ELEMENTOS DE GEOMETRÍA COMPLEJA EN ANÁLISIS

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- FIBRADOS : "Pullbacks"; grasmanianas, complexif$^{\underline{es}}$; cuantización geométrica; teoría de conexiones

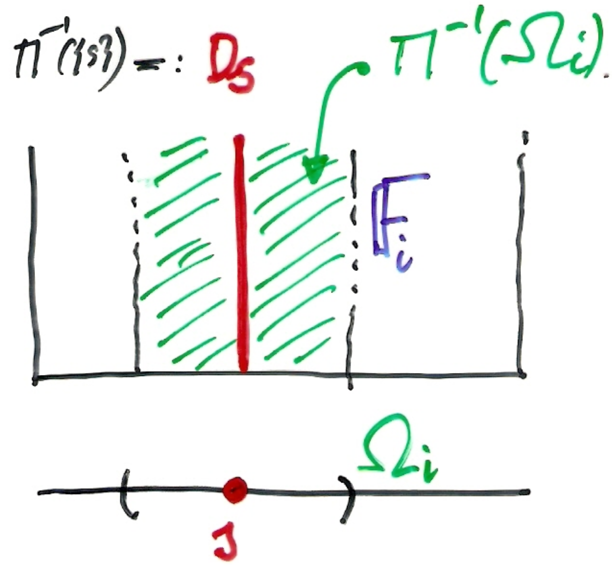
- REPRESENTACIONES : $\left\{ \begin{array}{l} \text{grupos BL} \\ \text{álgebras } C^* \end{array} \right.$

  Borel-Weil & Harish-Chandra
  Geometría de órbitas
  Op. completamente positivos

- NÚCLEOS REPRODUCTIVOS

FIBRADOS

$$\begin{array}{l} D \qquad \Pi^{-1}(\Omega_i) \stackrel{\sim}{=} \Omega_i \times \mathbb{F}_i \\ \Pi \big\downarrow \\ Z = \bigcup_i \Omega_i \end{array}$$



- Π Cuasi-Hermítico:

$$\exists \; s \in Z \longmapsto s^{-*} \in Z \;, (s^{-*})^{-*} = s \ni$$

(a) $\forall s \in Z, \quad (\cdot|\cdot)_{s,s^{-*}} : D_s \times D_{s^{-*}} \longrightarrow \mathbb{C}$ SESQUI

(b) $\overline{(\xi|\eta)_{s,s^{-*}}} = (\eta|\xi)_{s^{-*},s}$ $\quad \forall s \in Z;$ $\xi \in D_s,$ $\eta \in D_s$

(c) $s \longmapsto (\cdot|\cdot)_{s,s^{-*}}$ diff^{5}

- Π Hermítico: $^{-*} = id_Z$.

PULLBACK

$$\begin{tikzcd}
\gamma^{*}(\Pi_{\mathcal{H}}) \equiv D \arrow[r, dotted] \arrow[d] & I_n(\mathcal{H}) := \left\{ (S,x) : \begin{array}{l} x \in S \leq \mathcal{H} \\ \dim S = n \end{array} \right\} \arrow[d, "\Pi_{\mathcal{H}}"] \\
\underset{\text{VAR.}}{Z} \arrow[r, "\gamma", "\text{HOL}^{\text{a}}"'] & Gr(n;\mathcal{H}) \ni S
\end{tikzcd}$$

$(S,x) \longmapsto S$

• EJEMPLOS

(a) T$^{\text{a}}$ COWEN-DOUGLAS, Acta Math. 1978.

Hardy

$$\mathcal{H} := H^2(\mathbb{D}) = \left\{ \sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}$$

Núcleo reproductivo $\qquad f(\lambda) = \langle f, K_\lambda \rangle$

$$K(z,\lambda) := (1 - \bar{\lambda} z)^{-1} \; ; \qquad z, \lambda \in \mathbb{D}$$

$$Z = \mathbb{D} \xrightarrow{\gamma} Gr(1; H^2(\mathbb{D}))$$

$$\lambda \longmapsto \langle k_\lambda \rangle \qquad \text{"} \; k_\lambda = K(\cdot, \lambda)$$

