S-matrix solution of the Lippmann-Schwinger equation for regular and singular potentials

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- [I] arXiv:1609, next month
- [II] Long version, in preparation

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Overview

Lippmann-Schwinger equation (LS)

Scattering T-matrix $T(z)$, $\mathrm{Im}(z) \neq 0$, Two-body scattering

$$
\begin{aligned}
T(z) =& V - V R_0(z) T(z) \\
R_0(z) =& [H_0 - z]^{-1} \\
H_0 =& -\frac{1}{2\mu}\nabla^2 \\
H =& H_0 + V
\end{aligned}
$$

- Resolvent of H, $R(z)$:

$$R(z) = [H - z]^{-1}$$

$$R(z) = R_0(z) - R_0(z) T(z) R_0(z)$$

- Spectrum of H: $H|\psi_{\mathbf{p}}\rangle = E_p|\psi_{\mathbf{p}}\rangle$

Continuous spectrum: Povzner's result

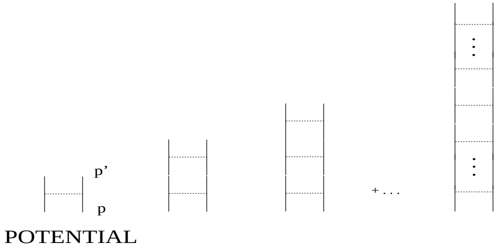
$$|\psi_{\mathbf{p}}\rangle = |\mathbf{p}\rangle - \lim_{\epsilon\to 0^+} R_0(E_p + i\epsilon)T(E_p + i\epsilon)|\mathbf{p}\rangle$$

Bound States: Poles in $T(z)$ for $z \in \mathbb{R}^-$

- **LS in momentum space**

For definiteness we consider *uncoupled spinless case* by now:

$$T(\mathbf{p}', \mathbf{p}, z) = V(\mathbf{p}', \mathbf{p}) - \int \frac{d^3q}{(2\pi)^3} V(\mathbf{p}', \mathbf{q}) \frac{1}{\frac{q^2}{2\mu} - z} T(\mathbf{q}, \mathbf{p}, z)$$



- LS in partial waves

$$T_\ell(p',p,z) = \frac{1}{2}\int_{-1}^{+1} d\cos\theta \, P_\ell(\cos\theta) T(\mathbf{p}',\mathbf{p},z)$$

$$\cos\theta = \hat{\mathbf{p}}' \cdot \hat{\mathbf{p}}$$

$$T(\mathbf{p}',\mathbf{p},z) = \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(\cos\theta) T_\ell(p',p,z)$$

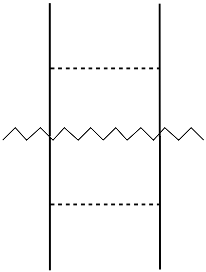
$$T_\ell(p',p,z) = V_\ell(p',p) + \frac{\mu}{\pi^2}\int_0^\infty dq q^2 \frac{V_\ell(p',q) T_\ell(q,p,z)}{q^2 - 2\mu z}$$

Convention: $V(\mathbf{p}',\mathbf{p}) \to -V(\mathbf{p}',\mathbf{p})$, $T(\mathbf{p}',\mathbf{p},z) \to -T(\mathbf{p}',\mathbf{p},z)$

• On-shell unitarity (extensively used later)

Propagation of real two-body states

$$p' = p$$
$$E_p = \frac{p^2}{2\mu}$$



$$\text{Im}T_\ell(p^2) = \frac{\mu p}{2\pi}|T_\ell(p^2)|^2 \ , \ p > 0$$
$$\text{Im}\frac{1}{T_\ell(p^2)} = -\frac{\mu p}{2\pi}$$

Unitarity cut for $p^2 > 0$

Criterion for Singular Potentials

$$V(r) \xrightarrow[r\to 0]{} \alpha r^{-\gamma}$$

$$\bar{\alpha} = \alpha + \ell(\ell+1)$$

Potential	Ordinary	Singular
γ	< 2	> 2
$\gamma = 2$	$\bar{\alpha} > 0$	$\bar{\alpha} \leq 0$

Ordinary/Regular Potentials:

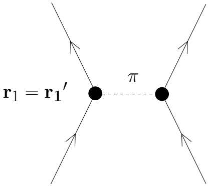
Standard quantum mechanical treatment
Boundary condition: $u(0) = 0$ and behavior at ∞
No extra free parameters

The One-Pion-Exchange (OPE) potential for the singlet NN interaction $(r > 0)$:

Yukawa potential

$$V(r) = -\tau_1 \cdot \tau_2 \left(\frac{g_A m_\pi}{2 f_\pi}\right)^2 \frac{e^{-m_\pi r}}{4\pi r}$$

Exchange of a pion between two nucleons



In many instances one has singular potentials

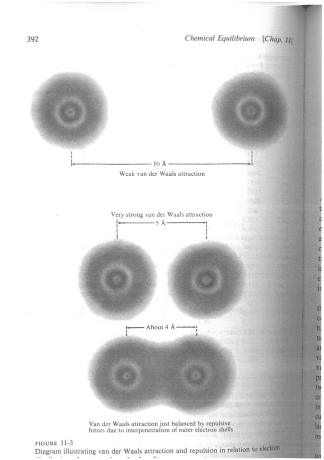
Multipole expansion

$$\Phi(\mathbf{x}) = \int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' = 4\pi \sum_{\ell,m} \frac{q_{\ell m}}{2\ell+1} \frac{Y_{\ell m}(\theta,\phi)}{r^{\ell+1}}$$

$$q_{\ell m} = \int Y^*_{\ell m}(\theta',\phi') {r'}^{\ell} \rho(\mathbf{x}') d^3x'$$

Van der Waals Force (molecular physics)

$$V(r) = -\frac{3}{2}\frac{\alpha_A \alpha_B}{r^6}\frac{I_A I_B}{I_A + I_B}$$



In QFT/EFT we treat composite objects as **point like**

Physical meaning for $r \to 0$?:

