Projections on uniformly convex spaces

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Projection norms

Theorem (Kadec and Snobar)

Every finite dimensional Banach space E is complemented in any superspace F with a projection $P : F \to E$ with

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Theorem

Let $1 \leq p \neq 2 < \infty$. *There exist a constant* M *depending only on* p *and a sequence of into isomorphisms*

$$T_n : \ell_p^{k(n)} \to \ell_p^n$$

with $\|T_n\|\|T_n^{-1}\| \leq M$ *such that there is no uniformly bounded sequence of projections* $P_n : \ell_p^n \to \ell_p^{k(n)}$.

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- Bourgain: $p = 1$.

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Through the rest our setting will be finite dimensional uniformly convex Banach spaces.

Uniformly convex spaces

We recall that F is said to be uniformly convex if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$ where as usual δ denotes the modulus of convexity of F, namely,

$$\delta(\varepsilon) = \inf\left\{1 - \frac{\| x + y \|}{2}; x, y \in F, \| x \| = \| y \| = 1, \| x - y \| = \varepsilon\right\}.$$

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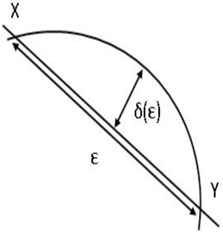

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$$\begin{cases} (p-1)\varepsilon^2 & \text{for } 1 < p \leq 2 \\ (p \cdot 2^{-p})\varepsilon^p & \text{for } 2 \leq p < \infty \end{cases}$$

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Theorem (S.)

Let F be a uniformly convex space of modulus of convexity δ of power type p, i.e. $\delta(\varepsilon) \geq K\varepsilon^p$, and dimension $n+1$ for $n+1$ large enough. Assume E is a subspace of F with $\dim E \leq n^{\frac{1}{p+2}}$. Then every projection $P : F \to E$ such that $\|P\| \leq \dim E$ satisfies:

$$\mathbb{P}_F \left\{ f \in B(F) : \|Pf\| \leq n^{-\frac{1}{(p+1)(p+2)}} \right\} \geq 1 - (n+1)e^{-Kn^{\frac{1}{p+1}}}$$

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Why is it happening this phenomena?

Key idea

Theorem (Gromov and Milman)

If F is a uniformly convex $n+1$-dimensional Banach space, with modulus of convexity δ and $\phi \in F^*$ with $\|\phi\| = 1$, then

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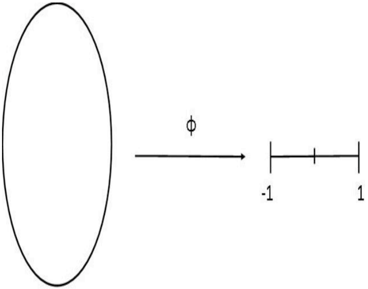

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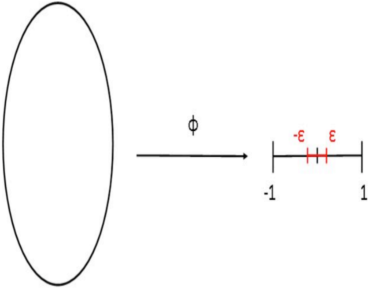

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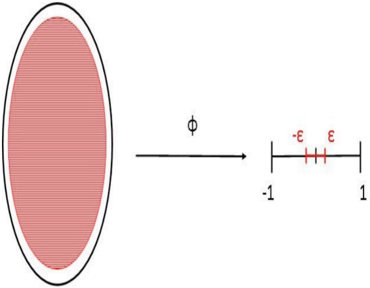

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Proof.

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3. Take $P : F \to E$ and consider $j_\varepsilon P : F \to \ell_\infty^N$. If $\|j_\varepsilon Pf\|$ is close to 0 then so does $\|Pf\|$.

□

The volumetric point of view

To sum up

For $F = (\mathbb{R}^{n+1}, \|\cdot\|_F)$ uniformly convex of power type p and a projection $P : F \to E$, the set

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- E is 1-complemented in $(\mathbb{R}^{n+1}, \|\cdot\|_G)$.
- $$1 \leq \frac{\mathrm{Vol}_{n+1}(B(F))}{\mathrm{Vol}_{n+1}(G)} \leq \left(1 - (n+1)e^{-Kn^{\frac{1}{p+1}}}\right)^{-1}.$$

Uniformly convex spaces are a Levy family

Consider $F_n = (\mathbb{R}^{n+1}, \| \cdot \|_{F_n})$ uniformly convex of power type p with $\mathbb{P}_{F_n}$, the normalized uniform volume element on B_{F_n}. For fixed $\varepsilon > 0$ and $A \subseteq B(F_n)$:

$$\mathbb{P}_{F_n}(A) \geq 1/2 \quad \Longrightarrow \quad \mathbb{P}_{F_n}(A_\varepsilon) \geq 1 - 2e^{-2Kn\varepsilon^p},$$

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The renormings are also a Levy family

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Theorem

Uniformly convex n-dimensonal Banach spaces of power type 2 contains a 2-isomorphic copy of ℓ_2^m *for* $m \geq cn$.

Coming back to the linear structure

Euclidean sections for the renorming

Theorem

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Theorem (Rudelson and Vershynin)

Let X be an n-dimensional Banach space and (T, d, μ) a metric probability space satisfying

$$\mu(A) \geq 1/2 \implies \mu(A_\varepsilon) \geq 1 - c_1 e^{-c_2 n \varepsilon^2}.$$

If (T, d) can be K-lipschitz embedded into X then X contains a 2-isomorphic copy of ℓ_2^m for $m \geq cn$.

Proposition (S.)

If F is uniformly convex of power type 2 and $(\mathbb{R}^{n+1}, \|\cdot\|_G)$ denotes the renorming given for a specific projection P then $(\mathbb{R}^{n+1}, \|\cdot\|_G)$ contains a 2-isomorphic copy of ℓ_2^m for

$$m \geq c\frac{n}{\|P\|^2}.$$

In particular $m \geq c\sqrt{n}$.

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$(B(F), \|\cdot\|_F, \mathbb{P}_F)$ and $(B(G), \|\cdot\|_G, \mathbb{P}_G)$ are metric probability spaces.

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$$\underline{\square}_1(B(F), B(G)) \to 0.$$

THANK YOU!