HYPERMEASURE THEORY

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Salobreña 2009

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1. Introduction

In the classical theory, measures and integrals corresponded to functionals (or vector valued operators) on a function algebra e.g. C(K) where K is compact; or L^{∞} . Replacing these commutative algebras by non-commutative C*-algebras gave birth to Non-Commutative Measure Theory.

But replacing C*-algebras by more general classes of Banach spaces gives rise to fruitful new insights. Weakly Compact Operators are a unifying theme running through Vector Measure Theory and its generalisations. This is the focus of my talk.

2. Weak compactness and measure theory

Let K *be a compact Hausdorff space and* X *a Banach space. Let*

$$T:C(K) \to X$$

be a bounded linear operator. When does there exist an X-*valued Baire measure* m *on* K *such that*, *for all* f *in* C(K),

$$Tf = \int f dm \ ?$$

Let B(K) be the algebra of bounded Baire measurable functions on K.

When does there exist an operator $T^{\infty}:B(K)\to X$ *such that this operator is an extension of* T *and, whenever* (f_n) *is a bounded, monotone increasing sequence in* B(K) *with pointwise limit f*, $T^{\infty}f_n \to T^{\infty}f$ *in the norm topology of* X ?

When X is one dimensional the answer is ‘always’. This is the classical Riesz Representation Theorem. But when X is an arbitrary Banach space the answer is: *when T is a weakly compact operator*.

3.Weakly compact operators

From now onward, A and X are Banach spaces. Let us recall some familiar facts:

(i) The weak topology for X, is the topology generated by all seminorms of the form $x \to |\varphi(x)|$, where $\varphi \in X^*$.

(ii) A subset $S \subset X$, is *weakly compact* if it is compact in the weak topology of X.

(iii) A linear map $T:A \to X$ is said to be *weakly compact* if it maps the closed unit ball of A into a weakly compact subset of X. Since all weakly compact sets are bounded in norm, it follows that T is a bounded operator.

For any bounded operator $R:A\to X$, the adjoint map $R^*:X^*\to A^*$ is defined by

$\langle R^*\varphi, a\rangle = \langle \varphi, Ra\rangle$ for each a in A and each φ in X^*. On repeating this construction we get $R^{**}:A^{**} \to X^{**}$. Since there is a canonical embedding of A into A^{**} and of X into X^{**}, we can regard R^{**} as an extension of R. One of the key characterisations of weakly compact operators is as follows:

Let $T:A\to X$ *be a bounded linear operator. Then* T *is a weakly compact operator if, and only, if the range of T^{**} is in* X, (*or more precisely, the canonical image of* X *in* X^{**}).

So when $T:A\to X$ is weakly compact, then T^{**} is continuous from A^{**}, equipped with the $\sigma(A^{**},A^*)$-topology, to X, equipped with the $\sigma(X, X^*)$-topology. i.e. T^{**} is weak* to weak continuous.

Now suppose that A = C(K), where K is compact Hausdorff. Then T induces an X-valued measure on the Baire sets of K which is additive with respect to the *norm* topology of X.
There is a "Right topology" for X , such that, a linear map from X into Y is weakly compact precisely when it is a continuous map from X , equipped with the Right topology, into Y , equipped with the norm topology.

4. Continuity from the Right topology to the norm topology

Let X and Y be a Banach spaces.
Let X_1 be the closed unit ball of X.

The Mackey topology for the dual pair (X**,X*) is the topology of uniform convergence on sets $K \subset X^*$, where K is absolutely convex and $\sigma(X^*,X^{**})$ compact. i.e. where K is a weakly compact, absolutely convex subset of the Banach space X* . We denote this topology by $\tau(X^{**},X^*)$; it is the finest locally convex topology for the dual pair (X**,X*). We identify X with its canonical embedding in X** and call the relative topology induced on X by $\tau(X^{**},X^*)$, the "Right topology" for X.

Theorem (see Peralta, Villanueva, Wright, Ylinen, also Ruess)

Let $T:X\to Y$ *be a linear map. Then the following conditions are equivalent.*

1) T is continuous from X, equipped with the Right Topology, into Y, equipped with the norm topology.

2) T is continuous from X_1*, equipped with the relative topology induced by the Right topology, into Y, equipped with the norm topology.*

3) T is weakly compact.

4) T is a bounded linear operator and $T^{**}:X^{**}\to Y^{**}$ *is continuous from* X^{**}*, equipped with the* $\tau(X^{**},X^{*})$ *topology, into* Y^{**} *equipped with the norm topology.*

5. GENERALISED NIKODYM THEOREMS

Let Z be a Banach space.
A sequence in Z, (z_n), is *weakly convergent* if $\lim\varphi(z_n)$ exists for each φ in Z*. The Banach space Z is said to be *weakly complete* if, whenever (z_n), is a weakly convergent sequence then there exists z in Z such that $\lim\varphi(z_n) = \varphi(z)$ for each φ in Z*.

Given a C*-algebra A, we recall that A* is always weakly complete.