(b) CUANTIZACIÓN GEOMÉTRICA

A. Odzijewicz, Comm. Math. Physics 1988, 1992.

SISTEMA MECÁNICO

$$Z \xrightarrow{\mathcal{K}} Gr(1;\mathcal{H}^K)$$

FASE clásico — Cuantización — FASE cuántico

K núcleo $\equiv$ amplitud probabilidades transición

Equivalentes $\begin{cases} K \\ \mathcal{H} \\ \mathcal{K} : Z \to Gr(1;\mathcal{H}) \end{cases}$

- Aplicación:

  Mecánica Schrödinger

  Oscilador armónico

  Partículas relativistas.

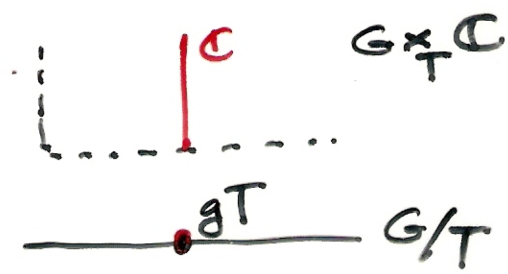
③ BOREL-WEIL [Harish Chandra] THEOREM

G Lie compact ; T maximal torus $\Rightarrow$ G/T complex [fixed]

- $\forall \mu \in \widehat{T}$, in $G \times \mathbb{C}$:

$$(g,z) \sim (g',z') \Longleftrightarrow \exists a \in T, \begin{cases} g' = g \cdot a \\ z' = \mu(a^{-1})\, z \end{cases}$$

$$L_\mu : G \times_T \mathbb{C} \to G/T$$
$$[(g,z)] \mapsto gT$$



<u>THEOREM</u> $\quad \mu \in \widehat{T}$ "antidominant" $\Rightarrow \Gamma_{hol}(L_\mu) \neq (0)$

& $\quad \widehat{G} \equiv \{ \mu \in \widehat{T} : \mu \text{ "antidominant"} \}$

- <u>Example</u>

$G = U(n)$, $\quad T = \mathbb{T}^n$

$G_{\mathbb{C}} = GL(n;\mathbb{C})$, $\quad \mathcal{Z}_{\mathbb{C}} := \left\{ \begin{pmatrix} \cdot & \cdots \\ 0 & \cdot \end{pmatrix} \right\}$

$G/T = G_{\mathbb{C}}/\mathcal{Z}_{\mathbb{C}}$ $\quad$ [since $G_{\mathbb{C}} = G\,\mathcal{Z}_{\mathbb{C}}$, $G \cap \mathcal{Z}_{\mathbb{C}} = T$]

$$\Longrightarrow \quad \widehat{U}(n) \equiv \{ (m_1, \dots, m_n) \in \mathbb{Z}^n : m_1 \leq \dots \leq m_n \}$$

T^a REPRESENTACIÓN

- A C^*-álgebra, $\varphi : A \to \mathbb{C}$

$$\varphi \rightsquigarrow \langle x|y\rangle_\varphi := \varphi(y^* x) \quad \forall x, y \in A^*$$

$$\rightsquigarrow \pi_\varphi : A \longrightarrow \mathcal{L}(\mathcal{H}_\varphi)$$

Repr. GNS

<u>TEOREMA</u> (Beltita/Ratiu, Adv. Math.).- 2007

$\pi_\varphi|_{U_A} : U_A \longrightarrow \mathcal{U}(\mathcal{H}_\varphi)$ puede "realizarse" como operador <u>multiplicación</u> sobre un espacio de Hilbert, de secciones real-analíticas de un fibrado vectorial.

- ¿HILBERT? $\equiv$ Núcleos reproductivos

- ¿HOLOMORFÍA?

