Local Lipschitz continuity of the diametric completion mapping

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The existence of nontrivial curves of constant width was already known to Euler [4]. They have since become an active subject of study, appearing in many different contexts and having surprising applications in different areas of mathematics and engineering.

The notion of diametrically maximal sets appeared later, at the beginning of last century, as a generalization of constant width sets [8]. A set in a metric space is *diametrically maximal* (or *complete*) if the addition of any point to the set increases the diameter.

It is well known that every bounded set C can be completed, in the sense that there is always a complete set D containing C such that $\operatorname{diam} C = \operatorname{diam} D$ (see, for example, the survey [2]). Such a set D is called a *completion* of C. In general, a set can have infinitely many completions. We denote by $\mathcal{H}$ the family of all nonempty, closed, bounded and convex sets in our space, endowed with the Hausdorff metric. The *diametric completion mapping*

$$\gamma : \mathcal{H} \to 2^{\mathcal{H}}$$

associates with each $C \in \mathcal{H}$ the family of all its completions. The question with which we are mainly concerned in this talk can be phrased in a simple manner: is γ a continuous function?

Every complete set satisfies the spherical intersection property and therefore it is an intersection of closed balls [2]. For this reason, γ is somehow related to the *ball hull mapping* β, which associates with every bounded set the intersection of all balls containing it. It is known that β need not be continuous, even in three dimensional spaces [11]. Hence, having in mind that

$$\gamma = \gamma \circ \beta \,,$$

one might expect a similar behavior of γ. To our surprise, this is not so and we show that γ is locally Lipschitz continuous in the wide class of normed spaces with Jung constant less than 2. This class includes the spaces with normal structure [5] and, in particular, the finite dimensional spaces.

The setting for the following is a normed real vector space $(X, \|\cdot\|)$. For a subset $C \subset X$, we let $\operatorname{diam}_{\|\cdot\|} C$ denote the diameter of C. Given a point $x \in X$, the radius of C with respect to x is defined as

$$r(x, C) = \sup\{\|x - y\| : y \in C\}\,.$$

If C is a bounded set, then C is included in the ball with center x and radius $r(x, C)$, for every $x \in X$. When $x \in C$, it is clear that

$$r(x, C) \leq \operatorname{diam} C$$

and the problem of covering a bounded set C by a ball with radius smaller than $\operatorname{diam} C$ is a classical subject in convex geometry and functional analysis.

In this latter context, a normed space X has *normal structure* if for each $C \in \mathcal{H}$ there is a point $x \in C$ such that $r(x, C) < d_C$. Different generalizations of this concept can be found in the literature. The Jung constant $J(X)$ is defined by

$$J(X) = \sup\{2r(C) : C \text{ convex, } \operatorname{diam} C = 1\},$$

where

$$r(C) = \inf\{r(x, C) : x \in X\}$$

is the radius of Chebyshev of C [1]. Notice that $J(X) < 2$ implies that every bounded set C can be included in a ball with radius smaller than $\operatorname{diam} C$ (now the center of the ball need not be in C). It is well known that $J(X) < 2$ does not imply normal structure: the space ℓ_∞ with the usual sup norm is an example.

We denote by ρ the Hausdorff metric on $\mathcal{H}$ that is induced by the norm and by Δ the Hausdorff metric on the family of all nonempty, closed and bounded sets in $2^{\mathcal{H}}$ that is induced by the metric ρ. Continuity in the following refers to these metrics.