THEOREM

Let A be a Banach space where A is weakly complete. Let* (T_n) *be a sequence of weakly compact operators mapping A into a Banach space Y. For each x in A** let* $(T_n{}^{**}x)$ *be a Cauchy sequence.*

Let $S\,a = \lim T_n a$ *for each a in A.*

Then

(i) S is weakly compact,

(ii) $S^{**}x = \lim T_n{}^{**}x$ *for each x in A**,*

(iii) Let (a_j) *be a sequence in A which converges to 0 in the Right topology. Then, as* $j \to \infty$, $\|T_n a_j\| \to 0$ *uniformly in n.*

(iv) Let (x_j) *be a sequence in A** which converges to 0 in the Mackey topology for the pair (A**, A*). Then, as* $j \to \infty$, $\|T_n{}^{**} x_j\| \to 0$ *uniformly in n.*

Key idea of proof: By using the main theorem of “Extending a result of Ryan on weakly compact operators”(Saito and Wright, Proc Edinburgh Math Soc) we find that the map $x \to (T_n x)$ is a weakly compact operator from A into c(X). It follows that when (a_j) is a sequence in A which converges to 0 in the Right topology then $\sup_n \|T_n a_j\|$ converges to 0 as $j \to \infty$.

THEOREM

Let A be a Banach space. Let (T_n) *be a sequence of weakly compact operators mapping A into a Banach space Y. For each x in A*** *let* $||T_n^{**}x|| \to 0$.

(i) *Let* (a_j) *be a sequence in A which converges to 0 in the Right topology. Then, as* $j \to \infty$, $||T_n a_j|| \to 0$ *uniformly in n.*

(ii) *Let* (x_j) *be a sequence in A*** *which converges to 0 in the Mackey topology for the pair (A**, A*). Then, as* $j \to \infty$, $||T_n^{**}x_j|| \to 0$ *uniformly in n.*

6. Pseudo weakly compact operators

When T is only sequentially continuous with respect to the Right topology, it is said to be *pseudo weakly compact*. When a Banach space X has the property that every pseudo weakly compact operator from X to another Banach space is weakly compact, then X is said to be *sequentially Right*. It turns out that every Banach space possessing Pelczynski's Property (V) must be sequentially Right.

By the Eberlein-Smulian Theorem weak compactness is, in some sense, a sequential property.

We know that $T:X\to Y$ is weakly compact if and only if it is continuous from X, equipped with the Right topology, into Y, equipped with the norm topology.

Clearly such an operator T is sequentially continuous from X, equipped with the Right topology, into Y, equipped with the norm topology. It is natural to ask if the converse is true.

Definition Let X and Y be Banach spaces. Let $T:X\to Y$ be a linear map such that, when $x_n\to 0$ in the Right topology then $||Tx_n||\to 0$. Then we call T *pseudo weakly compact*.

Example Let T be the identity map from L^1 onto L^1. Since L^1 is not reflexive, its unit ball is not weakly compact, see Theorem V.4.7 (D&S).

So T is not a weakly compact operator.

On the other hand, when $x_n \to 0$, in the Right Topology then $x_n \to 0$, in the $\tau((L^1)^{**},(L^1)^{*})$ -topology. So $x_n \to 0$, in the $\sigma((L^1)^{**},(L^1)^{*})$ -topology. Hence $x_n \to 0$ in the weak topology of L^1. But, by IV.8.14 (D&S), this implies that $x_n \to 0$ in the norm topology, so $\|Tx_n\| \to 0$. Thus T is pseudo weakly compact.

When X is a C*-algebra, then its second dual, X**, can be identified with the von Neumann envelope of X, when X is represented on its universal representation (Hilbert) space.

When the σ-strong* operator topology of X** is restricted to the unit ball of X, it coincides with the restriction of the Right topology to X_1. In an earlier note in JMAA, ("Multilinear maps on products of operator algebras", *JMAA* **292** (2004), 558-570), Ylinen and I introduced the notion of *quasi completely continuous* linear operators from a C*-algebra into a Banach space. It turns out that an operator from a C*-algebra into a Banach space is quasi completely continuous if, and only if, it is pseudo weakly compact.

For a linear operator T from a C*-algebra into a Banach space, the following are equivalent:

- T is weakly compact;
- T is quasi completely continuous;
- T is pseudo weakly compact.

It now makes sense to introduce the following definition:

Definition *A Banach space X is said to be sequentially Right if every pseudo weakly compact operator with domain X is weakly compact; in other words, if each operator on X which is sequentially continuous with respect to the Right topology is also continuous with respect to the Right topology.*

Proposition *Every closed complemented subspace of a sequentially Right Banach space is, itself, sequentially Right.*

Corollary *Every closed complemented subspace of a C^*-algebra is sequentially Right.*

Lemma *Let T be a linear map between two Banach spaces X and Y. Then T is Right-Right continuous if, and only if, it is bounded.*

Let X be a Banach space. A series $\sum x_n$ in X is called *weakly unconditionally Cauchy (w.u.C.)* if, for each ϕ in X*,

$$\sum|\phi(x_n)| \text{ is convergent.}$$

Lemma *Let X be a Banach space and $\sum x_n$ a w.u.C. series in X. Then (x_n) is a Right-null sequence in X.*

Let X and Y be Banach spaces and T a linear mapping from X into Y. We say that T is *unconditionally converging* if, for every w.u.C. series $\sum x_n$ in X, the series $\sum T(x_n)$ is unconditionally convergent.

Proposition *Every pseudo weakly compact operator between two Banach spaces is unconditionally converging.*

Let us recall that a Banach space X is said to have Pelczynski's *Property (V)* if, for every Banach space Y, every unconditionally converging operator from X to Y is weakly compact. We clearly have:

Corollary *Every Banach space satisfying property (V) is sequentially Right.*

Since every JB*-triple satisfies property (V) we obtain:
Every JB-triple is sequentially Right.*