De Broglie length
$\frac{1}{p} >> r_A \sim 2\text{Å}$

Nuclear physics

OPE is *singular attractive* for the Deuteron in NN scattering

${}^{2S+1}L_J$: ${}^3P_0$ NN partial wave

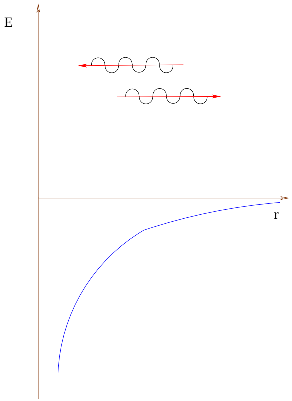
$$V(r) = \frac{m_\pi^2}{12\pi}\left(\frac{g_A}{2f_\pi}\right)^2\left[-4T(r)+Y(r)\right]$$
$$Y(r) = \frac{e^{-m_\pi r}}{r}$$
$$T(r) = \frac{e^{-m_\pi r}}{r}\left[1+\frac{3}{m_\pi r}+\frac{3}{(m_\pi r)^2}\right]$$

Triplet part of the OPE potential $V(r) \xrightarrow[r\to 0]{} \frac{g_A^2}{4\pi f_\pi^2} r^{-3}$

Quark Models, pNRQCD, pNRQED, QCD's EFTs, etc

Math: Taking $r \to 0$ for a singular potential

- Full range $r \in]0, \infty[$: Case, PR60,797(1950)

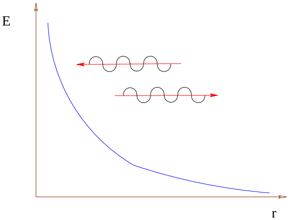


- Singular Attractive Potential

Near the origin the solution is the superposition of two oscillatory wave functions
One has to fix a relative phase, $\varphi(p)$ **How to do it?. Mess.**

Which are the appropriate boundary conditions? (Orthogonality of wave functions with different energy $\to d\varphi(p)/dp = 0$)
Case, PR60,797('50); Arriola, Pavón, PRC72,054002('05)

The potential does not determine uniquely the scattering problem Plesset, PR41,278(1932), Case, PR60,797(1950)



- Singular Repulsive Potential

There is only one finite (vanishing) reduced wave function at $r = 0$

The solution is fixed

Pavón Valderrama, Ruiz Arriola, Ann.Phys.323,1037('08)

Typically, the phenomenology is not accurate

E.g. this scheme *a la Case* does not fit well NN phase shifts.

Two points of view in NN scattering:

- Use a finite cutoff Λ fitted to data. Regularization dependence

It works phenomenologically
Entem,Machleidt, PRC68,041001(R)('03); Epelbaum,Gloeckle,Meißner, NPA747,362('05); Epelbaum,Krebs,Meißner, PRL115,122301('15)

- Take $r \to 0$ ($\Lambda \to \infty$) (Renormalized solutions)

Energy-independent boundary condition Arriola, Valderrama, PLB580,149('04); PRC74,054002('05); Case, PR60,797(1950)
Subtractive renormalization Frederico,Timoteo,Tomio, NPA653,209('99); Yang,Elster,Phillips, PRC80,044002('09)
Include one/zero counterterm Entem *et al.*,PRC77,044006('08)

These three-methods are equivalent. Not phenomenologically successful.

Low-energy EFT paradigm:
Contact terms are necessary to reproduce short-distance physics

They are allowed by symmetry
They are required to make loops finite. Nonrenormalizable QFT/EFT
They are expected to be relatively important because of power-counting

New exact equation in NR scattering theory

Yukawa potential,

$$V(\mathbf{q}) = \frac{2g}{\mathbf{q}^2 + m_\pi^2}$$

Singularity for $\mathbf{q}^2 = -m_\pi^2$

${}^1S_0$ potential: $({}^{2S+1}L_J)$ $\quad g = (g_A m_\pi/\sqrt{8} f_\pi)^2$

$$V(p) = \frac{g}{2p^2}\log(4p^2/m_\pi^2 + 1)$$

Left-hand cut (LHC) discontinuity **for On-shell scattering**

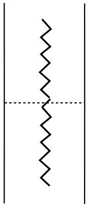
$$p^2 < -m_\pi^2/4 = L$$

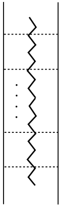
Born approximation

$$\Delta_{1\pi}(p^2) = \frac{V(p^2 + i0^+) - V(p^2 - i0^+)}{2i} = \mathrm{Im}V(p^2 + i0^+) = \frac{g\pi}{2p^2}$$

Full LHC Discontinuity , $p^2 = -k^2 < L$

$$2i\Delta(p^2) = T(p^2 + i0^+) - T(p^2 - i0^+)$$
$$\Delta(p^2) = \mathrm{Im}T(p^2 + i0^+)$$





The LS generates contributions with any number of pions to $\Delta(p^2)$, $p^2 < L$

$\Delta_{n\pi}(p^2)$ for $p^2 < -(nm_\pi^2/2)$

How to calculate $\Delta(A)$?

G.E. Brown, A.D. Jackson "The Nucleon-Nucleon interaction", North-Holland, 1976. Page 86: *"In practice, of course, we do not know the exact form of $\Delta(p^2)$ for a given potential ..."*

$$p = ik \pm \varepsilon \; , \; \varepsilon = 0^+ \; , \; p^2 = -k^2 < L$$

$$T(ik \pm \varepsilon, ik \pm \varepsilon) = V(ik \pm \varepsilon, ik \pm \varepsilon) + \frac{\mu}{2\pi^2} \int_0^\infty dq q^2 \frac{V(ik \pm \varepsilon, q) T(q, ik \pm \varepsilon)}{q^2 + k^2}$$

The last integral, so calculated, IS PURELY REAL!!

You can try to calculate numerically just the once iterated OPE

$$\frac{\mu}{2\pi^2} \int_0^\infty dq q^2 \frac{V(ik \pm \varepsilon, q) V(q, ik \pm \varepsilon)}{q^2 + k^2} \in \mathbb{R}$$

Notation: $A = p^2$

This is an example of:
Not all what you can calculate with a computer is the right answer!!

