Cardinal Invariants, Embeddings and Domination in Function Spaces

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Contents

Definitions

Every topological space in this presentation is assumed to be Tychonoff. The set of real numbers with the natural topology is denoted by $\mathbb{R}$ and the interval $[0, 1] \subset \mathbb{R}$ is represented by $\mathbb{I}$. For a space X the family of all subsets of X is denoted by $exp(X)$, the family of all open subsets of X is denoted by $\tau(X)$ and the family of all compact subspaces of X is denoted by $K(X)$. The space of all continuous functions from a space X into a space Y, endowed with the topology inherited from the product space Y^X, is denoted by $C_p(X, Y)$. On the other hand, $C_u(X)$ is the space of all continuous real-valued functions on a space X, with the topology of uniform convergence.

Definitions

A continuous bijection is called a condensation. A compact valued map $\varphi : Y \to exp(X)$ is called upper semicontinuous, abbreviated by usco, if for every $U \in \tau(X)$ the set $\{y \in Y : \varphi(y) \subset U\}$ is open in Y. An usco map $\varphi : Y \to exp(X)$ is onto if the family $\{\varphi(y) : y \in Y\}$ covers the space X. A space Y dominates a space X if there is a cover $\mathcal{C} = \{F_K : K \in \mathcal{K}(Y)\} \subset \mathcal{K}(X)$ of X such that $K \subset L$ implies $F_K \subset F_L$. A cover $\mathcal{F}$ of a space X is closed if every $F \in \mathcal{F}$ is closed in X; we call $\mathcal{F}$ closure-preserving if $\overline{\bigcup\{F : F \in \mathcal{F}'\}} = \bigcup\{\overline{F} : F \in \mathcal{F}'\}$ for any $\mathcal{F}' \subset \mathcal{F}$.

On progress

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(a) Metric and Domination indexes of function spaces.

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(b) Embeddings in spaces of the form $C_p(K)$ for some compact space K.

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(b) Embeddings in spaces of the form $C_p(K)$ for some compact space K.

(c) Cardinal invariants under closure-preserving covers of function spaces.

Indexes

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(c) *(**Guerrero**)The domination index of X is denoted by $dm(X)$ and defined by $dm(X) = min\{w(M) : M$ dominates $X\}$.*

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(b) *If K is fragmentable then $mi(K) \leq w(K)$*

A discrete version of wJNR vs σ-fagmentability

A problem of Arkhangelskii-Haydon

Is every Eberlein-Grothendieck scattered space σ-discrete?

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Partial answer

Every Eberlein-Grothendieck locally countable scattered space is σ-discrete.

Another question of Arkhangelskii

When is $C_p(X)$ σ-compact?

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(d) *The space X is finite.*

Generalizing

General approach

Given a topological property $\mathcal{P}$ assume that $C_p(X)$ is the union of a closure-preserving family $\mathcal{F}$ of closed subspaces and each element of $\mathcal{F}$ has $\mathcal{P}$. Does this imply that $C_p(X)$ has $\mathcal{P}$ or some related topological property?

1. Suppose that $C_p(X)$ is the union of a closure-preserving family of its pseudocompact subspaces. Must $C_p(X)$ be σ-pseudocompact?
2. Suppose that $C_p(X, \mathbb{I})$ is the union of a closure-preserving family of its pseudocompact subspaces. Must $C_p(X, \mathbb{I})$ be pseudocompact?
3. Suppose that $C_p(X, \mathbb{I})$ is the union of a closure-preserving family of its closed σ-compact subspaces. Does this imply that X is discrete?
4. Let X be a space, not necessarily compact, such that $C_p(X)$ is the union a closure-preserving family of its separable subspaces. Must $C_p(X)$ be separable?
5. Suppose that $C_p(X)$ is the union of a closure-preserving family of closed subspaces of cardinality $\mathfrak{c}$. Must $C_p(X)$ have cardinality $\mathfrak{c}$?

6. Suppose that $C_p(X)$ is the union of a closure-preserving family $\mathcal{F}$ of its second countable subspaces. Must X be countable? What happens if all the elements of $\mathcal{F}$ are closed in $C_p(X)$?
7. Suppose that $C_p(X)$ is the union of a closure-preserving family $\mathcal{F}$ of closed subspaces of countable tightness. Is it true that $t(C_p(X)) = \omega$?
8. Suppose that $C_p(X)$ is the union of a closure-preserving family of its closed metrizable subspaces. Must X be countable?
9. Suppose that X is compact and $C_p(X)$ is the union of a closure-preserving family of its metrizable subspaces. Must X be countable?

