Caos distribucional para operadores

Alfred Peris
Institut Universitari de Matemàtica Pura i Aplicada
Universitat Politècnica de València

VI Encuentro de Análisis Funcional y Aplicaciones
Salobreña, 15 al 17 de Abril de 2010

Definitions

- **Li-Yorke, 1975:** An uncountable subset $S \subset X$ of a metric space (X, d) is called a **scrambled set** for a dynamical system $f : X \to X$ if for any $x, y \in S$ with $x \neq y$ we have $\liminf_n d(f^n(x), f^n(y)) = 0$ and $\limsup_n d(f^n(x), f^n(y)) > 0$.
- **Schweizer-Smítal, 1994:** A dynamical system $f : X \to X$ with a scrambled set S is **distributionally chaotic** on S if, additionally, there is $\delta > 0$ so that for each $\varepsilon > 0$ and each pair $x, y \in S$ of distinct points we have

$$(1) \quad \liminf_n \frac{|\{k \leq n \ : \ d(f^k(x), f^k(y)) < \delta\}|}{n} = 0$$

and

$$(2) \quad \limsup_n \frac{|\{k \leq n \ : \ d(f^k(x), f^k(y)) < \varepsilon\}|}{n} = 1.$$

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- We recall that **the upper density** $\mathcal{D}(A)$ of a set $A \subset \mathbb{N}$ is defined by:

$$\mathcal{D}(A) = \limsup_n \frac{|A \cap \{1, \ldots, n\}|}{n}$$

- **Equivalent definition of distributional chaos:** A dynamical system $f : X \to X$ with a scrambled set S is distributionally chaotic on S if there is $\delta > 0$ so that for each $\varepsilon > 0$ and each pair $x, y \in S$ of distinct points we have

$$(1) \quad \mathcal{D}(\{k \in \mathbb{N} \ : \ d(f^k(x), f^k(y)) \geq \delta\}) = 1$$

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- Given a sequence $v = (v_n)_n$ of positive weights, we will consider the weighted ℓ^p-space $(1 \leq p < \infty)$:

$$X = \ell^p(v) := \{x \in \mathbb{K}^{\mathbb{N}} \ : \| \ x \ \| := \left(\sum_{j=1}^{\infty} | \ x_j \ |^p \ \ v_j \right)^{1/p} < \infty\}$$

- The **backward shift** $T = B : \ell^p(v) \to \ell^p(v)$

$$B(x_1, x_2, x_3, \ldots) := (x_2, x_3, x_4, \ldots)$$

is well-defined (equivalently, continuous) iff $\sup_n \frac{v_n}{v_{n+1}} < \infty$.

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Martínez-Giménez, Oprocha and Peris, 2009

If there is $r > 1$ such that, for any $m \in \mathbb{N}$ there exists an integer i such that

$$r \leq \frac{v_j}{v_{j+1}}, \quad j = i, \dots, i+m$$

then B exhibits distributional chaos.

Cao, Cui and Hou, 2009

Let X be a Banach space and let $T \in L(X)$ be an operator. T satisfies the *distributionally chaotic criterion*, if there is a constant $r > 1$ such that for any $m \in \mathbb{N}$, there exists $x_m \in X \setminus \{0\}$ satisfying:

(i) $\lim_{k\to\infty} \|T^k x_m\| = 0$ and

(ii) $\|T^i x_m\| \geq r^i \|x_m\|$ for $i = 1, 2, \dots, m$.

If T satisfies the distributionally chaotic criterion then T is distributionally chaotic.

Observation: In this case the spectral radius of T is strictly greater than 1.

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Cao, Cui and Hou, 2009 (Weakly distributionally chaotic criterion)

If for any sequence of positive numbers C_m increasing to ∞, there exist $\{x_m\}_{m\in\mathbb{N}}$ in X satisfying:

(a) $\lim_{n\to\infty} \|T^n x_m\| = 0$

(b) There is a sequence of positive integers N_m increasing to ∞, such that

$$\lim_{m\to\infty} \frac{1}{N_m} Card\{0 \le i < N_m : \|T^i(x_m)\| \ge C_m \|x_m\|\} = 1,$$

then T is distributionally chaotic.

Bermúdez, Bonilla, Martínez-Giménez and Peris, 2010

T satisfies the weakly distributionally chaotic criterion if and only if **(CDC)** there exist an increasing sequence of integers $B = (m_k)_k$ with $\mathcal{D}(B) = 1$ and a subset $X_0 \subset X$ satisfying

(a) $\lim_{n\to\infty} T^n x = 0$, $x \in X_0$, and

(b) $\lim_k \|T^{m_k}|_Y\| = \infty$, where $Y := \overline{\mathrm{span}(X_0)}$.

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We say that T is *dense distributionally chaotic* if we can find a dense scrambled set $S \subset X$ on which the conditions of distributional chaos happen.