• **GENERAL method:**
Analytic extrapolation of the LS from its integral expression

$$\mathfrak{f}(\nu) = \Delta v(\nu, k) + \frac{\theta(k - 2m_\pi - \nu)m}{2\pi^2} \int_{m_\pi+\nu}^{k-m_\pi} \frac{d\nu_1 \nu_1^2}{k^2 - \nu_1^2} \Delta v(\nu, \nu_1)\mathfrak{f}(\nu_1)$$

$$\Delta(A) = \frac{\mathfrak{f}(-k)}{2} \; , \; k = \sqrt{-A} \; , \; \text{IE}: \qquad -k + m_\pi < \nu < k - m_\pi$$

- The limits in the IE ARE FINITE
- The denominator never vanishes , $|\nu_1| \le k - m_\pi$ in the IE
- **NO FREE PARAMETERS**

Reason: Contact interactions (monomials) do not contribute to the discontinuity of $T(A)$
Short-distance physics is not resolved$\to$ Contact interactions

$$\mathfrak{f}(\nu) = \Delta v(\nu, k) + \frac{\theta(k - 2m_\pi - \nu) m}{2\pi^2} \int_{m_\pi + \nu}^{k - m_\pi} \frac{d\nu_1 \nu_1^2}{k^2 - \nu_1^2} \Delta v(\nu, \nu_1) \mathfrak{f}(\nu_1)$$

$$\Delta(A) = \frac{\mathfrak{f}(-k)}{2} \ , \ k = \sqrt{-A}$$

It can be applied to:

- Any local potential (spectral decomposition:)

$$V(\mathbf{p}', \mathbf{p}) = \frac{1}{\pi} \int_{\mu_0^2}^{\infty} d\mu^2 \frac{\eta(\mu^2)}{\mathbf{q}^2 + \mu^2} \ , \ \mathbf{q} = \mathbf{p}' - \mathbf{p}$$

- Higher partial waves, $\ell \geq 0$
- Coupled Channels
- Nonlocal potentials due to relativistic corrections

LS equation in the complex plane

Analytical properties of the potential

- Local potential, spectral decomposition:

$$V(\mathbf{q}^2) = \frac{1}{\pi}\int_{\mu_0^2}^{\infty} d\mu^2 \frac{\eta(\mu^2)}{\mathbf{q}^2+\mu^2} \ , \ \mathbf{q} = \mathbf{p}' - \mathbf{p}$$

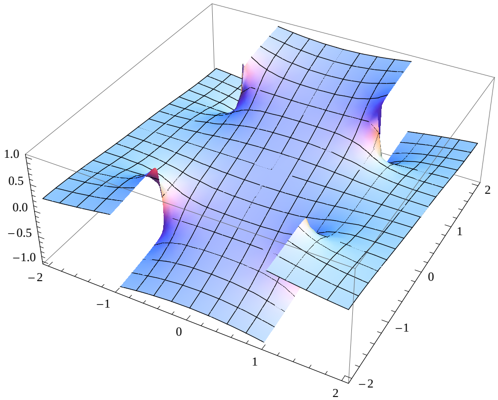
- S-wave projection:

$$\begin{aligned} v(p_1,p_2) &= \frac{1}{2\pi}\int_{-1}^{+1} dt \int_{\mu_0^2}^{\infty} d\mu^2 \frac{\eta(\mu^2)}{p_1^2+p_2^2-2p_1p_2t+\mu^2} \\ &= \frac{1}{4\pi p_1 p_2}\int_{\mu_0^2}^{\infty} d\mu^2 \eta(\mu^2) \\ &\times\left\{\log\left[\mu^2+(p_1+p_2)^2\right] - \log\left[\mu^2+(p_1-p_2)^2\right]\right\} \end{aligned}$$

Vertical cuts:

$$p_2 = \pm(p_1 \pm i\sqrt{m_\pi^2 + x^2})\ x \in \mathbb{R}$$

Analogously for p_1



$p_1 = m_\pi$. Branch points at $\pm(p_1 \pm i m_\pi)$

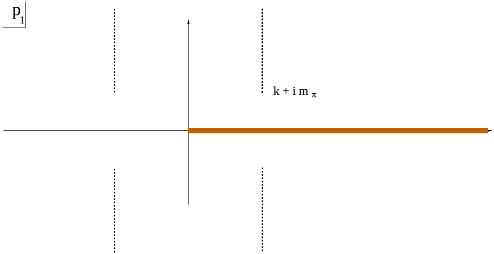
Deforming the integration contour in the LS equation

$k,\ k' \in \mathbb{R}$ in the half-off-shell T-matrix $t(k,k';{k'}^2/m)$,

$$t(k,k';\frac{{k'}^2}{m}) = v(k,k') + \frac{m}{2\pi^2}\int_0^\infty \frac{dp_1 p_1^2}{p_1^2 - {k'}^2} v(k,p_1) t(p_1,k';\frac{{k'}^2}{m}) \ ,$$

