Denting Points of Convex Sets and Weak-property (π) of Cones in Locally Convex Spaces.

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Workshop of the Functional Analysis and Applications Network
June 22th, 2021

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The notion of denting point, and dentable subset, was introduced by M. A. Rieffel in 1967. Perhaps it represented the greatest breakthrough in the theory of Radon-Nikodym and a corner stone in the geometry of Banach spaces. By the same time (1969), and independently, G. Choquet used the name "strongly extreme points", in the framework of topological vector spaces, for to describe the same property of denting points. Before giving the definition in normed spaces, we fix some notation and terminology.

- X is a normed space.
- Given $c \in X$, $B_\epsilon(c)$ is the open ball with center the point c and radius $\epsilon > 0$. That is $B_\epsilon(c) = \{x \in X : \|x - c\| < \epsilon\}$.
- Given $f \in X^*$(topological dual), the set $H_V := \{x \in X : f(x) < \lambda\}$ is called a open half space and we will use the abbreviation $H_V = \{f < \lambda\}$.
- If $C \subseteq X$, the set of the form $C \cap H_V$ is called a slice of the set C.
- If $C \subseteq X$, $co(C)$ is the the convex hull of the set C.

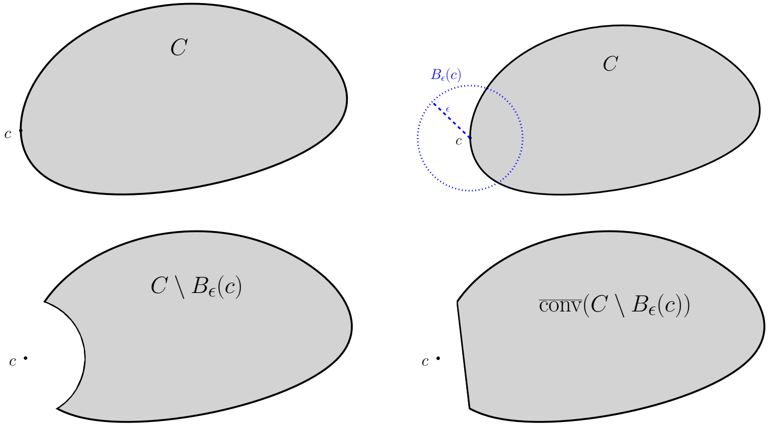
Definition 1

Let $(X, \|.\|)$ a normed space and $C \subseteq X$ a subset. The point $c \in C$ is called a denting point for C if, $\forall \epsilon > 0$,

$$c \notin \overline{co(C \setminus B_\epsilon(a_0))}^{\|\cdot\|}$$

We will write $c \in DP(C)$.

Definition Denting Point: $\forall \epsilon > 0, \quad c \notin \overline{co(C \setminus B_\epsilon(c))}^{\|.\|}$



Now, the Choquet's definition:

Definition 2 (Choquet, 1969)

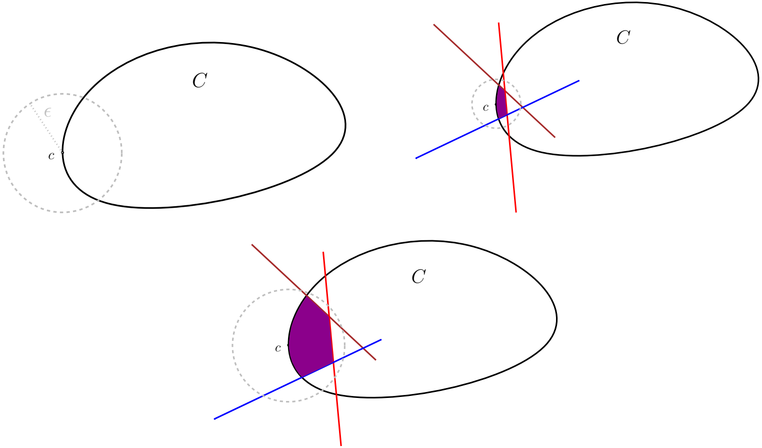
Let (X, τ) be a topological vector space with topological dual X^* and $A \subseteq X$ a subset (Choquet wrote "convex"). A point $a_0 \in A$ is called a Choquet's denting point of A if, for any τ-neighborhood V of 0_X, there are $f \in X^*$ and $\lambda \in \mathbb{R}$ such that $A \cap H_V \subseteq A \cap (a_0 + V)$, where $H_V := \{x \in X : f(x) < \lambda\}$ and $a_0 \in H_V$. We will write $a_0 \in ChDP(A)$.

