FEMOEA: a Fast and Efficient Multi-Objective Evolutionary Algorithm

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Abstract A multi-objective evolutionary algorithm which can be applied to many nonlinear multi-objective optimization problems is proposed. Its aim is to quickly obtain a fixed size set approximating the complete Pareto-front. It adapts ideas from different multi-objective optimization evolutionary algorithms, but also incorporates new devices. In particular, the search in the space is carried out on promising areas (hyperspheres) determined by a radius value. Such a radius is not a fixed figure, but it decreases as the optimization procedure evolves. This mechanism of a decreasing radius helps to maintain a balance between exploration and exploitation of the search space. Additionally, a new local search method which accelerates the convergence of the population towards the optimal Pareto-front, has been incorporated. Basically, this method is an extension of the local optimizer SASS and improves a given solution along a search direction (no gradient information is used). Finally, a termination criteria has also been proposed, which stops the algorithm if during three consecutive iterations the changes experimented in the candidate...
Pareto-front are negligible (in terms of the objective function values). To know how far two sets are from each other, a modification of the well-known Hausdorff distance is proposed. In order to analyze the algorithm performance, it has been compared to reference algorithms MOEA/D, NSGA-II and SPEA2 on a set of twenty instances, which includes, among others, the ZDT and DTLZ suites of benchmark problems. Several quality indicators have been considered, namely, hypervolume, average distance, additive epsilon indicator, spread and spacing. According to the computational results and statistical analysis performed, the new algorithm, named FEMOEA, outperforms, on average, MOEA/D, NSGA-II and SPEA2 in all the quality indicators.

Keywords Nonlinear multi-objective optimization · evolutionary algorithm · quality indicators · statistical analysis · computational study

1 Introduction

Multiobjective optimization problems are ubiquitous. Many real-life problems require taking several conflicting points of view into account. In fact, although the origins of multi-objective optimization literature are linked to utility theory, game theory, linear production theory and economics (see [29]), we now can find applications in many and diverse fields, such as portfolio optimization [21], jury selection [50], airline operations [22], radiation therapy [37], manpower planning [53] or reservoir management [2], among others. In [64], White mentions more than 500 applications between 1955 and 1986. Classical references on multi-objective optimization are the books [5,11,55,66,67]. Other more recent books are [19,25,42].

In this paper, we deal with the general nonlinear multi-objective optimization problem (MOP), which can be formulated as follows:

\[
\begin{aligned}
\min & \{f_1(y), \ldots, f_m(y)\} \\
n & y \in S \subseteq \mathbb{R}^n
\end{aligned}
\]  

(1)

where \(f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}\) are \(m\) real-valued functions. Let us denote by \(f(y) = (f_1(y), \ldots, f_m(y))\) the vector of objective functions and by \(Z = f(S)\) the image of the feasible region.

When dealing with multi-objective problems we need to clarify what ‘solving’ a problem means. In the following some widely known definitions are provided to explain the concept of solution of (1).

**Definition 1** A feasible vector \(y^* \in S\) is said to be efficient iff there does not exist another feasible vector \(y \in S\) such that \(f_l(y) \leq f_l(y^*)\) for all \(l = 1, \ldots, m\), and \(f_j(y) < f_j(y^*)\) for at least one index \(j\) (\(j \in \{1, \ldots, m\}\)). The set \(S_E\) of all the efficient points is called the efficient set or Pareto-set. If \(y_1\) and \(y_2\) are two feasible points and \(f_l(y_1) \leq f_l(y_2)\) for all \(l = 1, \ldots, m\), with at least one of the inequalities being strict, then we say that \(y_1\) dominates \(y_2\).

Efficiency is defined in the decision space. The corresponding definition in the criterion space is as follows:
**Definition 2** An objective vector \( z^* = f(y^*) \in Z \) is said to be non-dominated iff \( y^* \) is efficient. The set \( Z_N \) of all non-dominated vectors is called the non-dominated set or Pareto-front. If \( y_1 \) and \( y_2 \) are two feasible points and \( y_1 \) dominates \( y_2 \), then we say that \( f(y_1) \) dominates \( f(y_2) \).

Ideally, solving (1) means obtaining the whole efficient set, that is, all the points which are efficient, and its corresponding Pareto-front. However, for a majority of MOPs, it is not easy to obtain an exact description of the efficient set or Pareto-front, since those sets typically include an infinite number of points (usually a continuum set). To the extent of our knowledge, only two exact general methods, namely, two interval branch-and-bound methods (see [23,24]) have been proposed in literature which obtain an enclosure of those sets up to a pre-specified precision. Specifically, they offer a list of boxes (multi-dimensional intervals) whose union contains the complete efficient set (and their images the corresponding Pareto-front) as a solution. However, they are time consuming. Furthermore, they have large memory requirements, so that only small instances can be solved with them.

Contrarily, the use of (meta)heuristics may allow to obtain ‘good approximations’ of the Pareto-front, even for problems with more variables and objectives. By a good approximation we mean a discrete set of points covering the complete Pareto-front and evenly distributed over it. There is a plethora of methods with that purpose in literature. These include extensions of simulated annealing [13,44,51,57,60], tabu search [30,31,35,38], scatter search [12,45,61], ant systems [17,40,41,52] or particle swarm optimization [10,39,47], among others, to multi-objective programming. However, most of them are designed to deal with combinatorial MOPs (some exceptions are [35,45,52]).

Nonetheless, the most common approaches utilized in literature to cope with (1) is the use of multi-objective evolutionary algorithms (MOEAs). This is due to their ability to find multiple efficient solutions in one single simulation run. The numerous proposed variants have been surveyed, for instance, in [6,20,26,34,59]. For a more complete review of all the topics related to MOEAs see the book [9]. For an excellent selection of applications of MOEAs to real-world problems we refer the reader to [8].

In this paper, a new Fast and Efficient Multi-Objective Evolutionary Algorithm (FEMOEA), aimed at obtaining a good fixed size approximation of the Pareto-front is presented. To help FEMOEA to accelerate its convergence towards the optimal Pareto-front, two new devices have been incorporated: a new improving method, where no differentiability is assumed, and a new stopping rule. These two contributions can be included in any MOEA. Furthermore, it adapts some concepts from other evolutionary algorithms (EAs) devised to cope with single-objective optimization problems. It also includes ideas from other typical MOEAs, as the use of the crowding distance as a way to compute the estimation of the density of solutions during the selection procedure (see [15]). It is known that the crowding mechanism only works well for bi-objective problems [16]. However, we use this operator in this work because we want to focus on solving optimization problems with few objectives, usually
only two. The interest in this type of problems comes from the fact that, in practice, in real world problems, only two (maybe up to three) objectives are considered, and it is only for those bi- (or tri-)objective problems that obtaining the whole efficient set and its corresponding Pareto-front makes sense. As an example, the authors are familiar with continuous location theory, an area of operations research where only bi-objective problems have been considered so far [23,24]. Location theory is a prolific research area, to the point that it has now its own entry (90B85) in the Mathematics Subject Classification used by Mathematical Reviews and Zentralblatt für Mathematik. In fact, many optimization methods are evaluated by making use of location problems [56], mainly due to their high complexity and required computational effort.

FEMOEA could also be used to cope problems with more than three objectives, but the selection procedure should then be based on other measures of the density of solutions different from the crowding distance. However, if more objectives are to be considered, the efficient set and the Pareto-front would usually be so big that they will be useless to the decision-maker. That is why in those cases other multi-objective optimization techniques are used, which somehow incorporate preferences for the objectives in order to reduce the set of efficient solutions to be shown to the decision maker.

A comprehensive computational study is carried out in this paper to compare FEMOEA with the well-known NSGA-II [15] and SPEA2 [70] algorithms, which have become the reference algorithms in the multi-objective evolutionary computation community. Additionally, the algorithm MOEA/D [68] has been also included in the comparison, because it has proved to be very competitive. The implementation of those three algorithms in the platform j-Metal [18] has been used for the evaluation. Following the existing performance indicators in literature, the comparisons have been accomplished in terms of effectiveness, i.e. in terms of quality of the obtained approximations of the Pareto-front. The modus operandi has been to provide the algorithms with a budget in the number of function evaluations and to obtain a fixed size approximation of the Pareto-front. A set of 20 benchmark problems has been solved, and different quality indicators have been analyzed. As the statistical testing carried out shows, FEMOEA reaches, on average, better results than the other algorithms for all the quality indicators.