Theorem 1. *The diametric completion mapping γ is locally Lipschitz continuous in spaces with Jung constant less than 2.*

Let X satisfy $J(X) < 2$ and let τ satisfying $J(X)/2 < \tau < 1$. Consider two sets $C, C' \in \mathcal{H}$ and let $D_{CC'} = \max\{\operatorname{diam} C, \operatorname{diam} C'\}$ and $d_{CC'} = \min\{\operatorname{diam} C, \operatorname{diam} C'\}$. The idea of the proof is showing that, if C, C' satisfy the condition

$$\rho(C, C') \leq \beta\, d_{CC'}$$

with $\beta = \frac{1-\tau}{10\tau+2}$, then

$$\Delta(\gamma(C), \gamma(C')) \leq \frac{D_{CC'}}{\beta \, d_{CC'}} \; \rho(C, C') \; .$$

Notice that, when $D_{CC'} = 0$, the above constant (which has no meaning since $d_{CC'} = 0$ as well) can be replaced by 1. Therefore, we may assume that $D_{CC'} > 0$ and, using

$$\rho(C, C') \leq \beta \, d_{CC'}$$

together with

$$d_{CC'} \geq D_{CC'} - 2\rho(C, C') \, ,$$

we have

$$d_{CC'} \geq \frac{D_{CC'}}{1+2\beta} > 0$$

which yields

$$\begin{aligned}\Delta(\gamma(C),\gamma(C')) &\leq \frac{D_{CC'}}{\beta\, d_{CC'}}\,\rho(C,C') \\ &\leq \frac{1+2\beta}{\beta}\,\rho(C,C')\end{aligned}$$

an inequality that proves the local Lipschitz continuity of γ, having in mind that

$$\beta = \frac{1-\tau}{10\tau+2}$$

Let us say that a normed space X satisfies property (G) if every bounded set $C \subset X$ can be included in a ball with radius smaller than $\operatorname{diam} C$. This property is a particular case of the notion of relative normal structure [7]. It is natural to ask whether the condition $J(X) < 2$ (which is a uniform version of property (G)) can be replaced in Theorem 1 by property (G).

If $C \subset X$ is a bounded subset of a normed space X, the *wide spherical hull* $\eta(C)$ and the *tight spherical hull* $\theta(C)$ of C are defined as

$$\eta(C) = \bigcap_{x \in C} B(x, \operatorname{diam} C)$$

and

$$\theta(C) = \bigcap_{x \in \eta(C)} B(x, \operatorname{diam} C) \ .$$

These two mappings, which associate with each set an intersection of closed balls containing it, have been studied in connection with different questions in convexity, variational and functional analysis. However, some of their basic properties and possible applications to set covering, approximation and optimization problems have been hardly explored.

Corollary 2. *The wide spherical hull mapping* η *is locally Lipschitz continuous in spaces with Jung constant smaller than* 2*.*

The proof of this result is just a direct application of Theorem 1, together with the following characterization of $\eta(C)$ given in [9]:

$$\eta(C) = \bigcup \{D : D \in \gamma(C)\} \ .$$

In the general case, a similar result is available for convex bodies with nonempty interior (a similar result was proved for β in [11]). Given $C \in \mathcal{H}$, denote by $B(C, r) \subset \mathcal{H}$ the closed ball with center C and radius r. Recall that the inner radius r_C is the supremum of the radii of all balls contained in C.

Proposition 3. *Let $C \in \mathcal{H}$ be a set with inner radius $r_C > 0$. Then η and θ are locally Lipschitz continuous mappings in $B(C, r_C/2)$.*

More precisely, we show that defining

$$\lambda_C := 12(\operatorname{diam} C + r_C)/r_C,$$

then

$$\rho(\eta(D), \eta(D')) \leq \lambda_C \, \rho(D, D')$$

for every $D, D' \in B(C, r_C/2)$ with

$$\rho(D, D') < r_C/6 \, .$$

Analogously, there exists a positive constant μ_C satisfying

$$\rho(\theta(D), \theta(D')) \leq \mu_C \, \rho(D, D')$$

for every $D, D' \in B(C, r_C/2)$ with

$$\rho(D, D') < r_C/(4 + 2\lambda_C) \, .$$

The continuity of the diametric completion mapping γ can also be considered from the point of view of the space of equivalent norms. Indeed, if $\|\cdot\|$ and $|\cdot|$ are two different equivalent norms, they induce different mappings $\gamma_{\|\cdot\|}(\cdot)$ and $\gamma_{|\cdot|}(\cdot)$. It is quite natural to ask whether, for given $C \in \mathcal{H}$, the mapping