The two definitions (Rieffel and Choquet's) coincide in locally convex spaces (by Hahn-Banach's Theorem): $ChDP(A) = DP(A)$.

Definition 3 (Namioka, 1967)

Let $(X, \|.\|)$ be a normed space, $c \subseteq X$ be a subset. A point $c \in C$ is called a point of continuity of C if the identity map $id : (C, w) \to (C, \|.\|)$ is continuous at c. We will write $c \in PC(C)$.

Definition Point of Continuity: $id : (C, w) \to (C, \|.\|)$ is continuous at c



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Theorem 4 (B.-L. Lin, P.-K. Lin, and Troyanski, 1988)

Let $(X, \|.\|)$ be a Banach space, $A \subseteq X$ a bounded, closed and convex subset. Then

$$\left.\begin{aligned} a_0 &\in PC(A) \\ a_0 &\in ext(A) \end{aligned}\right\} \iff a_0 \in DP(A)$$

(Daniilidis, 2000) extended the theorem above to unbounded sets.

Theorem 5 (Daniilidis, 2000)

Let $(X, \|.\|)$ be a Banach space, $A \subseteq X$ a closed and convex subset. Then

$$\left.\begin{aligned} a_0 &\in PC(A) \\ a_0 &\in ext(A) \end{aligned}\right\} \iff a_0 \in DP(A)$$

The Theorem reformulated

$$(X, \|\cdot\|) \xrightarrow{\quad J \quad} (X^{**}, \|\cdot\|^{**})$$

A bounded, A convex $\Rightarrow$ $\overline{J(A)}^{w^*} := \widetilde{A}$ is w^*-compact and convex

X Banach, A closed $\Rightarrow$ $J(A) = \overline{J(A)}^{\|\cdot\|^{**}} := \widehat{A} \subseteq \widetilde{A}$

$a_0 \in \mathrm{ext}(A)$ + $a_0 \in PC(A)$ + A convex $\Rightarrow$ Theorem Lin-Lin-Troyanski $\Leftarrow$ $J(a_0) \in ext\left(\widehat{A}\right)$

Theorem Lin-Lin-Troyanski $\Rightarrow$ $J(a_0) \in \mathrm{ext}\left(\widetilde{A}\right)$

$\widetilde{A}$ w^*-compact and convex + $J(a_0) \in \mathrm{ext}\left(\widetilde{A}\right)$ $\Rightarrow$ *Bidual form Choquet's Lemma*

Bidual form Choquet's Lemma $\Rightarrow$ Base of slices for w-topology + $a_0 \in PC(A)$

$\Downarrow$

$a_0 \in DP(A)$

Warning

$$J(A) \subsetneq \overline{J(A)}^{\|\cdot\|^{**}} \subsetneq \overline{J(A)}^{w^*}$$

$$\left. \begin{array}{l} J(A) = \overline{J(A)}^{w^*} \\ \quad + \\ A \quad \text{bounded} \end{array} \right\} \Leftrightarrow \{ A \quad \text{weakly compact}$$

Theorem 6 (The "core" of Lin-Lin-Troyanski's proof)

Let $(X, \|.\|)$ be a normed space and $A \subseteq X$ a subset. Then

- $a_0 \in PC(A) \Longleftrightarrow id : (\widetilde{A}, w^*) \to (\widetilde{A}, \|.\|^{**})$ is continuous at $J(a_0)$
- if A is convex, then $\left.\begin{matrix} a_0 \in PC(A) \\ J(a_0) \in ext(\widehat{A}) \end{matrix}\right\} \Rightarrow J(a_0) \in ext(\widetilde{A})$

Thanks to the former reformulation of Lin-Lin-Troyanski's Theorem, we get "almost" immediately our first important result. It can be applied to some problems present in the literature (see, e.g., "Conclusions" in (Gong, 1995) and (García Castaño, Melguizo Padial, and Parzanese, 2021)).

