\pi$

$$\Gamma_\pi(t) = \Gamma_\pi(0)\left\{1 + \frac{1}{6} t \langle\, r^2 \,\rangle_s^\pi + \mathcal{O}(t^2)\right\}$$

• $\langle\, r^2 \,\rangle_s^\pi$ contributes 10% to the a_0^0 and a_0^2 scattering lengths from Roy equations+CHPT to two loops (2% of precision). It is a big contribution.

G. Colangelo, J. Gasser, H. Leutwyler, Nucl. Phys. B **603** (2001) 125 (CGL)

- It gives $\overline{\ell}_4$ that controls the departure of F_π from its value in the chiral limit

$$\langle\, r^2\,\rangle_s^\pi = \frac{3}{8\pi^2 F_\pi^2}\left(\overline{\ell}_4 - \frac{13}{12}\right)$$
$$F_\pi = F\left\{1 + \frac{m_\pi^2}{16\pi^2 F_\pi^2}\overline{\ell}_4\right\}$$

- Controversy between:

The ‘canonical’ value: $\langle\, r^2\,\rangle_s^\pi = 0.61 \pm 0.04\ \mathrm{fm}^2$
Donoghue, Gasser, Leutwyler, Nucl. Phys. B **343** (1990) 341
Based on solving the two-channel Muskhelishvili-Omnés equations

F. Ynduráin’s approach based on the Omnés representation of $\Gamma_\pi(t)$ and high energy QCD behavior
$\langle\, r^2\,\rangle_s^\pi = 0.75 \pm 0.07\ \mathrm{fm}^2$

- Change in the $\pi\pi$ scattering lengths:

$$\delta a_0^0 = +0.027\Delta_{r^2} \quad \delta a_0^2 = -0.004\Delta_r^2 \quad \langle\, r^2 \,\rangle_s^\pi = 0.61(1+\Delta_r^2)\ \mathrm{fm}^2$$

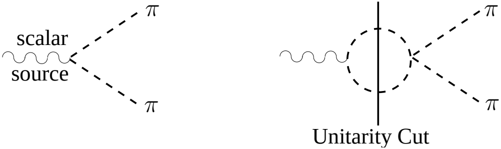
$\Delta_{r^2} = +0.23$

$$\delta a_0^0 = +0.006 \quad \delta a_0^2 = -0.001$$

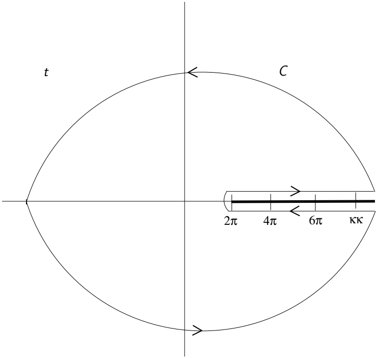
CGL:

$$a_0^0 = 0.220 \pm 0.005 \quad a_0^2 = -0.0444 \pm 0.0010\ \mathrm{fm}$$

precision 2%



$\Gamma_\pi(t)$ only has RHC or Unitarity cut
$\infty > t \geq 4m_\pi^2$



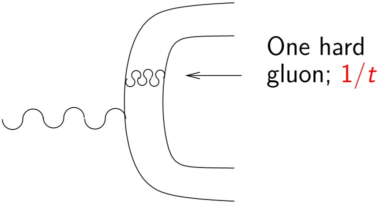
Dispersion relation:

$$\oint_C \frac{\Gamma_\pi(z)}{z-t} = 0$$

$$2\pi i \Gamma_\pi(t) = \int_{4m_\pi^2}^{\infty} ds \left(\frac{\Gamma_\pi(s+i\epsilon)}{s+i\epsilon-t} - \frac{\Gamma_\pi(s-i\epsilon)}{s-i\epsilon-t} \right)$$

Schwartz's reflection principle:
If $f(z)$ is real along an interval of the real axis and is analytic then:
$f(z^*) = f(z)^*$

$$\Gamma_\pi(t) = \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} ds \frac{\Im \Gamma_\pi(s)}{s-t}$$



Brodsky-Farrar QCD counting rules

The dispersion integral is convergent

Omnès representation

Let $F(t)$ an analytic function except for the presence of the RHC

1. Remove the zeroes and poles in $F(t)$ in the complex plane

$$g(t) = \frac{F(t)}{P(t)}$$

$$P(t) = \frac{F(0)}{s_1 \cdots s_n}(s_1 - t)(s_2 - t) \cdots (s_n - t)$$

2. Perform a dispersion relation of

$$f(t) = \log \frac{F(t)}{P(t)}$$

$$f(t+i\epsilon) - f(t-i\epsilon) = \log \frac{F(t+i\epsilon)}{P(t)} - \log \frac{F(t-i\epsilon)}{P(t)}$$

$$= 2i \, arg \frac{F(t+i\epsilon)}{P(t)} \qquad |F(t+i\epsilon)| = |F(t-i\epsilon)|$$