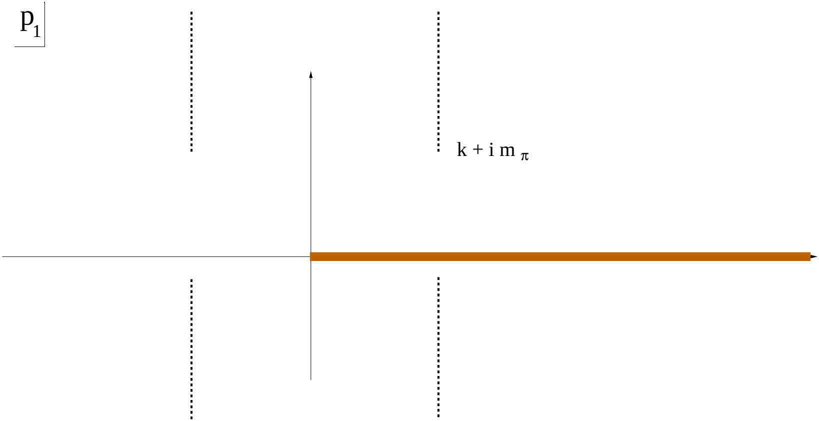
$v(k,p_1)$ implies the vertical cuts

$$p_1 = \pm(k \pm i\sqrt{m_\pi^2 + x^2})\ x \in \mathbb{R}$$



We add an increasing positive imaginary part to k

$$k = k_r + ik_i \ , \ k_i > 0$$



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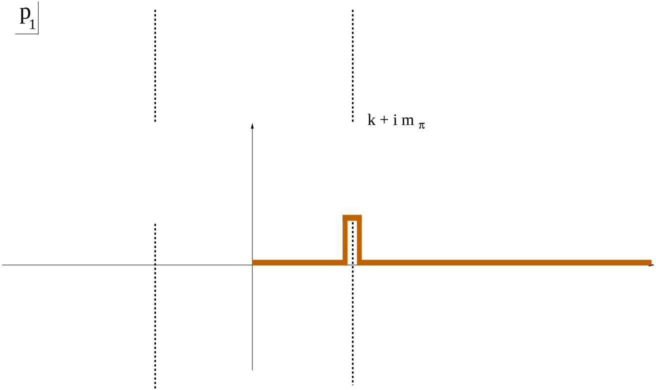
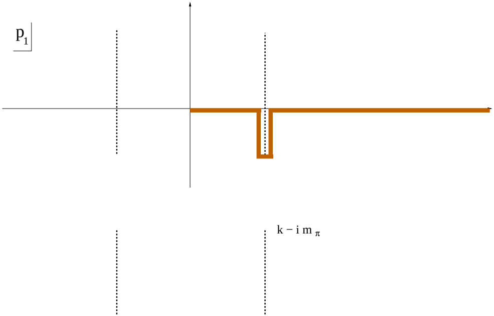
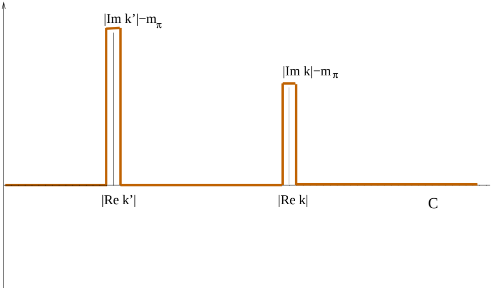


$k_r > 0 \ , \ k_i > m_\pi$
$k_r < 0 \ , \ k_i < -m_\pi$



$$k_r > 0 \;,\; k_i < -m_\pi$$
$$k_r < 0 \;,\; k_i > m_\pi$$

- $t(p_1, k'; {k'}^2/m)$ follows the same pattern in terms of k'.



Higher-order iterations

Twice-iterated LS:

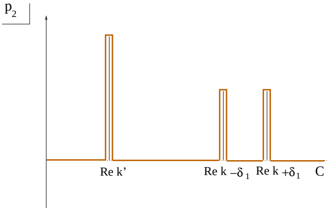
$$t(k,k';\frac{k'^2}{m}) = v(k,k') + \frac{m}{2\pi^2}\int \frac{dp_1 p_1^2}{p_1^2 - k'^2} v(k,p_1)v(p_1,k') + \left(\frac{m}{2\pi^2}\right)^2 \int \frac{dp_1 p_1^2}{p_1^2 - k'^2} v(k,p_1) \int \frac{dp_2 p_2^2}{p_2^2 - k'^2} v(p_1,p_2)v(p_2,k') + \ldots$$

New vertical additions (VA):

p_1 at $|\mathrm{Re}\, k|$

p_2 at $|\mathrm{Re}\, k| - \delta_1$, $|\mathrm{Re}\, k| + \delta_1$ for $|\mathrm{Im}\, p_1| > m_\pi$

But $|\mathrm{Im}\, k| - m_\pi > |\mathrm{Im}\, p_1|$ **every step reduces in m_π the extent of the vertical lines**



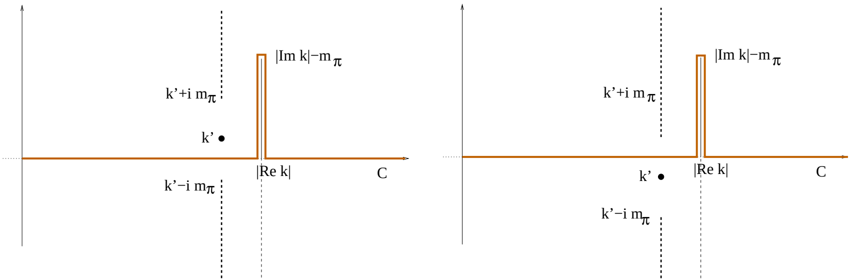
Analytical properties of $t(k, k'; k'^2/m)$

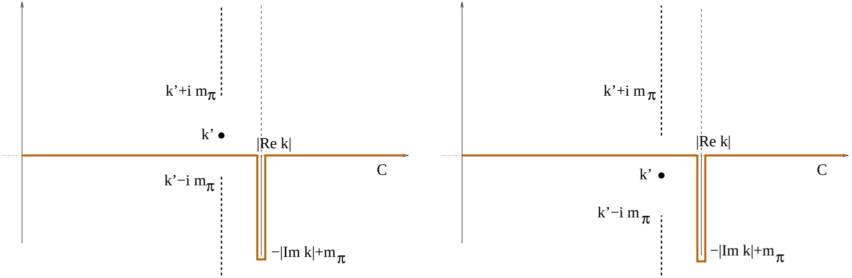
The energy pole gives rise to the RHC $(k'^2 > 0)$

Dynamics cuts: As a function k (k') the same vertical cuts as for the potential $v(k, k')$:

$$k = \pm(k' + \pm i\sqrt{m_\pi^2 + x^2})$$

$|\mathrm{Im}\, k| > m_\pi$, $|\mathrm{Im}\, k'| < m_\pi$





Intersection between the added vertical contour and the standard vertical cuts

Calculation of $\Delta(-k^2)$: Discontinuity across the LHC

On-shell scattering $t(k,k;k^2/m)$
LHC:

$$p=-p\pm i\sqrt{m_\pi^2+x^2}\longrightarrow p=\pm\frac{i}{2}\sqrt{m_\pi^2+x^2}$$