The rest of the paper is organized as follows. Our new algorithm FEMOEA is introduced in the following section, where special subsections are devoted to the new improving method and stopping rule. Section 3 presents the computational study. The test problems and the implementations employed are given first. Next, the quality indicators used in the comparative study are presented, as well as an explanation of the statistical testing performed. In addition, some results proving the usefulness of the improving method are provided along with the comparison with MOEA/D, NSGA-II and SPEA2 algorithms. In Section 4, our main conclusions are summarized and the research issues that we believe to be worth exploring in the future are highlighted.
2 Description of FEMOEA

FEMOEA is an evolutionary algorithm devised to cope with nonlinear multi-objective problems. Its main objective is to provide a good fixed size approximation of the Pareto-front, i.e., a fixed number of well-distributed and non-dominated solutions. To this aim, it combines ideas from typical algorithms for solving general multi-objective optimization problems: an external archive is utilized to store preferable non-dominated solutions [36,45], and the crowded comparison operator is applied to guide the algorithm towards a uniformly spread Pareto-front approximation [15]. Additionally, it also inherits some concepts from other evolutionary algorithms devised to cope with single-objective optimization problems, namely from UEGO algorithm [46]. In particular, it has adopted the concept of a decreasing radius, as a mechanism of maintaining a balance between exploration and exploitation of the search space. In addition, FEMOEA incorporates new mechanisms which help to accelerate the optimization process (efficiency) and improve the quality (effectiveness) of the solutions. The ‘improving method’ or the ‘termination criteria’ are two of those specific contributions.

The most important concept in FEMOEA is that of individual. An individual is defined by a center and a radius. The center is a solution and the radius is a positive number which determines the subregion of the search space covered by that individual. The main aim of the radius is to focus the searching operators on the corresponding subregions. The radius is a monotonous function that decreases as the optimization process moves forward, i.e., as the number of iterations increases. At any iteration of the algorithm, an individual can be created and its assigned radius only depends on this iteration number. Then, at each stage of the algorithm, several individuals with different radii can coexist simultaneously. The use of different radii throughout the optimization process allows, on the one hand, to identify regions in the search space with high quality solutions and, on the other hand, not to waste too much time on regions of the search space which are either already explored or do not provide high quality solutions [4]. This idea of a decreasing radius is a legacy of UEGO [46].

Apart from the center and the radius, an individual has two attributes which are related to the criterion space: the non-domination rank (d_{\text{rank}}) and the crowding distance (c_{\text{dist}}), see [15]. The non-domination rank indicates the number of individuals which dominate that particular individual. In this sense, a zero value means that such an individual is not dominated by any of the remaining ones in the current population. Regarding the second attribute, the crowding distance is an estimation of the density of solutions surrounding a particular solution in a population. In [15], Deb et al. proposed an algorithm which calculated the crowding distance of each point in a population \( P \). In that paper, the crowding distance was computed using the rectangular distance. However, in FEMOEA, the Euclidean distance has been considered, since it represents the crowding better than the rectangular one. For the sake of completeness, the algorithm proposed in [15] is depicted (see Algorithm
Algorithm 1 Crowding distance assignment($P$)

1: \( p = |P| \)
2: for \( i = 1 \) to \( p \)
3: \( c_{i}^{dist} = 0 \)
4: for \( l = 1 \) to \( m \)
5: \( P = \text{sort}(P, l) \)
6: \( c_{1}^{dist} = c_{p}^{dist} = \infty \)
7: for \( i = 2 \) to \( p - 1 \)
8: \( c_{i}^{dist} = c_{i}^{dist} + \left( \frac{f_{i}^{l-1} - f_{i}^{l+1}}{f_{l}^{\text{max}} - f_{l}^{\text{min}}} \right)^{2} \)
9: for \( i = 1 \) to \( p \)
10: \( c_{i}^{dist} = \sqrt{c_{i}^{dist}} \)

Sort using the \( l \)-th objective function value
In this way, boundary points are always selected

1). Notice that steps 8 to 10 have been modified to consider the Euclidean distance.

In Algorithm 1, \( f_{i}^{l} \) refers to the \( l \)-th objective function value of the \( i \)-th point in the set \( P \), and \( f_{l}^{\text{max}} \) and \( f_{l}^{\text{min}} \) refer to the maximum and minimum objective function values of the \( l \)-th objective function, respectively.

During the optimization process, two lists of individuals are kept by FEMOEA, whose maximum size \( M \) is the same for both lists and it is a given input parameter. Parameter \( M \) refers to the desired number of solutions in the final Pareto-front. The first list, named \textit{population list}, is composed of \( M \) diverse individuals with different attributes, i.e. various radii, non-domination ranks and crowding distances. FEMOEA is in fact a method for managing this list (i.e. creating, deleting and improving individuals). The second list, called \textit{external list}, can be understood as a deposit to keep non-dominated solutions. Notice that the number of non-dominated points may be fewer than \( M \) during the early stages of the optimization algorithm and hence, the \textit{external list} may contain fewer elements than the desired ones. In fact, it cannot be guaranteed that \( M \) non-dominated solutions have been found once the termination criteria have been satisfied, although this has always been the case in our computational experiments. When this is not the case, the \textit{external list} is then completed up to \( M \) elements with the most preferable solutions.

Definition 3 A solution \( i \) is preferable to a solution \( i' \), \( i \succ i' \), if

\[
\begin{align*}
&d_{i}^{rank} < d_{i'}^{rank}, \text{ or} \\
&d_{i}^{rank} = d_{i'}^{rank} \text{ and } c_{i}^{dist} > c_{i'}^{dist}.
\end{align*}
\]

The previous relation is known as crowded comparison operator (see [15]).

To accelerate the selection process, both lists are always sorted according to the crowded comparison operator, i.e. in ascending order according to non-domination rank, and in descending order of the crowding distance when several elements share the same non-domination rank.

In FEMOEA, each individual is intended to occupy an efficient solution. For this purpose, FEMOEA directs the individuals during the searching process towards the most suitable regions. Notice, therefore, that a particular
individual is not a fixed part of the search domain, but it can move through the space as the search proceeds. ‘Individuals-management’ is one of the core parts of FEMOEA. It consists of procedures for creating and selecting individuals during the whole optimization process. Additionally, FEMOEA includes an improving method, which has been logically separated from the individual-management. In this sense, FEMOEA can be considered a memetic algorithm [43]. Furthermore, this means that FEMOEA can easily be adapted to solve any multi-objective problem, only adapting the improving technique (an application to competitive facility location problems can be found in [1,48,49]).

2.1 The algorithm

A global description of FEMOEA is given in Algorithm 2. In the following, the different key stages in the algorithm are described:

- **Init_individual_lists**: In this procedure, as many individuals as the parameter $M$ indicates are created. The center of the individuals are randomly computed, while the radii will be the one associated at level 1. Such a radius coincides with the diameter of the search space, so the whole search area will be covered. The population_list is initialized from this set of individuals, while the external_list will consist only of the non-dominated individuals.

After this procedure, the FEMOEA main loop starts, which basically consists of three procedures: creating, improving and selecting individuals. This loop is executed until a stopping condition is fulfilled. For the problem at hand, the loop stops whenever a considerable improvement is not obtained among three consecutive Pareto-fronts (placed in external_list) or a number of maximum levels is achieved. The number of levels (cycles or generations) will be given by the input parameter $L$.

- **Create_new_individual(evals)**: For every individual in the population_list, evals/2 random trial points in the area defined by its radius are created. evals refers to the budget of function evaluations available for each existing

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**Algorithm 2 Algorithm FEMOEA**

1: Init_individual_lists
2: while termination criteria are not satisfied
3: Create_new_individual(evals)
4: if \(\text{length(population_list)} > M\)
5: Select_individual(population_list)
6: Improve_individual(population_list)
7: Update_external_list
8: if \(\text{length(external_list)} > M\)
9: Select_individual(external_list)
10: Improve_individual(external_list)
11: if \(\text{length(external_list)} < M\)
12: Compose_pareto
individual for creating a new offspring. For the problem at hand, we have set $evals = 20$.

Furthermore, for each new random candidate solution, the closest point (in the objective space) in the $external_list$ is calculated. Then, a new random point is computed in the segment joining the candidate solution with its closest point. Notice that the intermediate point can be placed outside the area covered by the original individual. If the intermediate point dominates the candidate solution, then it will be included in the $population_list$ as a new individual. On the contrary, if the candidate solution is the one which dominates the other, it will be the one inserted in the $population_list$. Additionally, if the two points are indeterminate (not one dominates the other), then both will be inserted as new individuals. The radius assigned to each new individual is the one associated with the current level $i$. The radius of an individual created at level $i$, $R_i$, is given by a decreasing exponential function, where $R_L$ and $R_1$ are the given (input parameters) smallest and largest radii. For a detailed description of how to compute the radius at each level of the algorithm see [46]. It is interesting to remark that a location in the search space can belong to different individuals with different radii. Individuals with small radii examine a relatively small area, their motion in the space is slower, but they are able to differentiate between efficient solutions which are very close. On the contrary, individuals with large radii study a somewhat bigger region, they may move great distances and discover new promising areas, which may be analysed conscientiously in later stages of the algorithm.