Theorem 7 (García Castaño, Melguizo Padial, and Parzanese, 2021)

Let $(X, \|.\|)$ be a normed space and $C \subseteq X$ a convex cone. Then

$$\left.\begin{array}{l} 0_X \in PC(C) \\ 0_{X^{**}} \in ext(\widehat{C}) \end{array}\right\} \Longleftrightarrow 0_X \in DP(C)$$

In order to generalize the former theorems to locally convex spaces, it needs to point out the following observations:

- in (B.-L. Lin, P.-K. Lin, and Troyanski, 1988) and successive works, the authors exhibit a counterexample for to show how much the hypothesis “Banach” space is essential.
- other kinds of "extreme points" (e.g., strong extreme points) are mentioned;
- the Bidual X^{**} is considered as well.
- the proof of (Daniilidis, 2000) is a “local” application of (B.-L. Lin, P.-K. Lin, and Troyanski, 1988). That is, it is considered the set $A_R := A \cap B_R(a_0)$, $R > 0$.

Denting points in LCS

Definition 8 (Bednarczuk and Song, 1998)

Let (X, τ) be a locally convex space and $A \subseteq X$ a subset. The point $a_0 \in A$ is called a denting point of A if, for any V τ-neighborhood of 0_X, we have

$$a_0 \notin \overline{co(A \setminus (a_0 + V))}^{\tau}$$

Def. in normed space

Let $(X, \|.\|)$ a normed space and $A \subseteq X$ a subset. The point $a_0 \in A$ is called a denting point of A if, for all $\epsilon > 0$, we have

$$a_0 \notin \overline{co(A \setminus B_{\epsilon}(a_0))}^{\|.\|}$$

Points of Continuity in LCS

Definition 9 (Bednarczuk and Song, 1998)

Let (X, τ) be a locally convex space and $A \subseteq X$ a subset. The point $a_0 \in A$ is called a point of continuity of A if, for any V τ-neighborhood of 0_X, we have

$$a_0 \notin \overline{(A \setminus (a_0 + V))}^w$$

Def. in normed space

Let $(X, \|.\|)$ a normed space and $A \subseteq X$ a subset. The point $a_0 \in A$ is called a point of continuity of A if, for all $\epsilon > 0$, we have

$$a_0 \notin \overline{(A \setminus B_\epsilon(a_0))}^w$$

Proposition 10

Let (X, τ) be a locally convex space, $A \subseteq X$ a subset. Then $DP(A) \subseteq PC(A)$.

Proposition 11 (Characterization Points of Continuity)

Let $A \subseteq X$ be a subset of a topological vector space (X, τ). Then the following statements are equivalent:

1. $a_0 \in PC(A)$;
2. *the identity map* $id : (A, w) \to (A, \tau)$ *is continuous at* a_0;
3. *every net in* A, *w-convergent to* a_0, *is* τ*-convergent to* a_0;
4. *for every* $V \in \tau(0_X)$ *there exists* $W_V \in w(0_X)$ *such that*
$$A \cap (a_0 + W_V) \subseteq A \cap (a_0 + V);$$
5. *(Bednarczuk and Song, 1998) for every* $V \in \tau(0_X)$,
$$a_0 \notin \overline{(A \setminus (a_0 + V))}^w.$$

Theorem 12 (Generalization of Daniilidis' Theorem (Parzanese, 2021))

Let (X, τ) be a locally convex space (or TVS) , $A \subseteq X$ a convex (or star-shaped at $a_0 \in A$) subset and $V \in \tau(a_0)$. Then

- $a_0 \in DP(A)$ if and only if $a_0 \in DP(A \cap V)$.
- $a_0 \in PC(A)$ if and only if $a_0 \in PC(A \cap V)$.

Definition 13

Let X be a vector space, $A \subseteq X$ a subset and $a_0 \in A$. The element a_0 is called a extreme point of A if, whenever $a_0 = (1-t)x + ty$, with $x, y \in A$ and $0 < t < 1$, we have $x = y = a_0$. We will write $a_0 \in ext(A)$.

Proposition 14 (Choquet, 1969)

Let X be a real vector space, $A \subseteq X$ a convex subset. Then the following statements are equivalent:

1. $a_0 \in ext(A)$;
2. *for* $x, y \in A$ *and* $a_0 = \frac{x+y}{2} \Rightarrow x = y = a_0$;
3. $A \setminus \{a_0\}$ *is convex.*

Theorem 15 (Choquet's Lemma (Choquet, 1969))

Let (X, τ) be a Hausdorff locally convex space, $A \subseteq X$ a convex and τ-compact subset. Then

$$ext(A) = DP(A)$$

Sketch of proof. Choquet's Lemma

Let (X, τ) be a Hausdorff locally convex space and $A \subseteq X$ a convex and τ-compact subset. Then $ext(A) \subseteq DP(A)$.