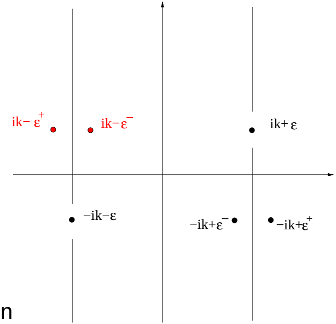
$$p^2=-\frac{1}{4}(m_\pi^2+x^2)\longrightarrow p^2\in]-\infty,L]\ ,\ L=-m_\pi^2/4$$

$$2i\Delta(-k^2)=t(ik+i\varepsilon,ik+i\varepsilon)-t(ik-i\varepsilon,ik+i\varepsilon)$$

$$=(-1)^\ell\left\{t(-ik+\varepsilon^-,ik+\varepsilon)-t(-ik+\varepsilon^+,ik+\varepsilon)\right\}$$

$$\varepsilon^-<\varepsilon<\varepsilon^+$$

Spared slide

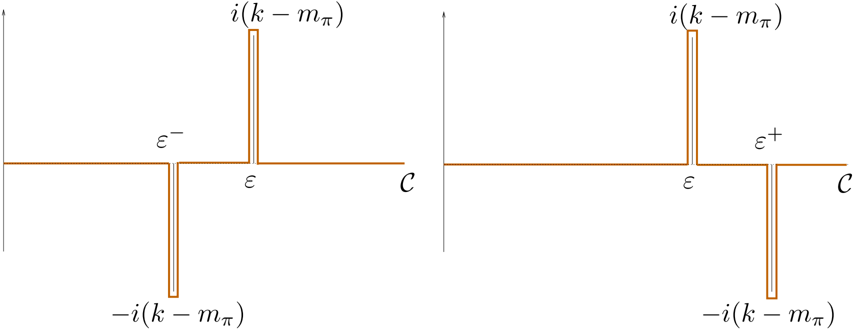


To explain the relation

$$
\begin{aligned}
2i\Delta(-k^2) &= t(ik+i\varepsilon, ik+i\varepsilon) - t(ik-i\varepsilon, ik+i\varepsilon) \\
&= 2i\mathrm{Im}\, t(ik+i\varepsilon, ik+i\varepsilon) \\
&= (-1)^\ell \left\{ t(-ik+\varepsilon^-, ik+\varepsilon) - t(-ik+\varepsilon^+, ik+\varepsilon) \right\}
\end{aligned}
$$

$$\varepsilon^- < \varepsilon < \varepsilon^+$$

$t(-ik+\varepsilon^-, ik+\varepsilon)$ $t(-ik+\varepsilon^+, ik+\varepsilon)$



$$
\begin{aligned}
&\operatorname{Im} t(-ik+\varepsilon^-, ik+\varepsilon) - \operatorname{Im} t(-ik+\varepsilon^+, ik+\varepsilon) \\
&= \operatorname{Im} v(i\nu+\varepsilon^-, ik+\varepsilon) - \operatorname{Im} v(i\nu+\varepsilon^+, ik+\varepsilon) \\
&+\theta(k-\nu-2m_\pi)\frac{m}{2\pi^2}\int_{-k+m_\pi}^{k-m_\pi}\frac{d\nu_1\nu_1^2}{k^2-\nu_1^2} \\
&\times\left[\operatorname{Im} v(i\nu+\varepsilon^-, i\nu_1+\varepsilon) - \operatorname{Im} v(i\nu+\varepsilon^+, i\nu_1+\varepsilon)\right] \\
&\times\left[\operatorname{Im} t(i\nu_1+\varepsilon-\delta, ik+\varepsilon) - \operatorname{Im} t(i\nu_1+\varepsilon+\delta, ik+\varepsilon)\right] \,.
\end{aligned}
$$

- **One needs to know**

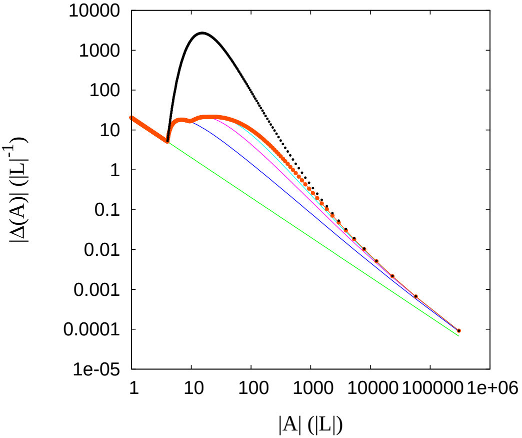
$$
\begin{aligned}
&\operatorname{Im} t(i\nu+\varepsilon^-, ik+\varepsilon) - \operatorname{Im} t(i\nu+\varepsilon^+, ik+\varepsilon) \\
&-k+m_\pi < \nu < k-m_\pi
\end{aligned}
$$

Proceeding in the same
Integral Equation $-k+m_\pi<\nu<k-m_\pi$:

$$\begin{aligned}\mathfrak{f}(\nu) \equiv& \mathrm{Im}\, t(i\nu+\varepsilon^-, ik+\varepsilon) - \mathrm{Im}\, t(i\nu+\varepsilon^+, ik+\varepsilon)\\ =& \mathrm{Im}\, v(i\nu+\varepsilon^-, ik+\varepsilon) - \mathrm{Im}\, v(i\nu+\varepsilon^+, ik+\varepsilon)\\ &+\theta(k-\nu-2m_\pi)\frac{m}{2\pi^2}\int_{\nu+m_\pi}^{k-m_\pi}\frac{d\nu_1\nu_1^2}{k^2-\nu_1^2}\\ &\times\left[\mathrm{Im}\, v(i\nu+\varepsilon^-, i\nu_1+\varepsilon) - \mathrm{Im}\, v(i\nu+\varepsilon^+, i\nu_1+\varepsilon)\right]\\ &\times\left[\mathrm{Im}\, t(i\nu_1+\varepsilon-\delta, ik+\varepsilon) - \mathrm{Im}\, t(i\nu_1+\varepsilon+\delta, ik+\varepsilon)\right]\ .\end{aligned}$$