$$\|\cdot\| \to \gamma_{\|\cdot\|}(C)$$

is continuous or even whether it has a Lipschitz behavior. Given a normed space $(X, \|\cdot\|)$ and $\varepsilon \geq 0$, we say that $|\cdot|$ is an ε-equivalent norm on X if, for every $x, y \in X$,

$$(1-\varepsilon)\|x-y\| \leq |x-y| \leq (1+\varepsilon)\|x-y\| \ . \tag{1}$$

Since we are dealing with more than one norm, we will specify in the notation which is the norm we are dealing with. For instance, the self-Jung constant $J^s_{|\cdot|}(X)$ relative to $|\cdot|$ is defined as

$$J^s_{|\cdot|}(X) = \sup\{2r_C(C) : C \text{ convex}, \operatorname{diam} C = 1\},$$

where $r_C(C) = \inf\{r(x, C) : x \in C\}$. However, ρ and dist will always refer to the norm $\|\cdot\|$.

Proposition 4. *Let $(X, \|\cdot\|)$ be a finite dimensional normed space and let $|\cdot|$ be an ε-equivalent norm. Then, for every $C \in \mathcal{H}$, there is a constant $M_{\|\cdot\|}(C)$ such that,*

$$\Delta_{\|\cdot\|}(\gamma_{\|\cdot\|}(C), \gamma_{|\cdot|}(C)) \leq \frac{2\varepsilon}{1-\varepsilon} M_{\|\cdot\|}(C)\ .$$

A (possibly not sharp) estimate for $M_{\|\cdot\|}(C)$ is

$$M_{\|\cdot\|}(C) = \frac{\operatorname{diam}_{\|\cdot\|} C}{1-\tau} \left(1 + \frac{\operatorname{diam}_{\|\cdot\|} C - r_{\|\cdot\|}(C)}{r_{\|\cdot\|}(C)} (2-\tau) \right),$$

where

$$r_{\|\cdot\|}(C) = \min\{\operatorname{inr}_{\|\cdot\|}(K) : K \in \gamma_{\|\cdot\|}(C)\}$$

and $\operatorname{inr}_{\|\cdot\|}(K)$ denotes the inner radius of K (the supremum of the radius of balls contained in K). If X is a normed space of infinite dimension and $J^s_{\|\cdot\|}(X) < 1$, the same result as in Proposition 4 holds for any convex set $C \in \mathcal{H}$ with nonempty interior. In this case, $r_{\|\cdot\|}(C)$ can be replaced directly by $\operatorname{inr}_{\|\cdot\|} C$.

Continuous selections of multivalued mappings have various applications in geometry of Banach spaces, convex sets, fixed point theory, approximation theory, and other fields. It is a natural question to ask whether the diametric completion mapping γ admits a continuous selection. We show that this is the case in finite dimensional, strictly convex spaces.

Theorem 5. *In a finite dimensional space X with strictly convex norm, the maximal volume completion ζ is a continuous selection for γ.*

We use a result of Groemer [6]. He showed that among the tight covers of C there is one of maximal volume, that this is a completion of C, and that it is uniquely determined if the norm is strictly convex.

One may ask whether general results on the existence of continuous selections can be applied in the case of the mapping γ. The main difficulty is that most of them are valid only for convex valued mappings. The diametric completion mapping γ is in general not convex, as we point out with the next result.

Proposition 6. *A finite dimensional normed space satisfies property (A) if and only if the set $\gamma(C)$ is convex, for every $C \in \mathcal{H}$.*

A normed space X has property A if every diametrically maximal set in X has constant width (Egglestone [3]). Our last result improves a previous result by Yost [12].

Proposition 7. *In a finite dimensional space X, the set of all norms satisfying property (A) is closed.*

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