- (X, τ) Hausdorff $\Rightarrow$ Hahn-Banach's Theorem for to strictly separate the point a and 0_X $\Rightarrow A \setminus \{0_X\}$ is convex.

Extreme points

Sketch of proof. Choquet's Lemma

Let (X, τ) be a Hausdorff locally convex space and $A \subseteq X$ a convex and τ-compact subset. Then $ext(A) \subseteq DP(A)$.

- (X, τ) Hausdorff $\Rightarrow$ Hahn-Banach's Theorem for to strictly separate the point a and 0_X $\Rightarrow A \setminus \{0_X\}$ is convex.
- (X, τ) Hausdorff $\Rightarrow (X, w)$ Hausdorff $\Rightarrow$ τ-compactness of $A \Rightarrow id : (A, \tau) \to (A, w)$ is an homeomorphism $\Rightarrow 0_X \in PC(A) \Rightarrow \forall U$ τ-open neighborhood of 0_X, there is W w-open neighborhood of 0_X such that $A \setminus W \supseteq A \setminus U$. $W = \bigcap_{i=1}^m H_i$, where H_i are τ-open half spaces containing $0_X \Rightarrow$

$$A \setminus U \subseteq A \setminus W = A \setminus \left(\bigcap_{i=1}^m H_i\right) = A \cap \mathscr{C}\left(\bigcap_{i=1}^m H_i\right) = A \cap \left(\bigcup_{i=1}^m \mathscr{C} H_i\right) =$$

$$= \bigcup_{i=1}^m (A \cap \mathscr{C} H_i) = \bigcup_{i=1}^m (A \setminus H_i) \subseteq A \setminus \{0_X\}$$

Sketch of proof. Choquet's Lemma

Let (X,τ) be a Hausdorff locally convex space and $A \subseteq X$ a convex and τ-compact subset. Then $ext(A) \subseteq DP(A)$.

- (X,τ) Hausdorff $\Rightarrow$ Hahn-Banach's Theorem for to strictly separate the point a and 0_X $\Rightarrow A \setminus \{0_X\}$ is convex.
- (X,τ) Hausdorff $\Rightarrow (X,w)$ Hausdorff $\Rightarrow$ τ-compactness of $A \Rightarrow id : (A,\tau) \to (A,w)$ is an homeomorphism $\Rightarrow 0_X \in PC(A) \Rightarrow \forall U$ τ-open neighborhood of 0_X, there is W w-open neighborhood of 0_X such that $A \setminus W \supseteq A \setminus U$. $W = \bigcap_{i=1}^m H_i$, where H_i are τ-open half spaces containing $0_X \Rightarrow$

$$A \setminus U \subseteq A \setminus W = A \setminus \left(\bigcap_{i=1}^m H_i\right) = A \cap \mathscr{C}\left(\bigcap_{i=1}^m H_i\right) = A \cap \left(\bigcup_{i=1}^m \mathscr{C} H_i\right) =$$

$$= \bigcup_{i=1}^m (A \cap \mathscr{C} H_i) = \bigcup_{i=1}^m (A \setminus H_i) \subseteq A \setminus \{0_X\}$$

- $A \setminus H_i$ convex, τ-compact, w-compact, for every $i = 1, ..., m$. Convex hull of a finite union of compact and convex sets is compact

$$\overline{co(A \setminus U)}^\tau \subseteq \overline{co(A \setminus W)}^\tau \subseteq \overline{co(A \setminus W)}^w = \overline{co\left(\bigcup_{i=1}^m (A \setminus H_i)\right)}^w =$$

$$= co\left(\bigcup_{i=1}^m (A \setminus H_i)\right) \subseteq A \setminus \{0_X\}$$

Therefore $0_X \notin \overline{co(A \setminus U)}^\tau$.

The following definition is "new". It allows us to enlightening a extra property that points of continuity must have in order to be denting points. The successive theorem will clarify, and precisely state, that property.