Additionally, both the non-domination rank and the crowding distance associated to each new individual are computed. The $population_list$ is then sorted according to the crowding comparison operator.

- $Select\_individual(list)$: If list reaches its maximum allowable capacity, a decision has to be made to determine which individuals should be kept and not removed. The selection strategy used in this work is based on the crowded comparison operator [15]. Then, the most preferable individuals will be selected, i.e. between two individuals with different non-domination rank we prefer the one with the lower rank. Otherwise, we prefer the point which is located in a region with the fewest number of points (i.e., the highest crowding distance).

- $Improve\_individual(list)$: This procedure applies the improving method to all the individuals on the list. As can be observed in Algorithm 2, this technique is applied to both the $population_list$ and the $external_list$ at different stages of the optimization process, i.e., steps 6 and 10, respectively. Once all the individuals in the input list have invoked the improving method, the improved individuals are reordered according to the crowded comparison operator.

Subsection 2.2 describes the improving method proposed in this paper.

- $Update\_external_list$: During the previous process, new non-dominated points may be generated. In Step 7 of Algorithm 2, the $external_list$ is updated by copying the non-dominated solutions of the $population_list$ to it. Of course,
this implies that the points on the external list dominated by the new ones have to be removed, and a reordering of the remaining ones according to the new values of the crowded comparison operator has to be performed.

- Compose pareto: The solution provided by the algorithm must include $M$ solutions since it is a requirement imposed by the user. If the number of solutions on the external list reaches this value, the Pareto-set presented as the final solution will be the one kept on that list. Notice that, on the external list, the non-dominated solutions which are better spread during the optimization process have been stored. However, it may happen that the number of non-dominated solutions found by the algorithm is smaller than $M$. In such a case, a joint list will be composed considering all the elements on the population list and the external list, and the $M$ most preferable solutions among them will be offered as a result by the algorithm.

2.2 The improving method

Most MOEAs include a mutation operator that alters the individuals of the population from its initial state. This can result in entirely new solutions being added to the population. With these new solution values, a multi-objective algorithm may be able to increase the population diversity and the probability to escape from local optima, helping then to push the population towards the true Pareto-set. Mutation occurs during evolution according to a user-definable mutation probability. However, the mutation operators are usually slow, requiring many function evaluations for convergence.

In this work, an improving method is suggested to accelerate the convergence of the population towards the optimal Pareto-front. Basically, the local method improves a given solution by making changes of different sizes along a search direction. In fact, the designed improving method is an extension of the local optimizer SASS, initially proposed by Solis and Wets in [54] to cope with single-objective optimization problems. Here, it has been adapted to work on multi-objective optimization problems. The proposed algorithm will be called MO_SASS throughout this paper. The way the heuristic MO_SASS works is described in Algorithm 3.

The algorithm MO_SASS can be applied to an arbitrary multi-objective optimization problem over a bounded subset of $\mathbb{R}^n$, although internally it assumes, as the original SASS does, that the range in which each variable is allowed to vary is the interval $[0,1]$. The new points are generated using a Gaussian perturbation $\xi \in \mathbb{R}^n$ over the search point $y$ and a normalized bias term $b \in \mathbb{R}^n$ to direct the search. In this way, given $y$, a first trial point, $y + \xi$ is considered, and if it dominates $y$, then $y + \xi$ replaces the initial point, but maintaining the same radius value. Otherwise, if $y$ and $y + \xi$ are indeterminate solutions, then $y + \xi$ is compared pairwise to the points on the external list. If it is dominated by any point from such a list, it is discarded; otherwise, it is stored in the external list. Notice that, as a consequence of this inclusion,
there may be dominated solutions in the external list. In such a case, those solutions are removed.

The coefficient values 0.4, 0.2 and 0.5 in steps 18, 24 and 26, used for updating the bias term \( b \) are retained from Solis and Wets's results [54]. The standard deviation \( \sigma \) specifies the size of the sphere that most likely contains the perturbation vector, whereas the bias term \( b \) locates the center of the sphere based on directions of past successes. The size of the standard deviation of the normalized perturbation \( \xi_{\text{aux}} \) is controlled by the repeated number of successes, \( \text{scnt} \), or failures, \( \text{fcnt} \). A success occurs when the new point dominates the initial one. The contraction (\( \text{ct} \)) and expansion (\( \text{ex} \)) constants are set by the user.

Algorithm 3 Algorithm MO_SASS\((y, \sigma_{ub}, bel)\)

1: Set \( ic = 1 \), \( y^{(ic)} = y \), \( b^{(ic)} = 0 \), \( \text{scnt} = 0 \), \( \text{fcnt} = 0 \), \( \sigma^{(0)} = \sigma_{ub} \), \( \sigma_{lb} = \max(\sigma_{ub}/1000, 10^{-5}) \)
2: Fix \( \text{ex} \), \( \text{ct} \), \( \text{Fcnt} \), \( \text{Maxfcnt} \), \( ic_{\text{max}} \)
3: while \( ic < ic_{\text{max}} \) and \( \text{fcnt} < \text{Maxfcnt} \)
4: \( \sigma^{(ic)} = \sigma^{(ic-1)} \)
5: if \( \text{scnt} > \text{Fcnt} \)
6: \( \sigma^{(ic)} = \text{ex} \cdot \sigma^{(ic-1)} \)
7: if \( \text{fcnt} > \text{Fcnt} \)
8: \( \sigma^{(ic)} = \text{ct} \cdot \sigma^{(ic-1)} \)
9: if \( \sigma^{(ic)} < \sigma_{lb} \)
10: \( \sigma^{(ic)} = \sigma_{ub} \) and \( b^{(ic)} = 0 \)
11: if \( \sigma^{(ic)} > \sigma_{ub} \)
12: \( \sigma^{(ic)} = \sigma_{ub} \)
13: Generate a multivariate Gaussian random vector \( \xi^{(ic)}_{\text{aux}} = N(b^{(ic)}, \sigma^{(ic)} I) \)
14: if \( y^{(ic)} + \xi^{(ic)} \) dominates \( y^{(ic)} \)
15: \( y^{(ic+1)} = y^{(ic)} + \xi^{(ic)} ; \text{scnt} = \text{scnt} + 1 ; \text{fcnt} = 0 \)
16: else
17: if \( bel = 0 \) and \( y^{(ic)} + \xi^{(ic)} \) is not dominated by any point on the external list
18: Include \( y^{(ic)} + \xi^{(ic)} \) in external list; \( \text{scnt} = 0 ; \text{fcnt} = \text{fcnt} + 1 ; b^{(ic+1)} = 0.4 \xi^{(ic)}_{\text{aux}} + 0.2 b^{(ic)} \)
19: else
20: if \( y^{(ic)} - \xi^{(ic)} \) dominates \( y^{(ic)} \)
21: \( y^{(ic+1)} = y^{(ic)} - \xi^{(ic)} ; \text{scnt} = \text{scnt} + 1 ; \text{fcnt} = 0 \)
22: else
23: if \( bel = 0 \) and \( y^{(ic)} - \xi^{(ic)} \) is not dominated by any point on the external list
24: Include \( y^{(ic)} - \xi^{(ic)} \) in external list; \( \text{scnt} = 0 ; \text{fcnt} = \text{fcnt} + 1 ; b^{(ic+1)} = b^{(ic)} - 0.4 \xi^{(ic)}_{\text{aux}} \)
25: else
26: \( b^{(ic+1)} = 0.5 b^{(ic)} ; \text{fcnt} = \text{fcnt} + 1 ; \text{scnt} = 0 \)
27: \( ic = ic + 1 \)
28: Return \( y^{(ic)} \)

As for the individual radius, it was mentioned that an individual has a radius associated to it which determines the subregion of the search space covered by that individual, in such a way that any single step taken by the improving method in a given individual is no longer than the radius of the individual. Since in MO_SASS the standard deviation \( \sigma \) specifies the size of
the sphere that most likely contains the normalized perturbation vector, its upper bound $\sigma_{ub}$ should have the same value than the normalized radius of the caller individual. That is why the parameter $\sigma_{ub}$ is also considered an input argument in MO_SASS.

It is worth mentioning that the use of MO_SASS allows, on the one hand, to push $y$ towards the true Pareto-set (steps 14-15 and 20-21 in Algorithm 3) and, on the other hand, to study its surrounding area to obtain indeterminate solutions (steps 17-18 and 23-24 in Algorithm 3). The inclusion of indeterminate points in the external_list may improve the quality of the final Pareto-front, but it increases the computational effort (the more elements on the list, the more computing time required to order it). Notice that MO_SASS is called (through the Improving method) by FEMOEA to improve both the population_list and the external_list, which may involve a large number of indeterminate points. Looking for a compromise between quality in the final Pareto-front and computational effort, Improving_method does not execute steps 17-18 and 23-24 when improving the external_list. The input parameter $bel$ tells MO_SASS whether the solution $y$ belongs to the population_list ($bel = 0$) or the external_list ($bel = 1$).

