COMPACTA IN BANACH SPACES

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Outline

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Gul'ko compacta

$$[0,1]^\Gamma \cap \Sigma(\Gamma) := \{f : \Gamma \to [0,1];\ \#\{\gamma \in \Gamma;\ f(\gamma) \neq 0\} \leq \aleph_0\}.$$

Theorem (Sokolov)

Let $K \subset [0,1]^\Gamma \cap \Sigma(\Gamma)$ be a compact space. K is a Gul'ko compactum if and only if there exists $\Gamma_1, \Gamma_2, \ldots \subset \Gamma$ such that, $\forall \gamma \in \Gamma$, $\forall k \in K$ and $\forall \epsilon > 0$, $\exists m \in \mathbb{N}$ such that $\gamma \in \Gamma_m$ and $\#\{\gamma \in \Gamma_m;\ |k(\gamma)| \geq \epsilon\} < \infty$.

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We take $\Gamma \subset \mathbb{R}$ uncountable, and $K_{\mathcal{A}} := \{\chi_A;\ A \in \mathcal{A}\} \subset \{0,1\}^\Gamma$, where $\mathcal{A}$ is adequate.

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Definition

A family $\mathcal{A}$ of subsets of Γ is called adequate if:

- $\forall \gamma \in \Gamma$, $\{\gamma\} \in \mathcal{A}$,
- given $A \in \mathcal{A}$ and $B \subset A$, then $B \in \mathcal{A}$,
- given $A \subset \Gamma$ such that $\forall F \subset A$ finite, $F \in \mathcal{A}$ then $A \in \mathcal{A}$.

Proposition

Let $\Gamma \subset \mathbb{R}$ uncountable and $\mathcal{A}$ an adequate family. TFAE:

(i) *Every $A \in \mathcal{A}$ is closed in Γ.*

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(iii) *The map* $\phi : \Gamma \to \Gamma_{\mathcal{A}}$ *defined by* $\phi(\gamma) = \{\gamma, *\}$ *is usco. Where* $\Gamma_{\mathcal{A}} := \Gamma \cup \{*\}$ *being* $* \notin \Gamma$ *an extra element,* $\tau_{\mathcal{A}}$ *is the topology on* $\Gamma_{\mathcal{A}}$ *given by: every* $\gamma \in \Gamma$ *is isolated, and* $S(*) := \{\{*\} \cup \Gamma \backslash A;\ A \in \mathcal{A}\}$ *is a subbasis of* $*$*.*

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(iv) *If* $\mathcal{B}$ *is a basis of* Γ *then* $\forall A \in \mathcal{A}$ *and* $\forall \gamma \in \Gamma$, $\exists B \in \mathcal{B}$ *such that* $\gamma \in B$ *and* $\#A \cap B < \aleph_0$.

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If some condition above holds, then $K_{\mathcal{A}}$ is Gul'ko compactum.

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Paired means that there exists a separately continuous mapping $f : K \to \mathcal{K}$ that separates points of K and $\mathcal{K}$.

Outline

Some tools

Definition

A PRI $(P_\alpha)_{\omega_0\leq\alpha\leq\mu}$ is subordinated to a set $\Gamma\subset X$ if $P_\alpha\gamma\in\{0,\gamma\}$ $\forall\gamma\in\Gamma$ and $\forall\alpha\in[\omega_0,\mu]$.

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Theorem

Let X a B. s. with a full-PG, $\Gamma \subset X$ countably supporting X^* and $\Delta \subset X^*$ countably supporting X. Then $\exists$ a SPRI subordinated to Γ and Δ.

Where, Γ countably supports X^* means that for all $x^* \in X^*$, $\#\{\gamma \in \Gamma;\ \langle \gamma, x^* \rangle\} \leq \aleph_0$.

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A set $\Gamma \subset X$ has the Amir-Lindenstrauss property if $\forall x^* \in X^*$ and $\forall c > 0$, the set $\{\gamma \in \Gamma; |\langle \gamma, x^* \rangle| > c\}$ is finite.