NÚCLEOS REPRODUCTIVOS

$$K \rightsquigarrow \Pi : D \to Z \ni$$

$\forall (s,t) \in Z$, $\quad K(s,t) : D_t \to D_s$ LINEAL ACOTADA

&

- Definido-positiva: $\forall n \geq 1$;

$t_j \in Z \wedge \xi_j \in D_{t_j^{-*}}$ $(j = 1, 2, \dots, n)$

$$\sum_{j,l=1}^{n} \left(K(t_l \mid t_j^{-*}) \xi_j \mid \xi_l \right)_{t_l, t_l^{-*}} \geq 0.$$

- $\forall s \in Z$, $\xi \in D_s$,

$$K_\xi := K(\cdot, s)\xi \in C^{(\infty)}(Z, D) \text{ ''}$$

SECCIÓN

$$\mathcal{H}_0^K := \mathrm{CL}\{K_\xi \mid \xi \in D\} \ \& \ \mathcal{H}^K := \overline{\mathcal{H}_0^K}$$

en $\left(K_\eta \mid K_\xi \right)_{\mathcal{H}^K} := \left(K(s^*, t)\eta \mid \xi \right)_{s^*, s}$

ASSUMPTION:

G_B Banach-Lie subgroup of G_A, $*$-stable

$\exists\ \mathcal{H}_X$ Hilbert (complex) ; $\pi_X : G_X \to \mathcal{L}(\mathcal{H}_X)$ holomorphic $*$-repr.

s.t. $\pi_B(g) = \pi_A(g)\big|_{\mathcal{H}_B} \quad \forall g \in G_B$

$P : \mathcal{H}_A \to \mathcal{H}_B$ orth. proj.

- Vector bundle $\quad G_A \times_{G_B} \mathcal{H}_B =: D^G$

$$(g, f) \sim (g', f') \quad \begin{cases} g' = g \cdot w \\ f' = \pi_B(w^{-1}) f \end{cases} \quad w \in G_B$$

$$\Pi^G : G_A \times_{G_B} \mathcal{H}_B \longrightarrow G_A / G_B$$

$$[(g, f)] \mapsto g G_B \mapsto \boxed{(g G_B)^{-*} := g^{-*} G_B}$$

$$K^G(g_1 G_B, g_2 G_B)([(g_2, f)])$$

$s = g_2 G_B$; $\xi = [(g_2, f)] \in D_s$

$$:= [(g_1, P(\pi_A(g_1)^{-1} \pi_A(g_2) f))]$$

$D^U := U_A \times_{U_B} H_B \xrightarrow{\Pi^U} U_A/U_B$; K^U etc.

TEOREMA (Beltita/G., JFA 2008).-

① $\exists\, \mathcal{H}^K \subseteq \mathcal{O}(G_A/G_B\,;\, D^G)$,

con $\mathcal{H}^K \underset{\iota(K)}{\simeq} \mathcal{H}_A \ni$

$$\begin{tikzcd}
\mathcal{H}_A \arrow[r, "\pi_A(g)"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{H}_A \arrow[d] \\
\mathcal{H}^K \arrow[r, "\mu_A(g)"'] & \mathcal{H}^K
\end{tikzcd}, \quad \forall g \in G_A$$

con $\mu_A(g)F(\cdot) := gF(g^{-1}\cdot)$, si $F \in \mathcal{H}^K$.

②

$$\begin{tikzcd}
U_A \underset{U_B}{\times} \mathcal{H}_B \arrow[r] \arrow[d, "\pi^U"'] & G_A \underset{G_B}{\times} \mathcal{H}_B \arrow[d, "\pi^G"] \\
U_A/U_B \arrow[r, hook] & G_A/G_B
\end{tikzcd}$$

Complexificación

REAL-ANALÍTICO

HOLOMORFO

U_A - equivariante

APLICACIONES

① $\mathcal{A}$ C^*-álgebra „ $\mathcal{A} \xrightarrow{\Phi} \mathcal{L}(\mathcal{H})$ CP

$\Phi \rightsquigarrow \pi_{\Phi} : \mathcal{A} \to \mathcal{L}(\mathcal{H})$

Stinespring (dilatación)

- Toda π_{Φ} admite "realización" como multiplicación, HILBERT fecciones.

② ÓRBITAS : $\rho : \mathfrak{A} \to \mathcal{L}(\mathcal{H})$ „

REPRES.