$\phi(t) = arg\frac{F(t)}{P(t)}$

- $\phi(t)$ is continuous
- $\phi(4m_\pi^2) = 0$

$$\log\frac{F(t)}{P(t)} = \gamma_0 + \frac{t}{\pi}\int_{4m_\pi^2}^{\infty} ds\frac{\phi(s)}{s - t - i\epsilon}$$

$$F(t) = P(t)\exp\frac{t}{\pi}\int_{4m_\pi^2}^{\infty} ds\frac{\phi(s)}{s(s - t - i\epsilon)}$$

With n the degree of $P(t)$ then

$$F(t) \to (-1)^n e^{i\phi(+\infty)}\ t^{n-\phi(+\infty)/\pi}$$

$\phi(+\infty) \to (n+1)\pi$ to guarantee that $F(t) \to 1/t$ (QCD)

Watson final state theorem

$$S = 1 + i2\rho(F + T)$$
$$SS^\dagger = 1$$

- Elastic case (only $\pi\pi$ intermediate state)
- Keep only linear terms in F. There is only one source!

$$\Im\Gamma_\pi(t) = \Gamma_\pi(t)\rho(t)T^*_{\pi\pi}(t)$$

Since the $\Im\Gamma_\pi(t)$ is real the phase of $F(t)$ ($\phi(t)$) and of $T_{\pi\pi}(t)$ ($\delta(t)$) are the same (modulo π)

Corolary

$t_{\pi\pi} = \rho T_{\pi\pi} = \sin\delta_\pi \, e^{i\delta_\pi}$, $\delta_\pi(4m_\pi^2) = 0$, δ_π is continuous and at most differs by modulo π from the phase of $T_{\pi\pi}$

This happens when δ_π crosses π ($\sin\delta_\pi < 0$)

$$F(t) = P(t) \exp \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} ds \frac{\phi(s)}{s(s-t)}$$

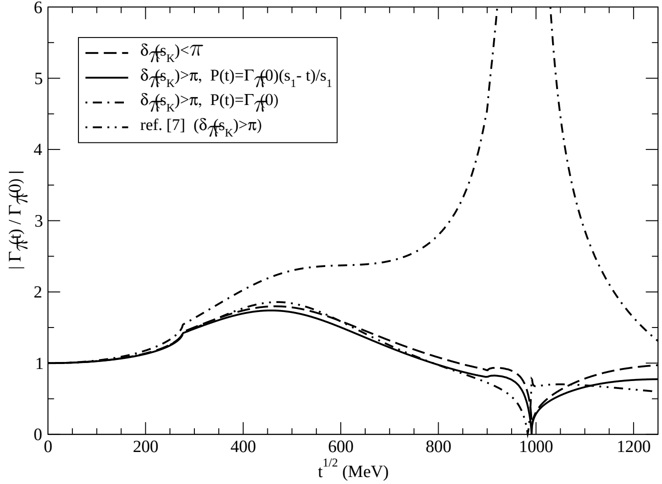
$$\phi(s) = \delta(s) \quad s < 4m_K^2$$

Ynduráin's method: F. Ynduráin, Phys. Lett. B **578** (2004) 99
The form factor is assumed to be free of zeroes:

$$\Gamma_\pi(t) = \Gamma_\pi(0) \exp\left[\frac{1}{\pi}\int_{4m_\pi^2}^{\infty} ds \frac{\phi(s)}{s(s-t)}\right]$$
$$\langle\, r^2 \,\rangle_s^\pi = \frac{6}{\pi}\int_{4m_\pi^2}^{\infty} ds \frac{\phi(s)}{s^2}$$

However, this result is very unstable and depends on the exact value of $\delta_\pi(4m_K^2)$ whether is larger or smaller than π

Actual values of $\delta_\pi(4m_K^2)$ are very close to π

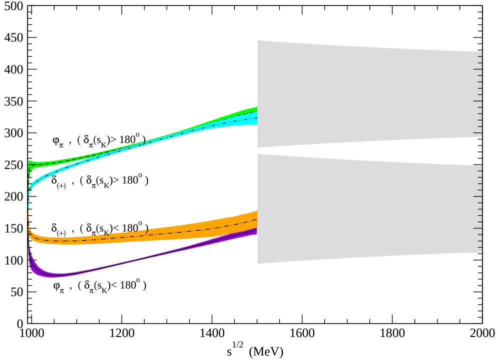

δπ(sK)<π
δπ(sK)>π, P(t)=Γπ(0)(s1- t)/s1
δπ(sK)>π, P(t)=Γπ(0)
ref. [7] (δπ(sK)>π)
| Γπ(t) / Γπ(0) |
t1/2 (MeV)
0
1
2
3
4
5
6
0
200
400
600
800
1000
1200