Definition 16 (Parzanese, 2021)

Let (X, τ) be a topological vector space with topological dual X^*, $A \subseteq X$ a subset. A point $a_0 \in A$ is called a weak Choquet's denting point of A if for any w-neighborhood W of 0_X there are $f \in X^*$ and $\lambda \in \mathbb{R}$ such that $A \cap H_W \subseteq A \cap (a_0 + W)$, where $H_W := \{x \in X : f(x) < \lambda\}$ and $a_0 \in H_V$. We will write $a_0 \in wChDP(A)$.

Theorem 17 (Caractherization weak Choquet's denting point (Parzanese, 2021))

Let (X, τ) be a locally convex space (TVS) with topological dual X^* and $A \subseteq X$ a subset. Then

1. $DP(A) \subseteq wChDP(A)$.
2. $a_0 \in wChDP(A)$ if and only if, for every w-neighborhood W of 0_X, we have

$$a_0 \notin \overline{co(A \setminus (a_0 + W))}^w$$

   (in other words, *the open slices of A containing a_0 form a neighborhood base of a_0 in the relative weak topology of A*).
3. $DP(A) = wChDP(A) \cap PC(A)$.

We recall that for a subset $A \subseteq X$ of a locally convex space, we have adopted the following notation:

$$\widehat{A} := \overline{J(A)}^{\beta} \subseteq (X^{**}, \beta) \quad \subseteq_{\text{since } (w^* \subseteq \beta)} \quad \widetilde{A} := \overline{J(A)}^{w^*} \subseteq (X^{**}, w^*)$$

Definition 18 (weak* denting point, (García Castaño, Melguizo Padial, and Parzanese, 2021))

Let (X, τ) be a locally convex space and $E \subseteq X^{**}$ a subset. We will say that $e_0 \in E$ is a weak* denting point of E if, for any w^*-neighborhood W^* of $0_{X^{**}}$, we have

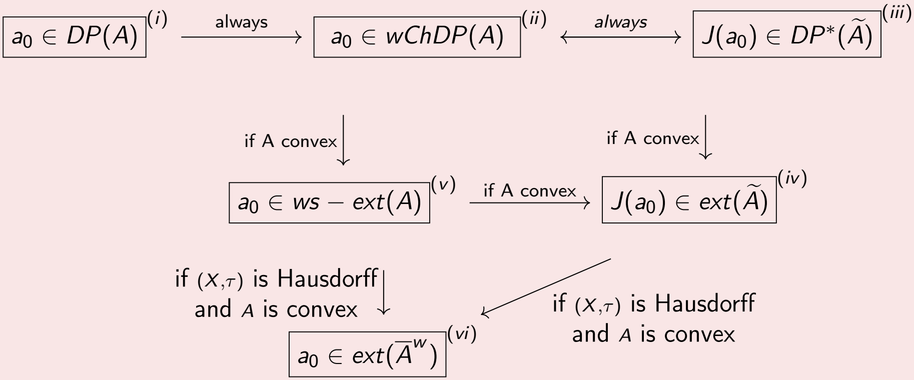
$$e_0 \notin \overline{co(E \setminus (e_0 + W^*))}^{w^*}$$

We will write $e_0 \in DP^*(E)$ and we recall that (X^{**}, w^*) is always a Hausdorff space.

Theorem 19 (Bidual form of Choquet's Lemma, (García Castaño, Melguizo Padial, and Parzanese, 2021))

Let (X, τ) be a locally convex space. Suppose $E \subseteq (X^{**}, w^*)$ is a convex and w^*-compact subset in the Hausdorff space (X^{**}, w^*). Then $DP^*(E) = ext(E)$.

Relationships (Parzanese, 2021)



Definition 20 (Points of Continuity in the Strong Bidual, (García Castaño, Melguizo Padial, and Parzanese, 2021))

Let (X, τ) be a locally convex space, $E \subseteq X^{**}$ a subset. We will say that $e_0 \in E$ is a weak* point of continuity of E if the identity map $id : (E, w^*) \to (E, \beta)$ is continuous at e_0. We will write $e_0 \in w^* - PC(E)$.