The stopping rules are determined by the maximum number of iterations ($ic_{max}$) and by the maximum number of consecutive failures ($Maxfcnt$). After a comprehensive computational study, they have been set to 400 and 20, respectively.

In order to study whether the introduction of MO_SASS into FEMOEA really helps to accelerate its convergence towards the optimal Pareto-front or, on the contrary, the obtained results are the consequence of randomness, another improving method has been designed. Algorithm 4 sketches its main structure. Basically, this method behaves like MO_SASS, although it tries to improve the initial solution $y$ by making random changes, instead of following a search direction. The actions carried out when indeterminate solutions are obtained as well as the termination criteria, are the same as the ones described for MO_SASS.

**Algorithm 4 Algorithm Random1($y, \sigma_{ub}, bel$)**

1: Set $ic = 1$, $y^{(ic)} = y$, $scnt = 0$, $fcnt = 0$
2: Fix $Scnt$, $Fcnt$, $Maxfcnt$, $ic_{max}$
3: while $ic < ic_{max}$ and $fcnt < Maxfcnt$
4: Generate a random vector $y_{ran}^{(ic)}$ in the area defined by the species radius $\sigma_{ub}$
5: if $y_{ran}^{(ic)}$ dominates $y^{(ic)}$
6: $y^{(ic+1)} = y_{ran}^{(ic)}$, $scnt = scnt + 1$, $fcnt = 0$
7: else
8: if $bel = 0$ and $y_{ran}^{(ic)}$ is not dominated by any point on the external_list
9: Include $y_{ran}^{(ic)}$ in external_list; $scnt = 0$, $fcnt = fcnt + 1$
10: else
11: $scnt = 0$, $fcnt = fcnt + 1$
12: $ic = ic + 1$
13: Return $y^{(ic)}$
2.3 The stopping rule

The termination criterion of most MOEAs in literature is only based on a number of function evaluations, i.e. the algorithms usually stop when a maximum is achieved [15, 45, 70]. However, although this stopping rule can be suitable to compare algorithms in terms of efficiency, it can be counterproductive for practical purposes. The number of function evaluations required to obtain good approximations of the Pareto-front is not known in advance and depends on the particular instance to be solved. Hence, whatever the number we choose, it may be too small for some problems and too high for others, to obtain the quality desired by the user.

In this work, a new stopping rule based on the well-known Hausdorff distance is proposed. Informally, it measures how far two sets are from each other. Mathematically, the modified Hausdorff distance

$$hd(F_1, F_2) = \frac{\sum_{a \in F_1} \min\{d(a, b) : b \in F_2\}}{\max\{d(a, a') : a, a' \in F_1\}} + \frac{\sum_{b \in F_2} \min\{d(a, b) : a \in F_1\}}{\max\{d(b, b') : b, b' \in F_2\}}$$

where $F_1$ and $F_2$ are two given discrete sets and $d(\cdot, \cdot)$ is a distance function (we have used the Euclidean distance).

The termination criteria proposed in this work establishes that the algorithm will finish if during three consecutive iterations, the changes experimented in the candidate Pareto-front are negligible (in terms of the objective function values), for a given tolerance $tol$ (for this work, $tol = 10^{-7}$), i.e. the algorithm stops at iteration $t$ provided

$$hd(\text{Pareto-front}_t, \text{Pareto-front}_{t-1}) < tol, \text{ and}$$

$$hd(\text{Pareto-front}_{t-1}, \text{Pareto-front}_{t-2}) < tol.$$ 

Notice that this stopping criterion allows the algorithm to terminate whenever a good approximation of the Pareto-front is obtained, or even if the algorithm is trapped, unable to converge to a better Pareto-front. This may allow the algorithm to save considerable CPU time in some instances.

As a safeguard, a second termination criterion, which can be based on the maximum number of function evaluations allowed, the maximum number of iterations permitted or even the maximum execution time admitted to provide the solution should be defined. For the particular case of FEMOEA, we have fixed a maximum number of iterations. This maximum value is represented by the input parameter $L$.

2.4 Input parameters

Five input parameters must be provided by the user:

- $M$: The number of solutions which must compose the final Pareto-front.
- $L$: The maximum number of levels (or iterations).
– $R_1$ and $R_L$: The radii of the individuals that are associated to the minimum and maximum level, respectively.
– $tol$: The tolerance associated with the termination criterion.

Notice that the only parameters which really need to be fine tuned are $R_L$ and $L$. The remaining ones are either a determination of the user based on his/her experience, requirements or needs (as occurs with the value of $M$ and $tol$), or a parameter associated to the particular problem to be handled ($R_1$).

3 Computational studies

All the computational results in this paper have been carried out on a 4-core processor HP ProLiant ML330 G6 to 2.00GHz and 7.8GB memory (using one of its cores).

3.1 Test problems

A thorough study has been conducted to investigate the performance of the analyzed algorithms. A set of 20 standard benchmark problems has been used. In particular, the so-called ZDT functions as well as the DTLZ family have been included in the performance assessment. Additionally, nine multi-objective test problems from literature have been considered. The mathematical formulation of those problems can be seen in the Appendix. 18 of them are bi-objective problems, and the other 2 are tri-objective problems.

The ZDT family of functions [69] has been selected because it is a broad and popular set of test functions for benchmarking the performance of multi-objective optimization methods. Each of these test functions contains a particular feature that is representative of a real world optimization problem that could cause difficulty in converging to the Pareto-front. The ZDT1 function has a convex Pareto-optimal front. The ZDT2 function has a non-convex Pareto-optimal front. The ZDT3 function adds a discreteness feature to the front. Its Pareto-optimal front consists of several noncontiguous convex parts. The introduction of a sine function in this objective function causes discontinuities in the Pareto-optimal front, but not in the decision space. The ZDT4 function has 21 local Pareto-optimal fronts and therefore is highly multi-modal. The ZDT6 function has a non-uniform search space: the Pareto-optimal solutions are non-uniformly distributed along the global Pareto-front, and also the density of the solutions is lowest near the Pareto optimal front and highest away from the front. In all the ZDT functions, two design variables have been chosen. The design variables range is the interval $[0, 1]$, except for the ZDT4 problem, where the second design variable lies on the interval $[-5, 5]$ (its first one is also included on the interval $[0, 1]$).

The DTLZ suite of benchmark problems, created by Deb et al. [16], differs from the majority of multi-objective test problems in that the problems are scalable to any number of objectives, $m$. Nine test problems are included
in this family, of which the first seven are solved in this work. DTLZ8 and DTLZ9 have side constraints that cannot be managed by the unconstrained proposed algorithm, hence their omission from this paper. Additionally, all DTLZ functions are scalable with respect to the number of design variables $n \geq m$, whose ranges lie in the interval $[0,1]$. For all the DTLZ functions discussed in this work, two objectives as well as two design variables have been adopted. For a deep discussion about fine details of these test problems, fitness landscapes, and Pareto optimal front geometries, the interested reader is referred to [9,33].

Viennet and Viennet2 are three-objective test problems. The Viennet Pareto-front is a connected region in criterion space, while the true Pareto-set of Viennet2 consists of disconnected areas in decision space. Furthermore, the optimal Pareto-front of Viennet2 is a single, convoluted three-dimensional Pareto curve. Both problems have two decision variables, whose ranges are $[-3,3]$ and $[-4,4]$, respectively. Again, the concerned reader is referred to [9] for a comprehensive analysis of these functions.

The behavior of the algorithms has also been tested with several functions whose fronts are non-connected and often referred to as discontinuous in literature. Apart from ZDT3, DTLZ7 and Viennet2, whose fronts are non-connected, the problems labeled as Deb, Kursawe and Poloni, have also been included in the study (see [33]). This subset of benchmark problems are bi-objective. Regarding the number of decision variables, it is equal to 2 for Deb and Poloni problems and to 3 for the other one. The corresponding ranges are $[0,1]$, $[-5,5]$ and $[-\pi,\pi]$, respectively. For the sake of completeness, four more bi-objective problems deeply referenced in literature, have been included, i.e. Deb1, Fonseca, Qv and Schaffer. For more information about these benchmark problems see [9,14,33,70,69]. For the study at hand, the decision variables are on the intervals $[0.1,1]$, $[-4,4]$, $[-5,5]$ and $[-10^5,10^5]$ respectively.