Continuity is restored by including a zero in the form factor at s_1 if $\delta(s_K) \geq \pi$

$$\Gamma_\pi(t) = \Gamma_\pi(0)\frac{s_1 - t}{s_1}\exp\left[\frac{1}{\pi}\int_{4m_\pi^2}^{\infty} ds\frac{\phi(s)}{s(s-t)}\right]$$

$$\langle\, r^2 \,\rangle_s^\pi = -\frac{6}{s_1} + \frac{6}{\pi}\int_{4m_\pi^2}^{\infty}\frac{\phi(s)}{s^2}ds$$

$\delta(s_1) = \pi \quad s_1 < 4m_K^2$



$\delta(4m_K^2) < \pi \quad \phi(\infty) \to \pi \qquad \delta(4m_K^2) \geq \pi \quad \phi(\infty) \to 2\pi$

$$\phi_{as} \simeq \pi \left(n \pm \frac{2d_m}{\log(s/\Lambda^2)} \right)$$

$d_m = 12/(33 - 2n_f) \simeq 1/2$
Λ is the scale QCD parameter
$n = 1, 2$ for $\delta_\pi(4m_K^2) < \pi, \ \geq \pi$

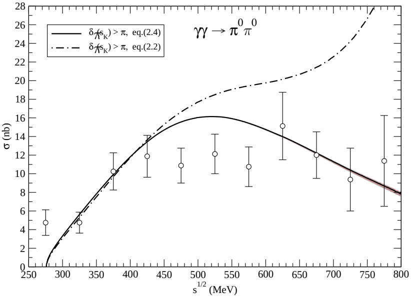
$$\langle\, r^2 \,\rangle_s^\pi = 0.63 \pm 0.05 \quad \text{fm}^2$$

Compatible with CGL $\langle\, r^2 \,\rangle_s^\pi = 0.61 \pm 0.04$ fm^2
Both methods are compatible

$\gamma\gamma \to \pi^0\pi^0$

L.Roca, J.A.O., Phys. Lett. B 659 (2008) 201

Similar problems appear in the literature for this reaction due to the use of the Omnés function with its critical behavior depending of $\delta(4m_K^2) > \pi$ or $< \pi$



Pennington, Phys. Rev. Lett. 97 (2006) 011601

Morgan,Pennington, Phys. Lett. B 272 (1991) 134

Donoghue,Holstein, Phys. Rev. D 48 (1993) 137

The amplitude $F(s)$ has RHC and LHC. The latter is given by $L(s)$

$$F(s) - L(s)$$

has only RHC

$$\Omega(s) = \left(1 - \theta(s_1)\frac{s}{s_1}\right) \exp\left[\frac{s}{\pi}\int_{4m_\pi^2}^{\infty} \frac{\phi_0(s')}{s'(s'-s)} ds'\right]$$

Twice-subtracted dispersion relation for

$$(F(s) - L(s))/\Omega(s)$$

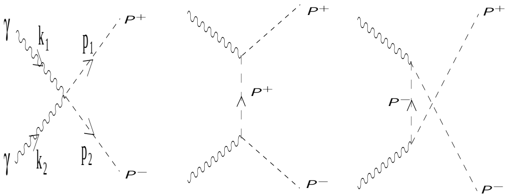
$$F(s) = L(s) + c_0 s\Omega(s) + \frac{s^2}{\pi}\Omega(s)\int_{4m_\pi^2}^{\infty} \frac{L_0(s')\sin\bar{\phi}_0(s')}{s'^2(s'-s)|\Omega(s')|} ds'$$

$$+ \theta(s_1 - 4m_K^2)\frac{\Omega_0(s)}{\Omega_0(s_1)}\frac{s^2}{s_1^2}(F_0(s_1) - L_0(s_1)) \ .$$

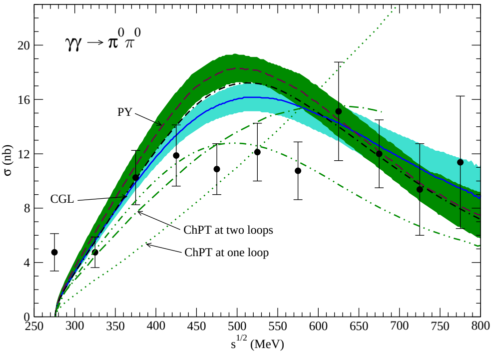
Low's theorem allows to fix c_0

F.E. Low, Phys. Rev. 96 (1954) 1428

The $\gamma\gamma \to \pi\pi$ amplitudes tend linearly to the Born term for $s \to 0$



Other sources for $L(s)$ are Axial vector (a_1) and vector (ρ, ω) resonances



The strong interactions affect the output

CGL: Colangelo, Gasser, Leutwyler Nucl. Phys. B 603 (2001) 125

PY: Peláez, Ynduráin, Phys. Rev. D 71 (2005) 074016

This reaction can be used to favor one or other strong amplitude for meson-meson S-wave

Projected precise experiments on $\gamma\gamma \to \pi^0\pi^0$, $\to \pi^+\pi^-$ for the near future (BELLE, Frascatti)