Theorem 21 (García Castaño, Melguizo Padial, and Parzanese, 2021)

Let (X, τ) be a Hausdorff locally convex space, infrabarrelled and $A \subseteq X$ a subset. Then

$$a_0 \in PC(A) \iff J(a_0) \in w^* - PC(\widetilde{A})$$

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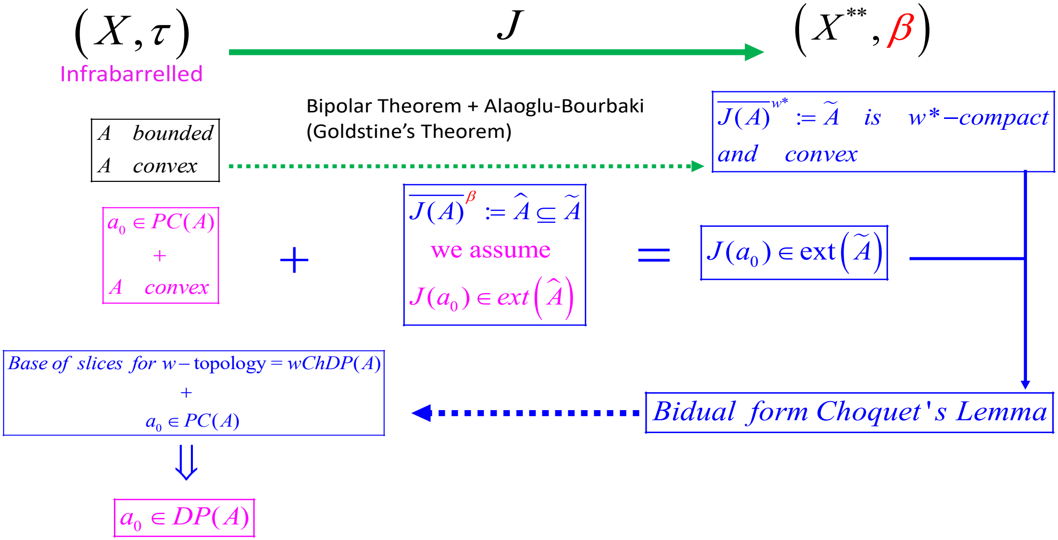
Theorem 22 ("Generalization" of Lin-Lin-Troyanski (García Castaño, Melguizo Padial, and Parzanese, 2021))

Let (X, τ) be a Hausdorff locally convex space, infrabarrelled and $A \subseteq X$ a τ-bounded and convex subset. If $a_0 \in ext(A)$, then

$$\left.\begin{array}{l} a_0 \in PC(A) \\ J(a_0) \in ext(\widehat{A}) \end{array}\right\} \Longleftrightarrow J(a_0) \in ext(\widetilde{A}) \Longleftrightarrow a_0 \in DP(A)$$



(X, τ)
Infrabarrelled
J
(X^{**}, β)
A bounded
A convex
Bipolar Theorem + Alaoglu-Bourbaki
(Goldstine's Theorem)
$\overline{J(A)}^{w^*} := \widetilde{A}$ is w^*-compact and convex
$a_0 \in PC(A)$ + A convex
+
$\overline{J(A)}^{\beta} := \widehat{A} \subseteq \widetilde{A}$
we assume
$J(a_0) \in ext(\widehat{A})$
=
$J(a_0) \in \text{ext}(\widetilde{A})$
Bidual form Choquet's Lemma
Base of slices for w − topology = $wChDP(A)$ + $a_0 \in PC(A)$
$a_0 \in DP(A)$

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Definition 23 (García Castaño, Melguizo Padial, and Parzanese, 2021)

Let (X, τ) be a topological vector space and $A \subseteq X$ a subset. We say that $a_0 \in A$ is a Bounded-Point of A if there is exists $U \in \tau(0_X)$ such that $A \cap (a_0 + U)$ is τ-bounded. We will write $a_0 \in BP(A)$.

Definition 24 (García Castaño, Melguizo Padial, and Parzanese, 2021)

Let (X, τ) be a topological vector space and $A \subseteq X$ a subset. We say that $a_0 \in A$ is a bounded point of continuity of A if there is $W \in w(0_X)$, i.e. a weak neighborhood of 0_X, such that $A \cap (a_0 + W)$ is τ-bounded. We will write $a_0 \in bPC(A)$.

In the next proposition resides the reason of choice for the name bounded point of continuity.