3.2 The algorithms and their implementations

In order to study its performance, FEMOEA has been compared to MOEA/D, SPEA2 and NSGA-II, three algorithms which represent the state of the art of MOEAs. The implementations provided by the framework jMetal [18] have been used.

jMetal is an object-oriented Java-based framework aimed at the development, experimentation, and study of metaheuristics for solving multi-objective optimization problems. jMetal provides a rich set of classes which can be used as the building blocks of multi-objective metaheuristics; taking advantage of code-reusing, the algorithms share the same base components, such as implementations of genetic operators and density estimators. jMetal includes several multi-objective metaheuristics and many problems usually included in performance studies. Additionally, it also provides quality indicators to performance assessing as well as a set of utilities that help in carrying out experimental studies.
The versions of MOEA/D, NSGA-II and SPEA2 obtained from jMetal are in Java. The algorithm FEMOEA has been implemented in C++.

The parameter setting for NSGA-II is the one used by Deb in [15]. The operators for crossover and mutation are SBX and polynomial mutation, with distribution indexes of $\eta_c = 20$ and $\eta_m = 20$, respectively. A crossover probability of $p_c = 0.9$ and a mutation probability $p_{mut} = 1/n$ (where $n$ is the number of decision variables) are used. Regarding SPEA2, the crossover and mutation operators are the same as those used in NSGA-II, using the same values concerning their application probabilities and distribution indexes. The parameters used for MOEA/D are the ones proposed in [68]. Regarding FEMOEA, we found that a good parameter setting is: $L = 30$ and $R_L = 0.005$. The parameter $R_1$ coincides with the diameter of the search space.

For all the algorithms, the number $M$ of points in the Pareto-front has been set to 100 for bi-objective problems and 300 for tri-objective instances. The same number of function evaluations employed in average (considering all the runs) by FEMOEA for a given problem was used in each of the runs of the other algorithms.

3.3 Effectiveness measures and statistical testing

Effectiveness measures the accuracy and convergence of obtained solutions (the quality of the final Pareto-fronts). Several indicators used in literature have been utilized. To quantify effectiveness, we have proceeded as in literature and all the algorithms have been run considering the same number of function evaluations. Since the analyzed algorithms are heuristics, every particular instance has been run a given number of times (100 in our computational studies) for each algorithm, and average values for every performance indicator have been computed.

According to [72] there exist three main methods for the assessment and comparison of Pareto-set approximations: the dominance ranking method (which allows collections of Pareto-set approximations from two or more stochastic optimizers to be directly compared statistically), the quality indicator method (the dominant method in literature, which maps each Pareto-front approximation to a number, and performs statistics on the resulting distribution(s) of numbers) and the attainment function method (which estimates the probability of attaining each goal in the objective space, and looks for significant differences between these probability density functions for different optimizers). In this paper both the dominance ranking and quality indicator approaches are followed. Before detailing them, some definitions are needed.

**Definition 4** A feasible vector $y^* \in S$ is said to be weakly efficient iff there does not exist another feasible vector $y \in S$ such that $f_l(y) \leq f_l(y^*)$ for all $l = 1, \ldots, m$. If $y_1$ and $y_2$ are two feasible points and $f_l(y_1) \leq f_l(y_2), l = 1, \ldots, m$, then we say that $y_1$ weakly dominates $y_2$, and will be denoted by $y_1 \preceq y_2$. 
Definition 5 We say that set $A$ weakly dominates set $B$, $A \preceq B$, provided that every point $y_2 \in B$ is weakly dominated by at least one point $y_1 \in A$.

Definition 6 We say that sets $A$ and $B$ are indifferent, $A \sim B$, iff $A \preceq B$ and $B \preceq A$.

Definition 7 We say that set $A$ is better than set $B$, denoted by $A \ll B$, iff $A \preceq B$ and $A \not\sim B$.

The corresponding definitions apply in the criterion space.

In the following subsections we will give the formulae of several unary quality indicators. A general formal definition follows.

Definition 8 A unary quality indicator is a function $I : \Omega \to \mathbb{R}$ which assigns each Pareto-front approximation set $PF_{ap} \in \Omega$ a real value $I(PF_{ap})$.

It is desired that whenever an approximation set $A$ of the Pareto-set is preferable to an approximation set $B$ with respect to weak Pareto dominance, the indicator value for $f(A)$ should be at least as good as the indicator value for $f(B)$. Such indicators are called Pareto compliant.

Definition 9 A quality indicator is said to be Pareto compliant iff for any pair of approximation sets $A, B$, $A \preceq B$ implies that the indicator assigns a better (or equal) indicator value to $f(A)$.

Notice that many of the indicators employed in literature are not Pareto compliant. They are usually designed to assess a single aspect of the approximation (its proximity to the true Pareto-front, its spread, its evenness, etc). They are still useful, since they may refine the preference structure of Pareto compliant indicators.

Assume that we want to compare the quality of the outcomes generated by $Q$ stochastic algorithms. For each algorithm $q$, $q \in \{1, \ldots, Q\}$, $e_q$ runs are performed, generating the approximation sets $PS_q^1, \ldots, PS_q^{e_q}$ (in the decision space). Let us denote by $SPS$ the set of all the approximation sets of the Pareto set, $SPS = \{PS_1^1, \ldots, PS_1^{e_1}, \ldots, PS_Q^1, \ldots, PS_Q^{e_Q}\}$.

In some of the indicators listed below the approximation sets of the Pareto-front need to be compared to the true Pareto-front. However, the true Pareto-front is not usually known or cannot be completely obtained. Then, a reference set $RS$ which approximates the true Pareto-front is used instead. In our studies, the reference set $RS$ has been obtained as follows. All the approximation sets in $SPS$ are combined, and then, the dominated points are removed from this union. The image of the remaining points forms the reference set.

Additionally, normalized objective values are used to allow different objectives to contribute equally to comparative indicator values. The standard normalization is

$$f_l(y)' = \frac{f_l(y) - z_l^{(\min)}}{z_l^{(\max)} - z_l^{(\min)}}.$$
where $z_l^{(\text{min})}$ (resp. $z_l^{(\text{max})}$) denotes the minimum (resp. maximum) value of $f_l$ when considering all the solutions in $SPS$.

In the following, the methods utilized in this paper for the assessment and comparison of Pareto set approximations are described.

### 3.3.1 Dominance ranking

There are several ways to assign each approximation set a rank on the basis of a dominance relation. In this paper the dominance ranking of a specific approximation set is related to the number of sets by which the set is dominated considering `$\prec$' the domination relationship [27].

**Definition 10** The dominance ranking of a set $PS_i \in SPS$ is given by

$$\text{rank}(PS_i) = 1 + |\{PS_j \in SPS : PS_j \prec PS_i\}|.$$  

The lower the rank, the better the approximation is. As the dominance ranking approaches rely on the concept of Pareto dominance and some ranking procedure only, they yield general statements about the relative performance of the algorithms which are independent of any preference information.

### 3.3.2 Quality indicators

The most commonly used quality indicator in literature is hypervolume [73, 63]. This Pareto compliant indicator measures the hypervolume of the portion of the criterion space that is weakly dominated by the approximation set. The higher the hypervolume, the better the approximation. In order to measure this quantity, a reference point that is dominated by all points is needed. For a given problem, the same reference point has to be used for all the algorithms and all the runs. In our computational studies, the point whose $l$-th component is the maximum of all the $l$-th components of points in $f(SPS)$ is considered. It is an approximation of the Nadir point obtained when considering all the approximations of the Pareto-front together.

Hypervolume can be thought of as a global quality indicator, in the sense that it assesses the approximation set as a whole. On the other hand, proximity indicators somehow measure the distance between the approximation set and the reference set. In this paper we have used two of those measures, namely, the average distance [13] and the unary additive epsilon indicator [71]. The former is not Pareto compliant, and is given by

$$D_{av}(f(PS_i)) = \frac{\sum_{b' \in RS} \min_{a' \in PS_i} d_{\infty}(f(a'), b')}{|RS|}$$

where

$$d_{\infty}(f(a'), b') = \max_{i=1,\ldots,m} \{|f_i(a') - b'^l| / z_l^{(\text{max})} - z_l^{(\text{min})}\}.$$
The latter is Pareto compliant and is computed as
\[ I_+(f(PS_i)) = \min_{\epsilon \in \mathbb{R}} \{ \forall b^r \in RS \exists a^i \in PS_i : \frac{f(a^i) - z_{l(\min)}}{z_{l(\max)} - z_{l(\min)}} - \epsilon \leq \frac{b^r - z_{l(\min)}}{z_{l(\max)} - z_{l(\min)}} \} \forall l \in \{1, \ldots, m\} \]
and gives the minimum distance by which \( f(PS_i) \) needs to be translated in each dimension in objective space such that \( RS \) is weakly dominated.