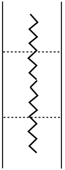
$$\Delta(k) =(-1)^\ell \frac{\mathfrak{f}(-k)}{2}$$

log-log plot for $^1S_0$ (Yukawa Pot.) $\Delta(A)$; $g_A = 6.80$

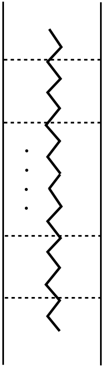


- $\Delta_{1\pi}$, $\Delta_{2\pi}$, $\Delta_{3\pi}$, $\Delta_{4\pi}$, Asymptotic sol. (dots) $|A| \gg m_\pi^2$
- Full solution $\Delta(A)$

- Two-nucleon reducible diagrams [II]; Guo,Ríos,JAO, PRC89,014002('14);



Similar size to the other NLO irreducible diagrams



- All pion lines must be put on-shell $\longrightarrow$ $A \leq -n^2 M_\pi^2/4$.
- As n increases their physical contribution fades away.
- This only occurs for the imaginary part!

Yukawa Potential: OPE $^1S_0$

- The OPE $^1S_0$ (Yukawa potential) is simple enough to derive suitable algebraic expression that can be analytically continued to obtain $\Delta(A)$:

$$\Delta_{1\pi}(p^2) = \frac{g\pi}{2p^2}\theta(L-A)$$

$$\Delta_{2\pi}(A) = \theta(4L-A)\left(\frac{g_A^2 m_\pi^2}{16 f_\pi^2}\right)^2 \frac{M_N}{A\sqrt{-A}} \log\left(\frac{2\sqrt{-A}}{m_\pi}-1\right)$$

$$\Delta_{3\pi}(A) = \theta(9L-A)\left(\frac{g_A^2 m_\pi^2}{4 f_\pi^2}\right)^3 \left(\frac{M_N}{4\pi}\right)^2 \frac{\pi}{4A}\int_{2m_\pi}^{2\sqrt{-A}-m_\pi} d\mu_1 \frac{1}{\mu_1(2\sqrt{-A}-\mu_1)}$$
$$\theta(\mu_1-2m_\pi)\int_{m_\pi}^{\mu_1-m_\pi} d\mu_2 \frac{1}{\mu_2(2\sqrt{-A}-\mu_2)}$$

$$\begin{aligned}\Delta_{4\pi}(A) =&\theta(16L-A)\left(\frac{g_A^2 m_\pi^2}{4f_\pi^2}\right)^4\left(\frac{M_N}{4\pi}\right)^3\frac{\pi}{4A}\int_{3m_\pi}^{2\sqrt{-A}-m_\pi} d\mu_1\frac{1}{\mu_1(2\sqrt{-A}-\mu_1)}\\ &\times\theta(\mu_1-3m_\pi)\int_{2m_\pi}^{\mu_1-m_\pi} d\mu_2\frac{1}{\mu_2(2\sqrt{-A}-\mu_2)}\\ &\times\theta(\mu_2-2m_\pi)\int_{m_\pi}^{\mu_2-m_\pi} d\mu_3\frac{1}{\mu_3(2\sqrt{-A}-\mu_3)}.\end{aligned}$$

This can be generalize for a diagram with n pions to

$$\begin{aligned}\Delta_{n\pi}(A) =&\theta(n^2L-A)\left(\frac{g_A^2 m_\pi^2}{4f_\pi^2}\right)^n\left(\frac{M_N}{4\pi}\right)^{n-1}\frac{\pi}{4A}\\ &\times\prod_{j=1}^{n-1}\theta(\mu_{j-1}-(n+1-j)m_\pi)\int_{(n-j)m_\pi}^{\mu_{j-1}-m_\pi} d\mu_j\frac{1}{\mu_j(2\sqrt{-A}-\mu_j)}\end{aligned}$$

with $\mu_0 = 2\sqrt{-A}$

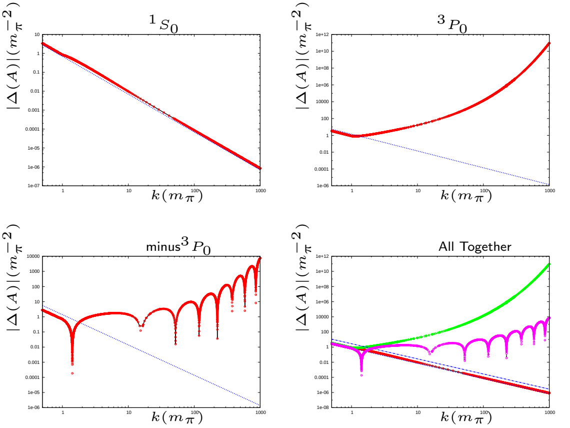
Yukawa potential

- Asymptotic solution for $k \gg m_\pi$

$$\frac{\mathfrak{f}'(\nu)}{\mathfrak{f}(\nu)} = -\lambda \frac{\theta(k - 2m_\pi - \nu)}{k^2 - (m_\pi + \nu)^2}$$

$$\begin{aligned} \Delta(A) &= \frac{\lambda \pi^2}{M_N A} e^{\frac{2\lambda}{\sqrt{-A}} \operatorname{arctanh}\left(1 - \frac{m_\pi}{\sqrt{-A}}\right)} \\ \lambda &= \frac{g M_N}{2\pi} \end{aligned}$$

$^3P_0$: singular attractive potential; m^3P_0:singular repulsive potential ($g \to -g$)



$k \to +\infty$:
$^3P_0$: "Exponential" growth
m$^3P_0$: Oscillatory-"Exponential" growth
$^1S_0$: Vanishes