Proposition 25 (García Castaño, Melguizo Padial, and Parzanese, 2021)

Let (X, τ) a topological vector space and $K \subseteq X$ a cone (not necessarily convex). Then $0_X \in bPC(K)$ if and only if $0_X \in BP(K) \cap PC(K)$.

The following two theorems show the use of bounded points and are general statements (the "Generalization" of Lin-Lin-Troyanski's Theorem is a direct application of them).

Theorem 26 (Parzanese, 2021)

Let (X, τ) be a locally convex space, $A \subseteq X$ a convex subset and $a_0 \in A$. Then:

$$\left.\begin{array}{l} a_0 \in BP(A) \\ a_0 \in PC(A) \\ J(a_0) \in ext\left(\widetilde{A}\right) \end{array}\right\} \Rightarrow \left.\begin{array}{l} a_0 \in bPC(A) \\ J(a_0) \in ext\left(\widetilde{A}\right) \end{array}\right\} \Rightarrow a_0 \in DP(A)$$

Theorem 27 (Parzanese, 2021)

Let (X, τ) be a Hausdorff locally convex space, infrabarrelled and $A \subseteq X$ a convex subset. Then:

$$\left.\begin{array}{l} a_0 \in BP(A) \\ a_0 \in PC(A) \\ J(a_0) \in ext(\widehat{A}) \end{array}\right\} \Rightarrow \left.\begin{array}{l} a_0 \in bPC(A) \\ J(a_0) \in ext(\widehat{A}) \end{array}\right\} \Rightarrow a_0 \in DP(A)$$

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Definition 28 (Song, 2003)

Let (X, τ) be a locally convex space and $K \subseteq X$ a cone (not necessarily convex). We say that K satisfies the weak property (π) if there are $f \in K^*$ and a real scalar $\alpha > 0$ such that $K \cap \{x \in X : f \leq \alpha\}$ is τ-bounded.

The definition above can be rephrased saying: "A cone have a bounded slice". The next propositions are generalizations showing again how the notion of bounded points can be fruitful. Roughly speaking: "If a convex cone have a bounded slice containing 0_X, then 0_X is a Denting point". Such a property is common in the literature about convex cones (see, e.g., Kountzakis and Polyrakis, 2006) and Optimization problems.

Proposition 29 (Parzanese, 2021)

Let (X,τ) a topological vector space and $K \subseteq X$ a cone (not necessarily convex). Then $0_X \in BP(K) \cap ChDP(K) \iff$ there are $f \in X^*$ and $\alpha > 0$ such that $K \cap \{x \in X : f(x) < \alpha\}$ is τ-bounded.

Proposition 30 (Parzanese, 2021)

Let (X,τ) a topological vector space and $K \subseteq X$ a cone (not necessarily convex). Then $0_X \in bPC(K) \iff 0_X \in BP(K) \cap PC(K)$.

Proposition 31 (Corollary, (Parzanese, 2021))

Let (X,τ) a topological vector space and $K \subseteq X$ a cone (not necessarily convex). Then $0_X \in bPC(K) \cap ChDP(K) \iff$ there are $f \in X^*$ and $\alpha > 0$ such that $K \cap \{x \in X : f(x) < \alpha\}$ is τ-bounded.

We state, separately, the following propositions about the existence of a base in a convex cone:

Proposition 32 (Parzanese, 2021)

Let (X, τ) a topological vector space and $K \subseteq X$ a convex cone. If there are $f \in X^*$ and $\alpha > 0$ such that $K \cap \{x \in X : f(x) < \alpha\}$ is τ-bounded (i.e., there exists a bounded slice of K containing 0_X), then $K^\# \neq \varnothing$ $(f \in K^\#)$ and the cone have a convex base.
($K^\# := \{g \in X^* : g(k) > 0, \forall k \in K \setminus \{0_X\}\}$)

Theorem 33 (Parzanese, 2021)

Let (X, τ) a locally convex space and $K \subseteq X$ a convex cone. Then $0_X \in bPC(K) \cap DP(K)$ $\iff$ there exists a τ-bounded slice of K containing 0_X $\iff$ K has a τ-bounded convex base.