Other two evenness/diversity indicators are used in the studies. None of them is Pareto compliant. They are the spread [15,45], another well-known indicator, and the spacing [58], given by
\[ TS(f(PS_i)) = \sqrt{\frac{1}{|PS_i|} \sum_{a^j \in PS_i} (D_j - \overline{D})^2} \]
where \( \overline{D} = \sum_{a^j \in PS_i} D_j / |PS_i| \) and \( D_j \) is the Euclidean distance in the criterion space between the solution \( f(a^j) \) and its nearest solution, i.e.,
\[ D_j = \min_{a^i \in PS_i} \{ \ell_2((f_1(a^i) - z_{1(\min)}^{\min}) / z_{1(\max)}^{\max} - z_{1(\min)}^{\min}, \ldots, f_m(a^i) - z_{m(\min)}^{\min}) / z_{m(\max)}^{\max} - z_{m(\min)}^{\min}) \} \).
It is related to the generational distance [62].

### 3.3.3 Statistical testing

We are dealing with stochastic algorithms. Thus, each run may produce a different result. In fact, in our studies, each algorithm \( q \) has been run a given number of times \( e_q \) for each problem, and the average values of each indicator for each problem are reported. But in order to have conclusive results, the following statistical analysis has also been performed for each indicator and problem. We have tested whether the means of the indicator for each of the algorithms are all equal (null hypothesis) or at least one of them is different from the other ones. To do it, we first test whether the corresponding values of the indicator for each algorithm follow a normal distribution, applying the Kolmogorov-Smirnov and Shapiro-Wilk tests. We assume normal distribution provided that at least one of those tests accepts normality. Then, if the distribution is normal for all the algorithms, we check the homocedasticity of the variables (i.e., whether the variances are equal), applying the Levene and Barlett tests. Again, we assume homocedasticity provided one of those tests accepts it. If so, an ANOVA test is performed; otherwise, a Welch test is done. In case one of the distributions is not normal, then the non-parametric Kruskal-Wallis test is performed to check the equality of the means.

We have always used a confidence level of 95% (i.e., we reject the null hypothesis provided the \( P \)-value is under 0.05). Notice that in case the null hypothesis is rejected, we conclude that not all the means are equal. This is marked by a ‘+’ symbol in all the following tables. Conversely, ‘-’ means that no statistical difference was found.
3.4 About the improving method

This section researches whether the designed improving method, i.e. Algorithm 3, really collaborates to approximate the solutions to the optimal Pareto-front or, on the contrary, a simple optimization technique based on random movements (Algorithm 4) is able to obtain similar results. To this aim, FEMOEA has been executed with both local searching methods, and a comparison has been carried out.

Since FEMOEA is a heuristic, different runs may provide different solutions. To take this effect into account, FEMOEA with every improving method, has been run 100 times for each test problem, and average values have been computed. In particular, the mean computing time ($\text{Av}(T)$) in seconds, the mean number of function evaluations ($\text{Av}(\text{eval})$), the mean hypervolume ($\text{Av}(\text{hyper})$), the mean $I_{\epsilon}^1$ indicator ($\text{Av}(I_{\epsilon}^1)$) and the mean spread ($\text{Av}(\text{Spr})$), have been calculated. Tables 1 and 2 summarize the obtained results for each benchmark problem. Additionally, the average values for the 20 problems have been computed and shown in the last line of the tables. As can be observed, the results obtained by FEMOEA are better when Algorithm 3 is considered as improving method. Notice that the number of consecutive successes is larger when such an algorithm is taken into account, i.e. the number of points which subsequently dominates the caller solution is greater. It means that Algorithm 3 uses more function evaluations, and hence more computing time than Algorithm 4, which usually finishes because the maximum number of failures is reached (see step 3 in Algorithm 4). This fact can be observed when comparing $\text{Av}(T)$ and $\text{Av}(\text{eval})$ columns.

However, in order to clearly show that the results of FEMOEA are affected by the selected improving method, Algorithm 4 has been modified by omitting Step 11. This allows to significantly reduce the number of counted consecutive failures, which obviously increases the number of attempts to achieve non-dominated solutions. Algorithm 4 without step 11 will be called ModAlg4. The results obtained by FEMOEA with ModAlg4 are shown on Table 3. As can be seen, the Pareto fronts provided by FEMOEA are better when Algorithm 3 is considered, in spite of ModAlg4 executing a larger number of function evaluations.

In what follows, only FEMOEA with Algorithm 3 is used in the comparative studies, and it will be denoted simply by FEMOEA.

3.5 Results for global indicators

The dominance ranking for all the outcomes was always 1. In this sense, the four algorithms are equally competitive. So we are led to the use of the quality indicators in order to be able to assess which algorithm is the best.

On Table 4, the hypervolume average results obtained by the different algorithms are given. The final column refers to the test analyzing the equality of the means (see Subsection 3.3.3). As we can see, none of the algorithms
Table 1 Results obtained by FEMOEA when Algorithm 3 is considered as improving method.

<table>
<thead>
<tr>
<th></th>
<th>Av($T$)</th>
<th>Av(eval)</th>
<th>Av(hyper)</th>
<th>Av($I^*_1$)</th>
<th>Av(Spr)</th>
</tr>
</thead>
<tbody>
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<td>ZDT1</td>
<td>2.60e+01</td>
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<td>6.61678e-01</td>
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<td>Average</td>
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<td>5.48609e-01</td>
<td>4.86e-03</td>
<td>3.14e-01</td>
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</table>

obtains the highest hypervolume in all the problems: FEMOEA obtains the best results (highlighted with gray background) in 9 out of the 20 problems, SPEA2 in 7 problems and both MOEA/D and NSGA-II in only 2 problems. As we can see, in all the cases the test is positive. Notice that FEMOEA is in average, considering all the problems, the algorithm providing the best results, followed by NSGA-II, SPEA2 and MOEA/D, in that order.

3.6 Results for proximity indicators

Table 5 shows the results for the average distance indicator. As with the hypervolume, the final column refers to the test analyzing the equality of the means. Again, none of the algorithms obtains the smallest average distance in all the problems, although MOEA/D obtains the best results in 9 out of the 20 problems, SPEA2 in 6 problems, FEMOEA in 5 instances and NSGA-II in 0. In all the cases the test is positive. Considering all the problems, FEMOEA obtains the smallest average, followed by SPEA2, NSGA-II and MOEA/D, in this order.

Similarly, Table 6 shows the average results for the additive epsilon indicator. FEMOEA obtains the best results in 7 out of the 20 problems, MOEA/D in
Table 2 Results obtained by FEMOEA when Algorithm 4 is considered as improving method.

<table>
<thead>
<tr>
<th></th>
<th>$\text{Av}(T)$</th>
<th>$\text{Av}(\text{eval})$</th>
<th>$\text{Av}(\text{hyper})$</th>
<th>$\text{Av}(I_{I_1}^2)$</th>
<th>$\text{Av}(\text{Spr})$</th>
</tr>
</thead>
<tbody>
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<td>ZDT1</td>
<td>1.92e+01</td>
<td>4.96e+05</td>
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</table>

6 instances, SPEA2 in 4 problems and NSGA-II in 3. Considering all the problems, FEMOEA obtains the smallest average, followed by NSGA-II, SPEA2 and MOEA/D. Notice that the test is negative for problems ZDT1, ZDT2, DTLZ2, DTLZ3, DTLZ7.

3.7 Results for dispersion indicators

Tables 7 and 8 show the results for the spread and spacing indicators, respectively. Having a look at the tables, similar conclusions as before can be inferred, i.e. none of the algorithms is the best one for the whole set of problems. Furthermore, FEMOEA overcomes MOEA/D, NSGA-II and SPEA2, obtaining a better spread for 9 out of 20 problems as well as a smaller spacing value for 10 out of 20 problems. The algorithm SPEA2 achieves the best results for 7 problems for the spread metric and for 6 instances for the spacing measure, while MOEA/D does it for 4 problems for both metrics. As can be seen, NSGA-II is defeated in all the cases. On average, FEMOEA obtains the smallest average for both indicators, followed by SPEA2, NSGA-II and MOEA/D, in this order. Finally notice that in all the cases the test is positive.
Table 3 Results obtained by FEMOEA when ModAlg4 is considered as improving method.

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<tr>
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<th>$Av(T)$</th>
<th>$Av$ (eval)</th>
<th>$Av$ (hyper)</th>
<th>$Av(I_{+}^\epsilon)$</th>
<th>$Av$ (Spr)</th>
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</table>

4 Conclusions and future research

In this work, a new multi-objective optimization algorithm, FEMOEA, has been proposed. Furthermore, a new technique (Algorithm 3) to improve the quality of the obtained approximation of the Pareto-front and a new stopping rule to reduce the computational effort have been also presented. These tools can be incorporated to any multi-objective optimization algorithm.