$\mathfrak{A} \begin{cases} \text{NUCLEAR} \\ \text{v NEUMANN inyectiva} \end{cases} \Rightarrow E_\rho : \mathcal{L}(\mathcal{H}) \to \rho(\mathfrak{A})'$

ESPERANZA

$\mathcal{S}(\rho) := \{ u\rho(\cdot)u^{-1} \mid u \in GL(\mathcal{H}) \}$ SEMEJANZA

$\mathcal{U}(\rho) := \{ \quad " \quad \mid u \in \mathcal{U}(\mathcal{H}) \}$ UNITARIA

- Entonces

$\mathcal{S}(\rho) \cong T(\mathcal{U}(\rho))$ COMPLEXIFICACIÓN de $\mathcal{U}(\rho)$.

c) GRASSMANIANS

$A = \mathcal{L}(\mathcal{H})$; $\mathcal{G} := G_A$, $\mathcal{U} := U_A$ .. $Gr(\mathcal{H})$

<u>Universal bundle</u> $\quad \mathcal{I}(\mathcal{H}) := \left\{ (S,x) \,\middle|\, \begin{matrix} S \subseteq \mathcal{H}, \\ x \in S \end{matrix} \right\}$

$$\Pi_{\mathcal{H}} : \mathcal{I}(\mathcal{H}) \longrightarrow Gr(\mathcal{H}) \qquad (S,x) \longmapsto S$$

<u>tautological bundle</u>

fiber at $S \equiv S$

- For $S_0 \in Gr(\mathcal{H})$, $p = p_{S_0} : \mathcal{H} \xrightarrow{\perp} S_0$ .. $B := \{p\}'$

$$Gr_{S_0}(\mathcal{H}) := \{ g S_0 \mid g \in \mathcal{G} \} = \{ u S_0 \mid u \in \mathcal{U} \} \equiv \mathcal{U}/\mathcal{U}_p$$

Put $\quad \mathcal{I}_{S_0}(\mathcal{H}) = \mathcal{I}(\mathcal{H}) \cap \left(Gr_{S_0}(\mathcal{H}) \times \mathcal{H} \right)$

<u>THEOREM</u> [Beltita-G 2009] Comp. Var & Op. Th.

(i) $\mathcal{G}/\mathcal{G}_p$ complexification of $Gr_{S_0}(\mathcal{H})$.

(ii)
$$\begin{array}{ccccc}
\mathcal{I}_{S_0}(\mathcal{H}) & \xrightarrow{\cong} & \mathcal{U} \times_{\mathcal{U}_p} S_0 & \hookrightarrow & \mathcal{G} \times_{\mathcal{G}_p} S_0 \\
\Pi_{\mathcal{H},S_0} \downarrow & & \vdots & & \downarrow \Pi^{\mathbb{C}}_{\mathcal{H},S_0} \\
Gr_{S_0}(\mathcal{H}) & \equiv & \mathcal{U}/\mathcal{U}_p & \longrightarrow & \mathcal{G}/\mathcal{G}_p
\end{array}$$

MORFISMO DE FIBRADOS

$$[d]\quad \begin{tikzcd} D \arrow[r, "\delta"] \arrow[d] & \tilde{D} \arrow[d] \\ Z \arrow[r, "\xi"] & \tilde{Z} \end{tikzcd} \qquad \theta = (\delta, \xi) \ni$$

$z \mapsto z^{-*}$ $\quad$ $\tilde{z} \mapsto \tilde{z}^{-*}$

$\xi(z^{-*}) = \xi(z)^{-*}$

(i) δ, ξ 'SMOOTH'

(ii) $[d]$ CONMUTATIVO

(iii) $\delta|_{D_z} : D_z \to \tilde{D}_{\xi(z)}$ LINEAL, ACOTADA

'PULLBACK' DE NÚCLEOS

$K = \tilde{K} \leftrightsquigarrow \tilde{D} \to \tilde{Z}$

$$\theta^* K(s,t) := \left(\delta|_{D_{s^{-*}}}\right)^{-*} \circ K(\xi(s), \xi(t)) \circ \delta|_{D_t} \, ;$$

es decir,

$$\begin{tikzcd} D_t \arrow[r, "\theta^* K(s,t)"] \arrow[d, "\delta"'] & D_s \\ \tilde{D}_{\xi(t)} \arrow[r, "K(\xi(s), \xi(t))"'] & \tilde{D}_{\xi(s)} \arrow[u, "(\delta|_{D_s})^*"'] \end{tikzcd} \qquad \forall s, t \in Z.$$