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- *In a fundamental b.o.s. $\{x_i, f_i\}_{i \in I}$, $\{f_i\}_{i \in I}$ countably supports X.*
- *(Amir and Lindenstrauss) In a WCG B. s. X, $\exists \Gamma \subset X$ linearly dense with the AL-property.*

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Remark

- *In a fundamental b.o.s.* $\{x_i, f_i\}_{i \in I}$, $\{f_i\}_{i \in I}$ *countably supports* X.
- *(Amir and Lindenstrauss) In a WCG B. s.* X, $\exists \Gamma \subset X$ *linearly dense with the AL-property.*
- *If* X *is WCG,* $\Gamma \subset X$ *has the AL-property and* $\{x_\lambda, f_\lambda\}_{\lambda \in \Lambda}$ *is a fundamental b.o.s, then* $\exists$ *a SPRI subordinated to* Γ *and* $\{f_\lambda\}_{\lambda \in \Lambda}$.

Biorthogonal systems in WCG Banach spaces

Theorem

Let X be a B.s., $K \subset X$ w-compact and $\{x_\lambda, f_\lambda\}_{\lambda\in\Lambda}$ be a fundamental b.o.s. Let $\Lambda^0 := \{\lambda \in \Lambda;\ \langle k, f_\lambda\rangle \neq 0,\ \textit{for some } k \in K\}$. Then $\exists(\Lambda^0_m)_{m\in\mathbb{N}}$ such that $\Lambda^0 = \cup_{m=1}^\infty \Lambda^0_m$ and $\|\sum_{i=1}^n f_{\lambda_i}\| \to +\infty$ for every $(\lambda_i)_{i=1}^\infty \in \Lambda^0_m$ and $\forall m \in \mathbb{N}$.

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Corollaries

- (Argyros and Mercourakis) For X a WCG B. s. and $\{x_\lambda, f_\lambda\}_{\lambda\in\Lambda}$ an M-basis.

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- (Argyros and Mercourakis) For X a WCG B. s. and $\{x_\lambda, f_\lambda\}_{\lambda\in\Lambda}$ an M-basis.
- (Argyros and Farmaki) Let X be a B.s., $K \subset X$ w-compact and $\{x_\lambda, f_\lambda\}_{\lambda\in\Lambda}$ an unconditional basis. Then $\Lambda^0 = \cup_{m=1}^{\infty}\Lambda^0_m$ such that $\{x_\lambda;\ \lambda \in \Lambda^0_m\} \cup \{0\}$ is w-compact $\forall m \in \mathbb{N}$.

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- (Jonhson) X a WCG B.s. with an unconditional basis $\{x_\lambda\}_{\lambda \in \Lambda}$. Then $\exists \Lambda = \cup_{m=1}^{\infty} \Lambda_m$ such that $\{x_\lambda;\ \lambda \in \Lambda_m\} \cup \{0\}$ is w-compact $\forall m \in \mathbb{N}$.

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- Argyros' example of a WCG space $C(K)$ with a subspace not WCG.

Theorem (Pták)

Let X be a Banach space. TFAE:

(i) *X is reflexive.*

(ii) $\forall$ *b.o.s* $\{x_n, f_n\}_{n\in\mathbb{N}}$ *with* $\{f_n\}_{n\in\mathbb{N}}$ *bounded,* $\|\sum_{n=1}^{k} x_n\| \to +\infty$.

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Proposition

Let $\{x_\lambda; f_\lambda\}_{\lambda\in\Lambda}$ be a total biorthogonal system. TFAE:

(i) *$\{x_\lambda\}_{\lambda\in\Lambda}$ has the AL-property.*

(ii) *$\{x_\lambda\}_{\lambda\in\Lambda} \cup \{0\}$ is weakly compact.*

Proposition

Let X be a B.s. and $\{x_\lambda; f_\lambda\}_{\lambda\in\Lambda}$ a b.o.s. If $\{x_\lambda\}_{\lambda\in\Lambda}$ has the AL-property, then $\|\sum_{i=1}^{k} f_{\lambda_i}\| \to +\infty$.