Bermúdez, Bonilla, Martínez-Giménez and Peris, 2010

If

1. T satisfies (CDC) and
2. there exists a dense subset $X_0 \subset X$ such that $\lim_{n\to\infty} T^n x = 0$, for each $x \in X_0$,

then T is dense distributionally chaotic and, moreover, there is a dense linear manifold $Y \subset X$ such that Y is a distributionally scrambled set for T.

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Suppose that

1. there exists a dense set X_0 such that $\lim_{n\to\infty} \|T^n x\| = 0$, $\forall x \in X_0$ and
2. there exists a eigenvalue λ with $|\lambda| > 1$.

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Definitions

- **Beauzamy, 1988:** A vector $x \in X$ is called **irregular** for an operator $T : X \to X$ provided that $\sup_n \|T^n x\| = \infty$ and $\inf_n \|T^n x\| = 0$. In particular, the line $S := \{\lambda x \ : \ \lambda \in \mathbb{K}\}$ is a scrambled set for T.
- **Prajitura, 2009:** An operator $T : X \to X$ is **completely irregular** if every $x \in X \setminus \{0\}$ is irregular. In particular, the full space $S = X$ is a scrambled set for T.
- Beauzamy (1988) constructs weighted forward shifts which are completely irregular, later slightly modified by Prajitura (2009). Smith (2008) also obtains completely irregular operators with some additional properties.

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The forward shift $F : X \to X$, $F(x_1, \dots) := (0, x_1, \dots)$ is well-defined (continuous) on $X = \ell^p(v)$ iff $\sup_n \frac{v_{n+1}}{v_n} < \infty$.

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Let $X := \ell^p(v)$, where $v = (v_n)_n$ is a sequence of weights satisfying that, for a suitable strictly increasing sequence $(n_k)_k$ of natural numbers, we have

- $k < v_{n_k}$,
- $v_j := 1/k!$, $n_k < j \leq 2n_k$,
- $v_j := (1 + 1/n_k)^{j-2n_k}/k!$, $2n_k < j \leq n_{k+1}$,

$k \in \mathbb{N}$. Then the forward shift $T : X \to X$ is a completely irregular operator such that the sequence $\{\|T^n z\| \ ; \ n \in \mathbb{N}\}$ is dense in $[0, \infty[$ for each $z \in X$, $z \neq 0$.

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Martínez-Giménez, Oprocha and Peris, 2009

Let $X := \ell^p(v)$, where $v = (v_n)_n$ is a sequence of weights defined as above for a rapidly strictly increasing sequence $(n_k)_k$ of natural numbers, then the forward shift $T : X \to X$ is a completely irregular operator, and exhibits distributional chaos on $S = X$.

Based on the Dvoretzky theorem on almost spherical sections

Shkarin, 2008

Let X be a separable infinite dimensional Banach space, let $\{b_k\}_{k\in\mathbb{N}}$ be a sequence of numbers in $[3,\infty)$ such that $b_k \to \infty$ as $k \to \infty$ and $\{n_k\}_{k\in\mathbb{N}}$ be a stricly increasing sequence of positive integers such that $n_0 = 0$ and $n_{k+1} - n_k \geq 2$ for each $k \in \mathbb{N}$. Then there exists a biorthogonal sequence $\{(y_k, f_k)\}_{k\in\mathbb{N}}$ in $X \times X^*$ such that

(B1) $\|y_k\| = 1$ for each $k \in \mathbb{N}$;

(B2) $span\{y_k : k \in \mathbb{N}\}$ is dense in X;

(B3) $\|f_{n_k}\| \leq b_k$ for each $k \in \mathbb{N}$;

(B4) $\|f_j\| \leq 3$ if $j \in \mathbb{N} \setminus \{n_k : k \in \mathbb{N}\}$;

(B5) for any $k \in \mathbb{N}$ and any numbers $c_j \in \mathbb{K}$ with $n_k + 1 \leq j \leq n_{k+1} - 1$

$$\frac{1}{2}\|\sum_{j=n_k+1}^{n_{k+1}-1} c_j y_j\| \leq (\|\sum_{j=n_k+1}^{n_{k+1}-1} |c_j|^2\|)^{\frac{1}{2}} \leq 2\|\sum_{j=n_k+1}^{n_{k+1}-1} c_j y_j\|$$

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Given a sequence $\{w_n\}_{n\in\mathbb{N}}$ in $\ell^2(\mathbb{N})$ there exists $T : X \to X$ satisfying that $Ty_0 = 0$ and $Ty_{n+1} = w_n y_n$ with $\|y_n\| = 1$ where $\{y_n\}$ is given by the result above. The operator $I + T$ is so that there exists a dense sequence $\{x_k\}_k$ in X such that $\lim_n (I + T)^n x_k = 0$ for all $k \in \mathbb{N}$ and $I + T$ admits dense orbits (it is **hypercyclic**).

Bermúdez, Bonilla, Martínez-Giménez and Peris, 2010

If $w_n = n^{-2/3}$ and $n_k = (k+1)!$, under the above conditions, $I + T$ is hypercyclic and distributionally chaotic on X, with a dense linear manifold that constitutes a distributional scrambled set.

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