TEOREMA U1 (Beltita/G., Rev. Ibero-Americana 2011).-

$\forall\, D \xrightarrow{\Pi} Z \qquad \exists\, \Delta_K = (\delta_K, \eta_K)$ "

con K $\qquad \delta_K = (\eta_K \circ \Pi, \hat{K})$

tq
$$\begin{tikzcd} D \arrow[r, "\delta_K"] \arrow[d] & \mathcal{I}(\mathcal{H}^K) \arrow[d] \\ Z \arrow[r, "\eta_K"] & \mathrm{Gr}(\mathcal{H}^K) \end{tikzcd}$$
$$s \longmapsto \hat{K}(D_s)$$

$\hat{K}(\xi) := K_\xi \quad (\xi \in D)$

& $\quad K = \Delta_K^* Q_{\mathcal{H}^K}$ con $\left\{ Q_{\mathcal{H}^K}(S_1, S_2) := P_{S_1}\big|_{S_2} \right.$.

Más aún, si K es continuo,

(1) $\forall s \in Z$, $\hat{K}\big|_{D_s}$ es $\left\{ \begin{array}{l} \text{INYECTIVO} \\ \text{RANGO CERRADO} \end{array} \right.$

(2) η_K LOCALMENTE SIMILAR $\left[\eta_K(s) \rightleftharpoons \eta_K(t) \text{ si } s \approx t \right]$

entonces Δ_K es CONTINUO

COROLARIO Dado $D \xrightarrow{\pi} Z$,

$\exists K$ para $D \xrightarrow{\pi} Z$, con $\exists K(s,s)^{-1}$ $\forall s$
NÚCLEO

$\Longleftrightarrow$ $\exists\ \xi : Z \longrightarrow Gr(\mathcal{H})$.
CUANTIZACIÓN

En tal caso,

$\xi = \xi_K$, $\mathcal{H} = \mathcal{H}^K$ ∧ $D \cong \{(s,x) \in Z \times \mathcal{H} : x \in \xi(s)\}$

Dem.-

En Teorema U1,

$$D \simeq \{(s,x) \in Z \times \mathcal{H}^K : x \in \overline{\hat{K}(D_s)}\} \blacksquare$$

- Aplicación a todos los núcleos clásicos

- EJEMPLO :

  Hardy " $H^2(\mathbb{D})$

  $K(z,\lambda) = (1-\bar{\lambda}z)^{-1}$ & $\xi(\lambda) = \langle K(\cdot,\lambda)\rangle$, $\lambda \in \overline{\mathbb{D}}$
  $\in Gr(1, H^2)$.

TEOREMA U2.-

- $G \times [D \xrightarrow{\Pi} Z] \longrightarrow [D \xrightarrow{\Pi} Z]$ HOLOMÓRFICAMENTE

(1) $Z = \{u \cdot s_0 \mid u \in G\}$ " $s_0 \in Z$

(2) $\{u \in G \mid u \cdot s_0 = s_0\}$ SUBGRUPO BANACH-LIE de G

- $\exists$ K núcleo $D \xrightarrow{\Pi} Z$ $\ni$

(3) $\forall u \in G;\ s, t \in Z;\ \xi \in D_s$.

$$K(t, u \cdot s)(u \cdot \xi) = u K(u^{-1} \cdot t, s)\xi$$

(4) $\xi \longmapsto K(t, \Pi(\xi))\xi$, $D \to D$

HOLOMORFA $\forall t \in Z$.