FEMOEA, with those two devices, has been compared to three algorithms widely referenced in literature, i.e. MOEA/D [68] NSGA-II [15] and SPEA2 [70]. The performance of these four algorithms has been analyzed considering several metrics. More precisely, global indicators, dispersion indicators and also proximity indicators have been computed. Results have shown that, on average, FEMOEA overcomes MOEA/D, NSGA-II and SPEA2 in all the metrics (see Tables 4, 5, 6, 7 and 8).

The effectiveness of FEMOEA is not only a consequence of the use of Algorithm 3 as Improve_individual procedure. On the contrary, FEMOEA is competitive by itself, and it is able to obtain better results even when random-like methods are considered to improve the solutions: compare the results on tables 2 and 3 (columns 4-6), where the hypervolume metric, the $I_{+}^\epsilon$ indicator and the spread measure obtained by FEMOEA with two random methods are
Table 4: Average hypervolume values. MOEA/D, NSGA-II and SPEA2 were run with the same number of function evaluations as FEMOEA.

<table>
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<tr>
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<th>FEMOEA</th>
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<th>NSGA-II</th>
<th>SPEA2</th>
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shown, with the corresponding results obtained by MOEA/D, NSGA-II and SPEA2 on Tables 4, 6 and 7, respectively. As can be observed, the average hypervolume value obtained by FEMOEA are better than those achieved by the other algorithms. For the mean $I_{\epsilon}^+$, FEMOEA is better than MOEAD.

Finally, for the spread metric, FEMOEA outperforms NSGA-II and MOEA/D.

In the future, FEMOEA will be adapted to solve constrained problems in order to be able to solve any kind of multi-objective optimization problem. Hitherto, little research has been done on the design of methods for the constraint handling in multi-objective optimization (see some examples in [7,15, 28,65]). Therefore, it is an important challenge to research on defining new mechanisms able to solve the different conflicting objectives subject to various constraints.

Furthermore, several parallel approaches based on the coarse-grain paradigm will be analyzed. In a coarse-grain model, each processing element executes the sequential algorithm independently of the remaining ones during most of the time. In most papers in literature, the idea is that different processing elements work with smaller and different individuals in such a way that, when merging all the individuals, a population similar to that of the sequential version can be obtained. However, for the problem at hand, we would also like to study the possibility of reducing the computational time by decreasing
Table 5 Average values for the average distance indicator. MOEA/D, NSGA-II and SPEA2 were run with the same number of function evaluations as FEMOEA.

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<th>SPEA2</th>
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Table 6 Average $I_\epsilon$ values. MOEA/D, NSGA-II and SPEA2 were run with the same number of function evaluations as FEMOEA.

<table>
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<th>NSGA-II</th>
<th>SPEA2</th>
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the number of iterations that a particular processing element can execute. Of course, to maintain the effectiveness of the sequential version, some informa-
Table 7  Average spread values. MOEA/D, NSGA-II and SPEA2 were run with the same number of function evaluations as FEMOEA.

<table>
<thead>
<tr>
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<th>FEMOEA</th>
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<th>SPEA2</th>
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Table 8  Average spacing values. MOEA/D, NSGA-II and SPEA2 were run with the same number of function evaluations as FEMOEA.

<table>
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<tr>
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<th>FEMOEA</th>
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<th>SPEA2</th>
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<td>7.08e-01</td>
<td>5.45e-01</td>
</tr>
</tbody>
</table>

The function must migrate from a processing element to another one according to...
migratory policy. Different migratory policies will be analyzed looking for a compromise between effectiveness and efficiency of the parallel versions.

References


40. C.E. Mariano and E. Morales. MOAQ: an ant-Q algorithm for multiple objective optimization problems. In W. Banzhaf, J. Daida, A.E. Eiben, M.H. Garzon, V. Honavar,


**Appendix**

**ZDT1**

\[
\begin{align*}
    f_1 &= x_1 \\
    f_2 &= g(x)[1 - \sqrt{\frac{x_1}{g(x)}}] \\
    g(x) &= 1 + (9\sum_{i=2}^{n} x_i) \\
\end{align*}
\]

where \(0 \leq x_i \leq 1\) with \(i = 1..n\). In our studies \(n = 2\)

**ZDT2**

\[
\begin{align*}
    f_1 &= x_1 \\
    f_2 &= g(x)[1 - (\frac{x_1}{g(x)})^2] \\
\end{align*}
\]
\[ g(x) = 1 + \left(9 \sum_{i=2}^{n} x_i^{n-1} \right) \]

where \(0 \leq x_i \leq 1\) with \(i = 1..n\). In our studies \(n = 2\)

**ZDT3**

\[ f_1 = x_1 \]
\[ f_2 = g(x)[1 - \sqrt{\frac{x_1}{g(x)}} - \frac{x_1}{g(x)} \sin(10\pi x_1)] \]
\[ g(x) = 1 + \left(9 \sum_{i=2}^{n} x_i^{n-1} \right) \]

where \(0 \leq x_i \leq 1\) with \(i = 1..n\). In our studies \(n = 2\)

**ZDT4**

\[ f_1 = x_1 \]
\[ f_2 = g(x)[1 - (\frac{x_1}{g(x)})^2] \]
\[ g(x) = 1 + 10(n - 1) + \sum_{i=2}^{n} [x_i^2 - 10\cos(4\pi x_i)] \]

where \(0 \leq x_1 \leq 1, -5 \leq x_i \leq 5\) with \(i = 2..n\). In our studies \(n = 2\)

**ZDT6**

\[ f_1 = 1 - \exp(-4x_1) \sin^6(6\pi x_1) \]
\[ f_2 = g(x)[1 - (x_1/g(x))^2] \]
\[ g(x) = 1 + \left(9 \sum_{i=2}^{n} x_i^{n-1} \right) \]

where \(0 \leq x_i \leq 1\) with \(i = 1..n\). In our studies \(n = 2\)

**DTLZ1**

\[ f_1(x) = \frac{1}{2} x_1 x_2 ... x_{m-1} (1 + g(x_m)) \]
\[ f_2(x) = \frac{1}{2} x_1 x_2 ... x_{m-1} (1 - x_{m-1}) (1 + g(x_m)) \]
... \[ f_{m-1}(x) = \frac{1}{2} x_1 (1 - x_2) (1 + g(x_m)) \]
\[ f_m(x) = \frac{1}{2} (1 - x_1) (1 + g(x_m)) \]
\[ g(x_m) = 100[x_{m+1} + \sum_{i=x_m}^{n} ((x_i - 0.5)^2 - \cos(20\pi(x_i - 0.5))] \]

where \(0 \leq x_i \leq 1\) and \(i = 1..n\). In our studies \(n = 2\) and \(m = 2\)

**DTLZ2**

\[ f_1(x) = (1 + g(x_m)) \cos(\frac{x_1}{2^1}) \cos(\frac{x_2}{2^2}) ... \cos(\frac{x_{m-1}}{2^{m-2}}) \cos(\frac{x_{m-1}}{2^{m-1}}) \]
FEMOA: a Fast and Efficient Multi-Objective Evolutionary Algorithm

\[ f_2(x) = (1 + g(x_m)) \cos \left( \frac{x_1 \pi}{2} \right) \cos \left( \frac{x_2 \pi}{2} \right) \ldots \cos \left( \frac{x_{m-2} \pi}{2} \right) \sin \left( \frac{x_{m-1} \pi}{2} \right) \]

\[ f_3(x) = (1 + g(x_m)) \cos \left( \frac{x_1 \pi}{2} \right) \cos \left( \frac{x_2 \pi}{2} \right) \ldots \sin \left( \frac{x_{m-1} \pi}{2} \right) \]

\[ \ldots \]

\[ f_{m-1}(x) = (1 + g(x_m)) \cos \left( \frac{x_1 \pi}{2} \right) \sin \left( \frac{x_{m-2} \pi}{2} \right) \]

\[ f_m(x) = (1 + g(x_m)) \sin \left( \frac{x_1 \pi}{2} \right) \]

\[ g(x_m) = \sum_{x \in X_m} (x_i - 0.5)^2 \]

where \( 0 \leq x_i \leq 1 \) and \( i = 1..n \). In our studies \( n = 2 \) and \( m = 2 \)

**DTLZ3**

\[ f_1(x) = (1 + g(x_m)) \cos \left( \frac{x_1 \pi}{2} \right) \cos \left( \frac{x_2 \pi}{2} \right) \ldots \cos \left( \frac{x_{m-2} \pi}{2} \right) \cos \left( \frac{x_{m-1} \pi}{2} \right) \]

\[ f_2(x) = (1 + g(x_m)) \cos \left( \frac{x_1 \pi}{2} \right) \cos \left( \frac{x_2 \pi}{2} \right) \ldots \cos \left( \frac{x_{m-2} \pi}{2} \right) \sin \left( \frac{x_{m-1} \pi}{2} \right) \]

\[ f_3(x) = (1 + g(x_m)) \cos \left( \frac{x_1 \pi}{2} \right) \cos \left( \frac{x_2 \pi}{2} \right) \ldots \sin \left( \frac{x_{m-1} \pi}{2} \right) \]

\[ \ldots \]

\[ f_{m-1}(x) = (1 + g(x_m)) \cos \left( \frac{x_1 \pi}{2} \right) \sin \left( \frac{x_{m-2} \pi}{2} \right) \]

\[ f_m(x) = (1 + g(x_m)) \sin \left( \frac{x_1 \pi}{2} \right) \]

\[ g(x_m) = 100[|x_m| + \sum_{i \in X_m} (x_i - 0.5)^2 - \cos(20\pi(x_i - 0.5))] \]

where \( 0 \leq x_i \leq 1 \) and \( i = 1..n \). In our studies \( n = 2 \) and \( m = 2 \)