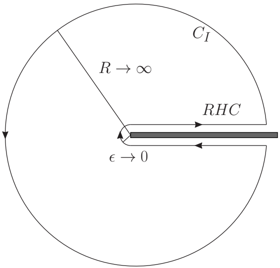
N/D method with non-perturbative $\Delta(A)$

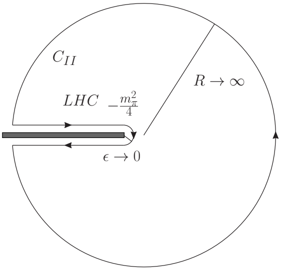
Once we now the exact $\Delta(A)$ for a given potential we can use S-matrix theory to solve the LS: N/D **method with the full** $\Delta(A)$

$$T_{J\ell S}(A) = \frac{N_{J\ell S}(A)}{D_{J\ell S}(A)}$$

$N_{J\ell S}(A)$ has Only LHC

$D_{J\ell S}(A)$ has Only RHC





Uncoupled Partial Waves

Exact knowledge of discontinuities

$$T_\ell(A) = \frac{N_\ell(A)}{D_\ell(A)}$$

$$\mathrm{Im}\frac{1}{T_\ell(A)} = -\rho(A) \equiv \frac{\mu\sqrt{A}}{2\pi} \quad A > 0 \text{ (RHC)}$$

$$\mathrm{Im} D_\ell(A) = -N_\ell(A)\rho(A) \quad A > 0 \text{ (RHC)}$$

$$\mathrm{Im} N_\ell(A) = D_\ell(A)\,\Delta(A) \quad A < L \text{ (LHC)}$$

(m_1, m_2) N/D equations for $D(A)$ and $N(A)$

$N/D_{m_1\,m_2}$

$$N(A) = \sum_{i=1}^{m_1} \nu_i (A-C)^{m_1-i} + \frac{(A-C)^{m_1}}{\pi} \int_{-\infty}^{L} dk^2 \frac{\Delta(k^2) D(k^2)}{(k^2-A)(k^2-C)^{m_1}}$$

$$D(A) = \sum_{i=1}^{m_2} \delta_i (A-C)^{m_2-i} - \frac{(A-C)^{m_2}}{\pi} \int_{0}^{\infty} dq^2 \frac{\rho(q^2) N(q^2)}{(q^2-A)(q^2-C)^{m_2}}$$

- $N(A)$ is substituted in $D(A)$
- Linear IE for $D(A)$ arises
- $D(0)=1$. To fix a floating constant in the ratio $T(A) = N(A)/D(A)$

Regular interactions

- N/D_{01}: **Regular solution for an ordinary potential**

Scattering is completely fixed by the potential

$$
\begin{aligned}
N(A) &= \frac{1}{\pi}\int_{-\infty}^{L} d\omega_L \frac{D(\omega_L)\Delta(\omega_L)}{(\omega_L - A)} \\
D(A) &= 1 - \frac{A}{\pi}\int_0^{\infty} d\omega_R \frac{\rho(\omega_R)N(\omega_R)}{(\omega_R - A)\omega_R} \\
&= 1 - \frac{i\mu\sqrt{A}}{2\pi^2}\int_{-\infty}^{L} d\omega_L \frac{\Delta(\omega_L)D(\omega_L)}{\sqrt{\omega_L}\left(\sqrt{\omega_L}+\sqrt{A}\right)}
\end{aligned}
$$

- **N/D_{11}: Additional subtraction in $N(A)$ is fixed in terms of scattering length**

$$
\begin{aligned}
D(A) &= 1 + ia\sqrt{A} + i\frac{M_N}{4\pi^2}\int_{-\infty}^{L} d\omega_L \frac{D(\omega_L)\Delta(\omega_L)}{\omega_L}\frac{A}{\sqrt{A}+\sqrt{\omega_L}} \\
N(A) &= -\frac{4\pi a}{M_N} + \frac{A}{\pi}\int_{-\infty}^{L} d\omega_L \frac{D(\omega_L)\Delta(\omega_L)}{(\omega_L - A)\omega_L}
\end{aligned}
$$

Effective Range Expansion (ERE)

$$
k\cot\delta(k) = -\frac{1}{a} + \frac{1}{2}rk^2 + \sum_{i=2} v_i k^{2i}
$$

- **N/D_{12}: Additional subtraction in $D(A)$, r is fixed**

$$
\begin{aligned}
D(A) &= 1 + ia\sqrt{A} - \frac{ar}{2}A - i\frac{M_N A}{4\pi^2}\int_{-\infty}^{L} d\omega_L \frac{D(\omega_L)\Delta(\omega_L)}{\omega_L} \\
&\times \left[\frac{\sqrt{A}}{(\sqrt{\omega_L}+\sqrt{A})\sqrt{\omega_L}} - \frac{i}{a\omega_L}\right] \\
N(A) &= -\frac{4\pi a}{M_N} + \frac{A}{\pi}\int_{-\infty}^{L} d\omega_L \frac{D(\omega_L)\Delta(\omega_L)}{(\omega_L - A)\omega_L}
\end{aligned}
$$

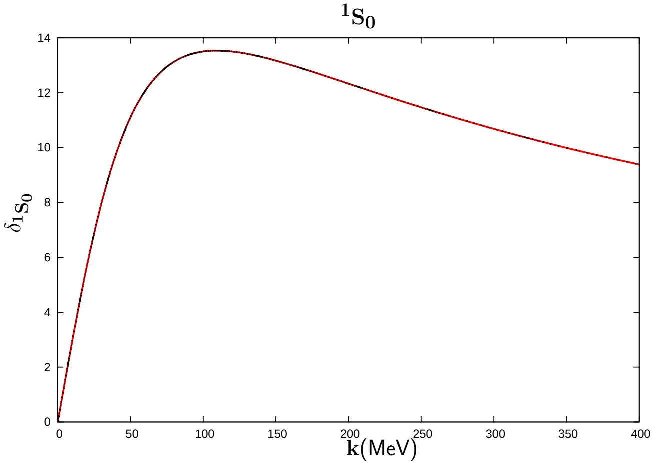
The results are just dependent on $\Delta(A)$ (input potential) and experimental ERE parameters