Entonces $\hat{K}, \xi_K$ holomorfas

& $K = \Delta_K^* \Phi_{H^K}$

CONEXIONES

DEF.- $\Phi : TD \to TD$ conexión:

(i) $\Phi \circ \Phi = \Phi$; (ii)
$$\begin{tikzcd} TD \arrow[r, "\Phi"] \arrow[d, "\tau_D"'] & TD \arrow[d, "\tau_D"] \\ D \arrow[r, "id_D"'] & D \end{tikzcd}$$

(iii) $\forall \xi \in D$,

Si $\Phi_\xi := \Phi|_{T_\xi D}$ entonces

$$\boxed{\operatorname{Ran} \Phi_\xi = \operatorname{Ker}(T_\xi \Pi)}$$

- $V_\xi D := \operatorname{Ker}(T_\xi \Pi)$ VERTICAL

  $H_\xi D := \operatorname{Ker} \Phi_\xi$ HORIZONTAL

$$T_\xi M = H_\xi D + V_\xi D$$

PULLBACK DE CONEXIONES

TEOREMA (2011).- $\forall\, \Delta = (\delta, \xi)$

$$\begin{array}{ccc} D & \xrightarrow{\delta} & \tilde{D} \\ \downarrow & & \downarrow \\ Z & \xrightarrow[\xi]{} & \tilde{Z} \end{array} \ni$$

- $\delta : D_1 \to \tilde{D}_{\xi(1)}$ DIFEO
- $\tilde{\Phi}$ conexión $\tilde{D} \to \tilde{Z}$,

$\exists!\ \Phi$ CONEXIÓN $D \to Z$ t.q.

$$\begin{array}{ccc} TD & \xrightarrow{T\delta} & T\tilde{D} \\ \Phi \downarrow & & \downarrow \tilde{\Phi} \\ TD & \xrightarrow[T\delta]{} & T\tilde{D} \end{array} \quad " \quad \Phi \underset{\text{DEF}}{\equiv} \Delta^* \tilde{\Phi}$$

Dem.- $\forall\, \xi \in D$,

$$\Phi_\xi := \left(T_\xi \delta \big|_{V_\xi D}\right)^{-1} \circ \tilde{\Phi}_{\delta(\xi)} \circ T_\xi \pi \qquad \blacksquare$$

FIBRADO HOMOGÉNEO

$$A \xrightarrow{E} B$$

C^*-álg. ESPERANZA

¿ $T(G_A \times_{G_B} \mathcal{H}_B)$?

- $T(G) = G \rtimes_{Ad} \mathfrak{g} : (g, X)\cdot(a, X') = (ga, Ad(a^{-1})X + X')$

- $G_B \xrightarrow{\pi_B} \mathcal{B}(\mathcal{H}_B) \rightsquigarrow TG_B \xrightarrow{T\pi_B} \mathcal{B}(\mathcal{H}_B \oplus \mathcal{H}_B)$

Con

$$T\pi_B : (b, Y) \longmapsto \begin{pmatrix} \pi_B(b) & \pi_B(b)\, d\pi_B(Y) \\ 0 & \pi_B(b) \end{pmatrix}$$

KRIEGL–MICHOR

- Entonces

$$T(G_A \times_{G_B} \mathcal{H}_B) \equiv TG_A \times_{TG_B} T\mathcal{H}_B$$

$$\equiv (G_A \rtimes_{Ad} \mathfrak{g}_A) \times_{G_B \rtimes_{Ad} \mathfrak{g}_B} (\mathcal{H}_B \oplus \mathcal{H}_B)$$

$$[(a, X), (f, h)] \xrightarrow[\Phi]{} [(a, E(X)), (f, h)]$$

CONEXIONES ASOCIADAS A NÚCLEOS

$$K \text{ para } \underset{\text{CONEXO}}{D \longrightarrow Z} \rightsquigarrow \mathcal{H}^K =: \mathcal{H}$$

&

$$\begin{array}{ccc} D & \xrightarrow{\delta_K} & I_{S_0}(\mathcal{H}) = GL(\mathcal{H}) \times_{\{p\}'} S_0 \\ \downarrow & & \\ Z & \xrightarrow[\xi_K]{} & Gr_{S_0}(\mathcal{H}) = GL(\mathcal{H})/\{p\}' \end{array}$$

Con $p = p_{S_0} : \mathcal{H} \longrightarrow S_0$ "

$$E_p = p(\cdot)p + (1-p)(\cdot)(1-p).$$

Como antes $E_p \rightsquigarrow \Phi_{\mathcal{H}}$

DEF.- $\Phi^K := \Delta_K^* \Phi_{S_0}$