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WCD Banach spaces

Let S be a family of all subsets of $\mathbb{N}$, and $L_S := \overline{X \cap_{n \in s} K_n}^{w^*}$, for $s \in S$. Let $\Phi : X^* \to 2^X$ defined by

$$\sup_{x \in L_s} |\langle x, x^* \rangle| = \sup_{x \in L_s \cap \Phi(x^*)} |\langle x, x^* \rangle|, \qquad \forall s \in S.$$

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The mapping Φ is a full-PG on a WCD Banach space.

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Theorem

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Corollary

Every WLD Banach space is DENS, i. e., $\operatorname{dens} X = w^* - \operatorname{dens} X^*$.

Outline

A strictly convex norm on $c_0(\Gamma)$

Let $\Gamma \subset \mathbb{R}$ uncountable and $(\Gamma_n)_{n\in\mathbb{N}}$ a countable basis of Γ. We introduce the following norm on $c_0(\Gamma)$,

$$|||x||| := \left(\sum_{n=1}^{\infty} 2^{-n} \|x \restriction_{\Gamma_n}\|_{\infty}^{2}\right)^{\frac{1}{2}}.$$

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Let $x, y \in c_0(\Gamma)$ with $|||x||| = |||y||| = 1$ and $x \neq y$. It is possible to find $k \in \mathbb{N}$, s. t. $\|x \restriction_{\Gamma_k} \|_\infty \neq \|y \restriction_{\Gamma_k} \|_\infty$ or $\|(x + y) \restriction_{\Gamma_k} \|_\infty < \|x \restriction_{\Gamma_k} \|_\infty + \|y \restriction_{\Gamma_k} \|_\infty$.

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Outline

Some remarks on Krein's theorem

We consider a result on quantification of Krein's theorem:

Theorem (Fabian, Hájek, Montesinos and Zizler)

Let X a B.s. and $M \subset X$ bounded. If $\overline{M}^{w^*} \subset X + \epsilon B_{X^{**}}$ (M is ϵ-WK), for some $\epsilon > 0$. Then $\overline{conv(M)}^{w^*} \subset X + 2\epsilon B_{X^{**}}$.

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We make a variation on the concept of ϵ-WK.

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Let X a B.s. and $M \subset X$ bounded, we say that M is ϵ-weakly self compact (ϵ-WSK) if for some $\epsilon > 0$,

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We deal with a version of Krein's Theorem considering M ϵ-WSK.

Upper envelopes

Let $x^{**} \in X^{**}$, we introduce the concept of the $w(X^*, M)$-usc envelope of x^{**} as follows

$$\hat{x}^{**}_M := \inf\{f;\ f : B_{X^*} \to \mathbb{R}, f \text{ is } w(X^*, M)\text{−continous and } f \geq x^{**}|_{B_{X^*}}\}.$$

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We prove that

$$\mathrm{hgraph}(\hat{x}^{**}_M) = \overline{\mathrm{hgraph}(x^{**})}^{w(X^*,M)\times R}.$$

Proposition

Let $x^{**} \in X^{**}$ and $M \subset X$ bounded, then

1. $\hat{x}^{**}_M(x^*) = \inf\{\langle x, x^*\rangle + \lambda;\ x \in M, \lambda \in \mathbb{R} \textit{ with } x + \lambda \geq x^{**} \textit{ on } B_{X^*}\}.$
2. $\hat{x}^{**}_M(x^*) = \lim_{N \in \mathcal{N}_M(x^*)} \sup\langle x^{**}, N\rangle,\ \forall x^* \in B_{X^*}.$
3. $\hat{x}^{**}_M(x^*) = \inf\{\langle x, x^*\rangle + \|x^{**} - x\|;\ x \in M\},\ \forall x^* \in B_{X^*}.$

Quantitative Krein's Theorem

Theorem

Let X a B.s. and $M \subset X$ bounded. If M is ϵ-WSK, for some $\epsilon \geq 0$, then $\overline{\mathrm{conv}\,(M)}^{w^*} \subset \mathrm{conv}\,(M) + (2\epsilon + \delta)B_{X^{**}}$.

Outline

Flat sets and ℓ_p-generating

The original idea comes from:

Proposition (Godefroy, Kalton and Lancien)

A separable B. s. X is isomorphic to a subspace of c_0 if and only if it has an equivalent C-LKK* norm for some $C \in (0,1]$.