- **N/D_{22}: Additional subtraction in $N(A)$, v_2 is fixed**

$$
\begin{aligned}
D(A) &= (1-\frac{2v_2}{r}A)(1+ia\sqrt{A})-\frac{ar}{2}A \\
&\quad +i\frac{M_N}{4\pi^2}A\int_{-\infty}^{L}d\omega_L\frac{D(\omega_L)\Delta(\omega_L)}{\omega_L^2} \\
&\times \left[\frac{A}{\sqrt{A}+\sqrt{\omega_L}}+i\frac{2}{ra^2\omega_L}(1+ia\sqrt{\omega_L})(1+ia\sqrt{A})\right] \\
N(A) &= -\frac{4\pi a}{M_N}+A\frac{8\pi a v_2}{M_N r}+\frac{A}{\pi}\int_{-\infty}^{L}d\omega_L\frac{D(\omega_L)\Delta(\omega_L)}{\omega_L^2} \\
&\times \left[\frac{A}{(\omega_L-A)}+\frac{2}{ra\omega_L}(1+ia\sqrt{\omega_L})\right]
\end{aligned}
$$

The more subtractions are included the more perturbative N/D is with respect to $\Delta(A)$. $\Delta_{n\pi}(A)$ contributes for $A < n^2 L$

Example: Regular case. $^1S_0$ Yukawa potential



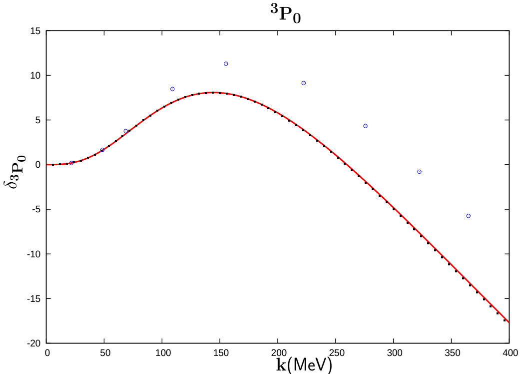
N/D_{01}; **LS** (black dots)

Analytical properties determine the solutions for singular potentials

Attractive singular interaction: $^3P_0$

$\mathbf{N/D_{12}}$ $T(A) = 0$ (N/D_{11} does not converge)

At least one parameter is needed The scattering volume is fixed



N/D_{12};
LS (black dots);
Phase shifts: Granada analysis

We compare with

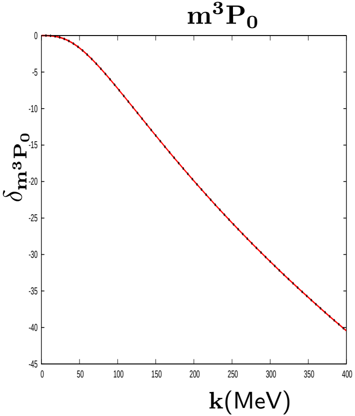
LS renormalized with one contact term:

$$V(p_1, p_2) \to V(p_1, p_2) + C_1 p_1 p_2$$

Repulsive singular interaction: $^3P_0$

$\mathbf{N/D_{11}}$; No free parameters ; $T(0) = 0$

Repulsive Singular Potential: LS is insensitive to all C_i



N/D_{12};
LS (black dots);

$T(A)$ in the complex plane

• As a **bonus** the non-perturbative-Δ N/D method allows to calculate $T(A)$ for $A \in \mathbb{C}$ in the 1st/2nd Riemann sheet

This is not trivial with LS

Look for and study resonances, virtual states and bound states

For bound states one does not need to solve the full-off-shell LS equation or Schrödinger equation

Bound State $A = (ik)^2$

Binding energy of near threshold bound state, $g_A = 7.45$

One does not need to solve Schrödinger equation

Poles of $T(A) \leftrightarrow$ zeros of $D(A)$

- As a **bonus** the non-perturbative-Δ N/D method allows to calculate $T(A)$ for $A \in \mathbb{C}$ in the 1st/2nd Riemann sheet

This is not trivial with LS

Binding energy of near threshold bound state, $g_A = 7.45$

One does not need to solve Schrödinger equation

Poles of $T(A) \leftrightarrow$ zeros of $D(A)$

$A = (ik)^2$	N/D_{01}	N/D_{11}	Schrödinger
$\Delta_{1\pi}$		2.02	
$\Delta_{2\pi}$		2.18	
$\Delta_{3\pi}$		2.21	
$\Delta_{4\pi}$	0.89	2.22	
Non-perturbative	2.22	2.22	2.22

- **Anti-bound (virtual) state for $^1S_0$**

$$\begin{aligned} T_{II}^{-1}(A) &= T_I^{-1}(A) + 2i\rho(A) \\ &= \frac{D_I + N_I\, 2i\rho(A)}{N_I}\ ,\ \text{Im}\sqrt{A} \geq 0 \end{aligned}$$

Look for zero of $D_{II}(A)$. $E = A/M_N =$

N/D_{11}:
-0.070 (LO) , -0.067 (NLO,NNLO) MeV

For the other $N/D_{m_1 m_2}$: -0.066 MeV always

G.E. Brown, A.D. Jackson "The Nucleon-Nucleon interaction", North-Holland, 1976. Page 86: *"In practice, of course, we do not know the exact form of $\Delta(p^2)$ for a given potential and the N/D equations do not represent a practical alternative to the exact solution of the LS equation for potential scattering..."*

Now (2016), this statement is superseded

Conclusions

- A new non-singular IE allows to calculate the exact $\Delta(A)$ in potential scattering for a given potential
- One can calculate the scattering amplitude for regular/singular potentials from its analytical/unitarity properties.
- Any proper solution for singular potentials can be found with this method
- We reproduce the LS outcome with/without one counterterm
- It can be straightforwardly used in the whole complex plane (bound states, resonances, virtual states)

- See Entem's talk about how to go beyond LS+one counterterm for an attractive singular potential.

- Including as well higher order chiral NN potentials.