A fundamental result

Theorem of Terada-Yayima.

Terada and Yajima established that if Z is a Čech-complete space and $\mathcal{F}$ is a closure-preserving closed cover of Z then some element $F \in \mathcal{F}$ must have non-empty interior. Since $C_u(X)$ is always Čech-complete, we have the following result which is crucial for understanding what happens when $C_p(X)$ has a closure-preserving cover by nice subspaces.

Proposition

For an arbitrary X, if $\mathcal{C}$ is a closure-preserving closed cover of $C_p(X)$ or $C_p(X, \mathbb{I})$ then there exists $C \in \mathcal{C}$ such that $U \subset C$ for some non-empty open subset U of the space $C_u(X)$.

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Corollary

If X is a space and $\mathcal{C}$ is a closure-preserving closed cover of $C_p(X)$ or $C_p(X, \mathbb{I})$ then some $C \in \mathcal{C}$ contains a homeomorphic copy of $C_p(X)$.

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Corollary Answer to problem 5

Suppose that $\mathcal{P}$ is a hereditary topological property and either $C_p(X, \mathbb{I})$ or $C_p(X)$ has a closure-preserving closed cover $\mathcal{C}$ such that every $C \in \mathcal{C}$ has $\mathcal{P}$. Then $C_p(X)$ also has $\mathcal{P}$.

Answer to problem 7

Corollary

For any infinite cardinal κ consider the following list $\mathbb{L}_0$ of topological properties:
$\mathbb{L}_0 = \{$*weight* $\leq \kappa$, *network weight* $\leq \kappa$, *i-weight* $\leq \kappa$, *diagonal number* $\leq \kappa$, *character* $\leq \kappa$, *pseudocharacter* $\leq \kappa$, *tightness* $\leq \kappa$, *spread* $\leq \kappa$, *hereditary Lindelöf number* $\leq \kappa$, *hereditary density* $\leq \kappa$, κ*-monolithicity*$\}$.
If a property $\mathcal{P}$ belongs to the list $\mathbb{L}_0$ and either $C_p(X, \mathbb{I})$ or $C_p(X)$ has a closure-preserving closed cover $\mathcal{C}$ such that every $C \in \mathcal{C}$ has $\mathcal{P}$ then $C_p(X)$ also has $\mathcal{P}$.

Answer to problem 8

Corollary

Consider the following list $\mathbb{L}_1$ of topological properties:
$\mathbb{L}_1 = \{$*metrizability, Fréchet-Urysohn property, small diagonal, hereditary realcompactness, Whyburn property, being perfect, being functionally perfect*$\}$.
If a property $\mathcal{P}$ belongs to the list $\mathbb{L}_1$ and either $C_p(X, \mathbb{I})$ or $C_p(X)$ has a closure-preserving closed cover $\mathcal{C}$ such that every $C \in \mathcal{C}$ has $\mathcal{P}$ then $C_p(X)$ also has $\mathcal{P}$.

Example

If a property $\mathcal{P}$ is not hereditary and $C_p(X)$ has a closure-preserving closed cover by subspaces that have $\mathcal{P}$ then $C_p(X)$ does not necessarily have $\mathcal{P}$. Indeed, Tkachuk proved that if K is the Cantor set then $C_p(K)$ has a countable family $\{F_n : n \in \omega\}$ of closed sets such that $\bigcup_{n\in\omega} F_n = C_p(K)$ and every F_n has a countable π-base but $C_p(K)$ does not have a countable π-base. It is easy to see that the family $\{F_n : n \in \omega\}$ is closure-preserving, so countable π-weight is not preserved by closed closure-preserving unions. However, for the properties which are closed-hereditary we have the following result.

Closed-hereditary Properties

Theorem

Given a space X and a closed-hereditary property $\mathcal{P}$, if $C_p(X, \mathbb{I})$ has a closed closure-preserving cover $\mathcal{C}$ such that every $C \in \mathcal{C}$ has $\mathcal{P}$ then $C_p(X, \mathbb{I})$ also has the property $\mathcal{P}$.