**DTLZ5**

\[ f_1(x) = (1 + g(x_m)) \cos \left( \frac{\theta_1 \pi}{2} \right) \cos \left( \frac{\theta_2 \pi}{2} \right) \ldots \cos \left( \frac{\theta_{m-2} \pi}{2} \right) \cos \left( \frac{\theta_{m-1} \pi}{2} \right) \]

\[ f_2(x) = (1 + g(x_m)) \cos \left( \frac{\theta_1 \pi}{2} \right) \cos \left( \frac{\theta_2 \pi}{2} \right) \ldots \cos \left( \frac{\theta_{m-2} \pi}{2} \right) \sin \left( \frac{\theta_{m-1} \pi}{2} \right) \]

\[ f_3(x) = (1 + g(x_m)) \cos \left( \frac{\theta_1 \pi}{2} \right) \cos \left( \frac{\theta_2 \pi}{2} \right) \ldots \sin \left( \frac{\theta_{m-1} \pi}{2} \right) \]

\[ \ldots \]

\[ f_{m-1}(x) = (1 + g(x_m)) \cos \left( \frac{\theta_1 \pi}{2} \right) \sin \left( \frac{\theta_{m-2} \pi}{2} \right) \]

\[ f_m(x) = (1 + g(x_m)) \sin \left( \frac{\theta_1 \pi}{2} \right) \]

\[ g(x_m) = \sum_{x \in X_m} (x_i - 0.5)^2 \]

\[ \theta_i = \frac{i}{\text{max}(i) \times \text{max}(i) + 1} + 1 + 2(1 + 2f(r)x_i) \]

where \( 0 \leq x_i \leq 1 \) and \( i = 1..n \). In our studies \( n = 2 \) and \( m = 2 \)

**DTLZ6**

\[ f_1(x) = (1 + g(x_m)) \cos \left( \frac{\theta_1 \pi}{2} \right) \cos \left( \frac{\theta_2 \pi}{2} \right) \ldots \cos \left( \frac{\theta_{m-2} \pi}{2} \right) \cos \left( \frac{\theta_{m-1} \pi}{2} \right) \]
\[ f_2(x) = (1 + g(x_m)) \cos(\frac{\theta_1}{2}) \cos(\frac{\theta_2}{2}) \cos(\frac{\theta_3}{2}) \cdots \cos(\frac{\theta_{m-1}}{2}) \sin(\frac{\theta_{m-1}}{2}) \]
\[ f_3(x) = (1 + g(x_m)) \cos(\frac{\theta_1}{2}) \cos(\frac{\theta_2}{2}) \cos(\frac{\theta_3}{2}) \cdots \sin(\frac{\theta_{m-1}}{2}) \]
\[ \cdots \]
\[ f_{m-1}(x) = (1 + g(x_m)) \cos(\frac{\theta_1}{2}) \cos(\frac{\theta_2}{2}) \cos(\frac{\theta_3}{2}) \cdots \sin(\frac{\theta_{m-1}}{2}) \]
\[ f_m(x) = (1 + g(x_m)) \sin(\frac{\theta_1}{2}) \]
\[ g(x_m) = \sum_{x_i \in X_m} x_i 0.1 \]
\[ \theta_i = \frac{\pi}{2}(1 + 2 f_i(x_i)) \]
\[ \theta_i = 1, 2, \ldots, (m - 1) \]
\[ \text{where } 0 \leq x_i \leq 1 \text{ and } i = 1..n. \text{ In our studies } n = 2 \text{ and } m = 2 \]

**DTLZ7**

\[ f_1(X_1) = x_1, \]
\[ f_2(X_2) = x_2 \]
\[ f_{m-1}(X_{m-1}) = x_{m-1} \]
\[ f_m(X_m) = (1 + g(x_m)) h(f_1, f_2, \ldots, f_{m-1}, g) \]
\[ g(X_m) = 1 + \frac{g}{|X_m|} \sum_{x_i \in X_m} x_i \]
\[ h(f_1, f_2, \ldots, f_{m-1}, g) = \sum_{i=1}^{m-1} \frac{f_i}{1 + g(1 + \sin(3\pi f_i))} \]
\[ \text{where } 0 \leq x_i \leq 1 \text{ and } i = 1..n. \text{ In our studies } n = 2 \text{ and } m = 2 \]

**Viennet**

\[ f_1(x, y) = 0.5(x^2 + y^2) + \sin(x^2 + y^2) \]
\[ f_2(x, y) = \frac{(3x+2y+4)^2}{8} + \frac{(x-y+1)^2}{2} + 15 \]
\[ f_3(x, y) = \frac{1}{x+y+1} - 1.1 \exp(-x^2 - y^2) \]
\[ \text{where } -3 \leq x, y \leq 3 \]

**Viennet2**

\[ f_1(x, y) = \frac{(x-2)^2}{2} + \frac{(y+1)^2}{12} + 3 \]
\[ f_2(x, y) = \frac{(x+y-3)^2}{40} + \frac{(x-y+2)^2}{8} - 17 \]
\[ f_3(x, y) = \frac{(x+2y-1)^2}{16} + \frac{(2y-x)^2}{16} - 13 \]
\[ \text{where } -4 \leq x, y \leq 4 \]

**Deb**

\[ f_1(x, y) = x \]
where \(0 \leq x, y \leq 1\), \(q = 4\) and \(\alpha = 2\)

**Deb1**

\[
f_1(x, y) = x
\]

\[
f_2(x, y) = g(y)
\]

\[
g(y) = 2 - \exp\left(-\left(\frac{y - 0.2}{0.004}\right)^2\right) - 0.8 \exp\left(-\left(\frac{y - 0.6}{0.4}\right)^2\right)
\]

where \(0.1 \leq x, y \leq 1\)

**Fonseca**

\[
f_1(x) = 1 - \exp\left(-\sum_{i=1}^{n}(x_i - \frac{1}{\sqrt{n}})^2\right)
\]

\[
f_2(x) = 1 - \exp\left(-\sum_{i=1}^{n}(x_i + \frac{1}{\sqrt{n}})^2\right)
\]

where \(-4 \leq x_i \leq 4\) with \(i = 1..n\). In our studies \(n = 3\)

**Kursawe**

\[
f_1(x) = \sum_{i=1}^{n-1}(-10e^{-0.2/(\sqrt{x_i^2 + x_{i+1}^2})})
\]

\[
f_2(x) = \sum_{i=1}^{n}(|x_i|^{0.8} + 5 \sin x_i^4)
\]

where \(-5 \leq x_i \leq 5\) with \(i = 1..n\). In our studies \(n = 3\)

**Poloni**

\[
f_1(x, y) = -[1 + (A_1 - B_1)^2 + (A_2 - b_2)^2]
\]

\[
f_2(x, y) = -(x + 3)^2 + (y + 1)^2
\]

\[
A_1 = 0.5 \sin(1) - 2 \cos(1) + \sin(2) - 1.5 \cos(1.5)
\]

\[
A_2 = 1.5 \sin(1) - \cos(1) + 2 \sin(2) - 0.5 \cos(2)
\]

\[
B_1 = 0.5 \sin(x) - 2 \cos(x) + \sin(y) - 0.5 \cos(y)
\]

\[
B_2 = 1.5 \sin(x) - \cos(x) + 2 \sin(y) - 0.5 \cos(y)
\]

where \(-\pi \leq x, y \leq \pi\)

**Qv**

\[
f_1(x) = \left(\frac{1}{n} \sum_{i=1}^{n}(x_i^2 - 10 \cos(2\pi x_i)) + 10\right)^{1/4}
\]

\[
f_2(x) = \left(\frac{1}{n} \sum_{i=1}^{n}((x_i - 1.5)^2 - 10 \cos(2\pi(x_i - 1.5)) + 10\right)^{1/4}
\]
where $-5 \leq x_i \leq 5$ with $i = 1..n$. In our studies $n = 2$

Schaffer

\[ f_1(x) = x^2 \]
\[ f_2(x) = (x - 2)^2 \]

where $-10^5 \leq x \leq 10^5$