Results

Theorem 34 (García Castaño, Melguizo Padial, and Parzanese, 2021)

Let (X, τ) be a locally convex space and $K \subseteq X$ a convex cone. The following facts (1-5) are equivalent:

1. K satisfies the weak property (π);
2. $0_X \in bPC(K) \cap DP(K)$;
3. $0_X \in bPC(K) \cap ws - ext(K)$;
4. $0_X \in bPC(K)$ and $0_{X^{**}} \in ext(\widetilde{K}) = ext(K^{**})$;
5. $0_X \in bPC(K)$ and K^* is β-quasi-generating in (X^*, β).

If (X, τ) is Hausdorff, then the above statements imply

6. $0_X \in DP(K) \cap ext(K)$.

Results

Theorem 35 (García Castaño, Melguizo Padial, and Parzanese, 2021)

Let (X, τ) be a locally convex space, infrabarrelled and $K \subseteq X$ a convex cone. The following facts (1-5) are equivalent:

1. K satisfies the weak property (π);
2. $0_X \in bPC(K) \cap DP(K)$;
3. $0_X \in bPC(K) \cap ws - ext(K)$;
4. $0_X \in bPC(K)$ and $0_{X^{**}} \in ext(\widehat{K})$;
5. $0_X \in bPC(K)$ and K^* is β-quasi-generating in (X^*, β).

If (X, τ) is Hausdorff, then the above statements imply

6. $0_X \in DP(K) \cap ext(K)$.

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(Song, 2003)
Consider $\ell^1 = \{x = (\xi_k)_{k\in\mathbb{N}} : \sum_{k=1}^{\infty} |\xi_k| < \infty\}$ endowed with the topology (τ) generated by the family $\mathscr{F}$ of seminorms $\{p_n\}_{n\in\mathbb{N}}$, where $p_n(x) := \sum_{k=1}^{n} |\xi_k|$.

- (ℓ^1, τ) is a Fréchet space.
- It is NOT a locally bounded space.
- Consider the convex cone $C = \left\{x \in \ell^1 : \xi_k \geq 0 \text{ for all } k \in \mathbb{N}\right\}$. It is τ-closed and $0_x \in ext(C)$ (i.e., is a pointed cone).
- $C^\# = \varnothing \quad (C^\# := \{f \in X^* : f(c) > 0, \forall c \in C \setminus \{0_X\}\})$. Therefore the cone DO NOT have a base.
- $0_X \in PC(C)$ but $0_X \notin BP(C)$. Despite this, $0_X \in DP(C)$!!!

The space (ℓ^1, τ) is "very particular". Observe that, fixed $x \in \ell^1$, we have $lim_n p_n(x) = \|x\|_1$ and $\{p_n(x)\}$ is a monotone increasing family of seminorms (pseudometrizable spaces).

Recall the

Theorem (Choquet's Lemma)

Let (X, τ) be a Hausdorff locally convex space, $A \subseteq X$ a convex and τ-compact subset. Then $ext(A) = DP(A)$.

We have seen how some kind of "boundedness" (e.g. bounded points) was essential to apply Choquet's Lemma. On the other hand, the former counterexample, dealing with a convex cone, DO NOT requires the boundedness for to conclude $ext(C) \Rightarrow DP(C)$.

Questions

1. Can we “generalize” Choquet’s Lemma in “some way”?
2. Which kind of hypothesis we have to impose on the space X (e.g., completeness, quasi-completeness, Fréchet space, etc.) or just on the considered subset (e.g., compactness)?
3. Can we extend “some conclusions” to unbounded sets, as in the case of cones?
4. Are there any counterexamples?

We don't know if the following conjectures are true (searching counterexamples or a proof validating/negating them!):

Conjecture 1

Let (X, τ) be a Hausdorff locally convex space (or Fréchet space?) and $A \subseteq X$ a *w − complete* and convex subset. Then

$$a_0 \in ext(A) \iff a_0 \in DP(A)$$

Proof. We get $a_0 \in PC(A)$, by the w-completeness. And after... ???

Conjecture 2 (maybe Conjecture 1 is true but just for cones)

Let (X, τ) be a Hausdorff locally convex space (or Fréchet space?) and $C \subseteq X$ a *w − complete* (convex) cone. Then

$$0_X \in ext(C) \iff 0_X \in DP(C)$$

Summary

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Denting Points of Convex Sets and Weak-property (π) of Cones in Locally Convex Spaces.

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Workshop of the Functional Analysis and Applications Network
June 22th, 2021

Thank You.