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Corollary

For any infinite cardinal κ consider the following list $\mathbb{L}_2$ of topological properties:
$\mathbb{L}_2 = \{$*Lindelöf number* $\leq \kappa$, *Lindelöf Σ index* $\leq \kappa$, *extent* $\leq \kappa$, *Nagami number* $\leq \kappa$ *domination index* $\leq \kappa\}$.
If a property $\mathcal{P}$ belongs to the list $\mathbb{L}_2$ and either $C_p(X, \mathbb{I})$ or $C_p(X)$ has a closure-preserving closed cover $\mathcal{C}$ such that every $C \in \mathcal{C}$ has $\mathcal{P}$ then $C_p(X, \mathbb{I})$ also has $\mathcal{P}$.

Closed-hereditary Properties

Corollary

Consider the following list $\mathbb{L}_3$ *of topological properties:*
$\mathbb{L}_3 = \{$*K-analyticity, Lindelöf* Σ*-property, normality, sequentiality*$\}$.
If a property $\mathcal{P}$ *belongs to the list* $\mathbb{L}_3$ *and either* $C_p(X, \mathbb{I})$ *or* $C_p(X)$ *has a closure-preserving closed cover* $\mathcal{C}$ *such that every* $C \in \mathcal{C}$ *has* $\mathcal{P}$ *then* $C_p(X, \mathbb{I})$ *also has* $\mathcal{P}$.

Special cases

Corollary

If $C_p(X, \mathbb{I})$ has a closure-preserving closed cover $\mathcal{C}$ such that every $C \in \mathcal{C}$ is realcompact then $C_p(X)$ is realcompact.

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Corollary, Answer to problem 3

Given a space X, if $C_p(X)$ has a closure-preserving closed cover by Čech-complete subspaces, then X is discrete.

Moreover

Theorem

Given a space X, if $C_p(X, \mathbb{I})$ has a closure-preserving closed cover by σ-countably compact subspaces, then $C_p(X, \mathbb{I})$ is countably compact.

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Theorem

If $C_p(X,\mathbb{I})$ has a closure-preserving closed cover $\mathcal{C}$ such that every $C \in \mathcal{C}$ is σ-compact then X is discrete.

Quotient-preserved Properties

Theorem

Given a space X and a property $\mathcal{P}$ that is preserved by quotient images, if $C_p(X, \mathbb{I})$ has a closed closure-preserving cover $\mathcal{C}$ such that every $C \in \mathcal{C}$ has $\mathcal{P}$ then $C_p(X, \mathbb{I})$ also has $\mathcal{P}$.

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Corollary

For any infinite cardinal κ consider the following list $\mathbb{L}_4$ of topological properties:
$\mathbb{L}_4 = \{\kappa$-stability, weak functional tightness $\leq \kappa$, functional tightness $\leq \kappa\}$.
If a property $\mathcal{P}$ belongs to the list $\mathbb{L}_4$ and $C_p(X, \mathbb{I})$ has a closure-preserving closed cover $\mathcal{C}$ such that every $C \in \mathcal{C}$ has $\mathcal{P}$ then $C_p(X, \mathbb{I})$ also has $\mathcal{P}$. If $\mathcal{P}$ is κ-stability then $C_p(X)$ has $\mathcal{P}$.

Lindelöf and Lindelöf Σ.

When we consider closure-preserving closed covers of $C_p(X)$ whose elements are either Lindelöf or Lindelöf Σ, it follows that $C_p(X, \mathbb{I})$ must have the respective property. However, we strongly suspect that in this case the whole space $C_p(X)$ must be Lindelöf or Lindelöf Σ respectively. It does not seem that easy to verify, even for the spaces with a unique non-isolated point. Still, we present some positive results in this direction; they are often generalizations of some well-known theorems about the properties of a space X for which $C_p(X)$ is either Lindelöf or Lindelöf Σ.

Lindelöf degree

Proposition

Given a space X and an infinite cardinal κ, suppose that $C_p(X)$ has a closure-preserving closed cover $\mathcal{C}$ such that $l(C) \leq \kappa$ for every $C \in \mathcal{C}$. Then any discrete family of non-empty open subsets of X has cardinality at most κ.

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Corollary

Suppose that κ is an infinite cardinal and X is a paracompat space such that $C_p(X)$ has a closure-preserving closed cover $\mathcal{C}$ with $l(C) \leq \kappa$ for every $C \in \mathcal{C}$. Then $l(X) \leq \kappa$; in particular, if X is metrizable then $w(X) \leq \kappa$.

Generalizing Asanov's Theorem

Lemma

For an arbitrary space X and an infinite cardinal κ, if $l(C_p(X,\mathbb{I})) \leq \kappa$, then $t(X^n) \leq \kappa$ for every $n \in \omega$.

Generalizing Asanov's Theorem

Lemma

For an arbitrary space X and an infinite cardinal κ, if $l(C_p(X, \mathbb{I})) \leq \kappa$, then $t(X^n) \leq \kappa$ for every $n \in \omega$.

Corollary

Given a space X, if $C_p(X)$ admits a closure-preserving closed cover $\mathcal{C}$ such that $l(C) \leq \kappa$ for every $C \in \mathcal{C}$, then $t(X^n) \leq \kappa$ for $n \in \mathbb{N}$.

Spaces X with a unique non-isolated point

Corollary

For a space X with a unique non-isolated point the following conditions are equivalent:

(a) $C_p(X)$ *is Lindelöf;*

(b) $C_p(X)$ *has a closure-preserving cover by Lindelöf subspaces;*

(c) *the space X is Lindelöf and $t(X^n) \leq \omega$ for any $n \in \mathbb{N}$.*

Lindelöf Σ

Proposition

If X is a Lindelöf Σ-space and $C_p(X)$ has a closure-preserving closed cover by Lindelöf Σ-subspaces then $C_p(X)$ is a Lindelöf Σ-space.

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Proposition

If X is a Lindelöf Σ-space and $C_p(X)$ has a closure-preserving closed cover by Lindelöf Σ-subspaces then $C_p(X)$ is a Lindelöf Σ-space.

Example (**Okunev**)

There exists a σ-compact space X such that $C_p(X)$ is not Lindelöf but some σ-compact set Q is dense in $C_p(X)$.

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Corollary

There exists a σ-compact space X such that $C_p(X)$ is not Lindelöf but there exists a closure-preserving cover of $C_p(X)$ by its σ-compact subspaces.

Generalizing results of Arhangel'skii and Tkachuk

Theorem

Assume that X is a space and $C_p(X)$ has a closure-preserving closed cover by its Lindelöf Σ-subspaces. Then $C_p(X)$ is ω-monolithic.

Generalizing results of Arhangel'skii and Tkachuk

Theorem

Assume that X is a space and $C_p(X)$ has a closure-preserving closed cover by its Lindelöf Σ-subspaces. Then $C_p(X)$ is ω-monolithic.

Corollary

Assume that X is a space for which either ω_1 is a caliber of it or the spread of $C_p(X)$ is countable. If $C_p(X)$ has a closed closure-preserving cover of $C_p(X)$ by Lindelöf Σ-subspaces, then X is cosmic.

General closure-preserving covers of $C_p(X)$

Theorem

Given a space X and a topological property $\mathcal{P}$ that is invariant under continuous images, if either $C_p(X)$ or $C_p(X, \mathbb{I})$ admits a closure-preserving (not necessarily closed) cover $\mathcal{C}$ such that each $C \in \mathcal{C}$ has $\mathcal{P}$ then $C_p(X, \mathbb{I})$ contains a dense subspace that has $\mathcal{P}$.

For σ-additive properties

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Corollary

For any infinite cardinal κ consider the following list $\mathbb{M}_0$ of topological properties:
$\mathbb{M}_0 = \{$*network weight* $\leq \kappa$, *spread* $\leq \kappa$, *Lindelöf number* $\leq \kappa$, *Lindelöf* Σ *index* $\leq \kappa$, *hereditary density* $\leq \kappa\}$.
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(c) $C_p(X)$ *has a dense subspace with the property* $\mathcal{P}$.
(d) $C_p(X, \mathbb{I})$ *has a dense subspace with the property* $\mathcal{P}$.

Answer to problem 4

Corollary

For any infinite cardinal κ consider the following list $\mathbb{M}_1$ of topological properties:
$\mathbb{M}_1 = \{$*k-separability, caliber κ, point-finite cellularity $\leq \kappa$, density $\leq \kappa$*$\}$.
If a property $\mathcal{P}$ belongs to the list $\mathbb{M}_1$ then the following conditions are equivalent:

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(c) *$C_p(X)$ has the property $\mathcal{P}$.*

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Answer to Problem 1

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For any space X, the following conditions are equivalent:

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For any space X, the following conditions are equivalent:

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(b) *$C_p(X)$ has a closure-preserving cover by closed pseudocompact subspaces.*

(c) *$C_p(X)$ is σ-pseudocompact.*

Answer to Problem 2

Corollary

For any space X, the following conditions are equivalent:

Answer to Problem 2

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For any space X, the following conditions are equivalent:

(a) $C_p(X, \mathbb{I})$ *has a closure-preserving cover by pseudocompact subspaces.*

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For any space X, the following conditions are equivalent:

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Corollary

For any space X, the following conditions are equivalent:

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(c) $C_p(X, \mathbb{I})$ *is pseudocompact.*

For compact spaces

Corollary

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If K is a compact space then the following conditions are equivalent:

(a) $C_p(K)$ *has a closure-preserving cover by its* σ*-compact subspaces.*

(b) $C_p(K, \mathbb{I})$ *has a closure-preserving cover by its* σ*-compact subspaces.*

(c) K *is Eberlein compact.*

For ω-perfect classes

Corollary

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For ω-perfect classes

Corollary

If $\mathcal{P}$ is a ω-perfect class and X is a compact space then the following conditions are equivalent:

(a) *$C_p(X)$ has a closure-preserving cover by subspaces that belong to $\mathcal{P}$.*

For ω-perfect classes

Corollary

If $\mathcal{P}$ is a ω-perfect class and X is a compact space then the following conditions are equivalent:

(a) *$C_p(X)$ has a closure-preserving cover by subspaces that belong to $\mathcal{P}$.*

(b) *$C_p(X, \mathbb{I})$ has a closure-preserving cover by subspaces that belong to $\mathcal{P}$.*

For ω-perfect classes

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If $\mathcal{P}$ is a ω-perfect class and X is a compact space then the following conditions are equivalent:

(a) *$C_p(X)$ has a closure-preserving cover by subspaces that belong to $\mathcal{P}$.*

(b) *$C_p(X, \mathbb{I})$ has a closure-preserving cover by subspaces that belong to $\mathcal{P}$.*

(c) *$C_p(X)$ belongs to $\mathcal{P}$.*

For Talagrand and Gul'ko compact spaces

Corollary

Suppose that X is a compact space and $\mathcal{P}$ is either K-analitycity or Lindelöf Σ-property. Then the following conditions are equivalent:

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(c) $C_p(X)$ *has* $\mathcal{P}$.

Open problems, Lindelöf and Lindelöf Σ properties

- Suppose that X is a space such that $C_p(X)$ is the union of a closure-preserving family of its closed Lindelöf subspaces. We know that in this case $C_p(X, \mathbb{I})$ is a Lindelöf space. But must the whole $C_p(X)$ be Lindelöf?
- Suppose that X is a space such that $C_p(X)$ is the union of a closure-preserving family of its closed Lindelöf Σ-subspaces. We know that in this case $C_p(X, \mathbb{I})$ is a Lindelöf Σ-space. But must the whole $C_p(X)$ be Lindelöf Σ? The answer is not clear even if X has a unique non-isolated point.

Open problems, Lindelöf and Lindelöf Σ properties

- Suppose that $C_p(C_p(X))$ is the union of a closure-preserving family of its closed Lindelöf Σ-subspaces. Must the space $C_p(C_p(X))$ be Lindelöf Σ?
- Suppose that X is a space such that $C_p(X)$ has the Baire property and can be represented as the union of a closure-preserving family of its closed Lindelöf Σ-subspaces. Must X be countable?
- Suppose X is a space such that $s(X) \leq \omega$ and $C_p(X)$ is the union of a closure-preserving family of its closed Lindelöf Σ-subspaces. Must X have a countable network?
- Suppose X is a space such that $s(X) \leq \omega$ and $C_p(X, \mathbb{I})$ is a Lindelöf Σ-space. Must X have a countable network?

Open problems, K-analyticity and and Fréchet-Urysohn property

- Suppose that X is a space such that $C_p(X)$ is the union of a closure-preserving family of its closed K-analytic subspaces. We know that in this case $C_p(X, \mathbb{I})$ is a K-analytic space. But must the whole $C_p(X)$ be K-analytic?
- Suppose that X is a space such that $C_p(X)$ is the union of a closure-preserving family of its closed sequential subspaces. We know that in this case $C_p(X, \mathbb{I})$ must be sequential. But must the whole $C_p(X)$ be sequential?
- Suppose that X is a space such that $C_p(X, \mathbb{I})$ is sequential. Must $C_p(X, \mathbb{I})$ (or equivalently $C_p(X)$) be Fréchet-Urysohn?

Open problems, Weight

- Is the space $C_p(\mathbb{I})$ representable as the union of a closure-preserving family of its second countable subspaces?

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