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Where,

Definition

$\|\cdot\|$ on a B. s. X is C-LKK* for some $C \in (0,1]$ if

$$\limsup_{n\to\infty} \|x^* + x_n^*\| \geq \|x^*\| + C \limsup \|x_n^*\|,$$

for every $x^* \in X^*$ and every w^*-null sequence $(x_n^*)_{n\in\mathbb{N}}$ in X^*.

Asymptotically p-flat set

We deal with $\ell_p(\omega_1)$-generation by introducing the following concept:

Definition

Let $(X, \|\cdot\|)$ be a B.s., $p \in (1, +\infty]$ and $q = \frac{p}{p-1}$. $M \subset X$ is $\|\cdot\|$-asymptotically p-flat if M is bounded and $\exists C > 0$ such that $\forall f \in X^*$ and every w^*-null sequence $(f_n)_{n\in\mathbb{N}} \subset X^*$, it holds

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$$\|f\|_M = \sup\{|\langle x, f\rangle|;\ x \in M\}.$$

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Observe that if $\|\cdot\|$ is C-LKK*, then B_X is $\|\cdot\|$-asymptotically ∞-flat.

Examples

- Every limited set $M \subset X$ (i.e., $\lim_{n\to\infty} \|f_n\|_M = 0$ $\forall (f_n) \subset X^*$ w^*-null sequence) is asymptotically p-flat $\forall p \in (1, +\infty]$.

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- If $(X, \|\cdot\|)$ has modulus of smoothness of power type p, then B_X is $\|\cdot\|$-asymptotically p-flat.
- Let $L_p(\Omega, \Sigma, \mu)$ with positive measure μ. B_{L_p} is $\|\cdot\|_p$-asymptotically p-flat for $p \in (1, 2)$, and $\|\cdot\|_p$-asymptotically 2-flat for $p \in [2, +\infty)$.

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- If X is superreflexive, B_X is asymptotically p-flat, for some $p \leq 2$.

The Asplund setting

Theorem

Let X be an Asplund B.s. with $\operatorname{dens}(X) = \#\omega_1$ and let $p \in (1, +\infty)$. TFAE:

(i) *X is WCG and $\exists M \subset X$ linearly dense and asymptotically p-flat set, (asymptotically ∞-flat).*

(ii) *X is generated by $\ell_p(\omega_1)$, (by $c_0(\omega_1)$).*

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As a consequece, we have

Corollary

For $p \in (1, +\infty)$, any subspace of $\ell_p(\omega_1)$ is $\ell_p(\omega_1)$-generated. Every subspace of $c_0(\omega_1)$ is $c_0(\omega_1)$-generated.

Innerly asymptotically p-flat sets

In order to deal with the non Asplund setting is necessary to introduce a more restrictive concept:

Definition

Let X be a B.s., $p \in (1, +\infty]$ and $q = \frac{p}{p-1}$. $M \subset X$ is innerly asymptotically p-flat if M is bounded and $\exists C > 0$ such that $\forall f \in X^*$ and every w^*-null sequence $(f_n)_{n\in\mathbb{N}} \subset X^*$, it holds

$$\limsup_{n\to\infty} \|f + f_n\|_M^q \geq \|f\|_M^q + C \limsup_{n\to\infty} \|f_n\|_M^q.$$

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An innerly asymptotically p-flat set has a certain Asplund behavior.

Proposition

Let X be a B.s. with (B_{X^*}, w^*) angelic. Then, for all $p \in (1, +\infty]$, every asymptotically p-flat set $M \subset X$ is an Asplund set.

The general setting

Theorem

Let X be a B.s. with $\mathrm{dens}(X) = \#\omega_1$ and $p \in (1, +\infty)$. TFAE:

(i) *X is WLD and $\exists M \subset X$ bounded, linearly dense and innerly asymptotically p-flat (∞-flat).*

(ii) *X is generated by $\ell_p(\omega_1)$, (by $c_0(\omega_1)$).*

Fixing $c_0(\Gamma)$ by operators

We provide alternative proofs for the following results:

Theorem (Dunford, Pettis and Pelczýnski)

Let X be a B.s., $T : c_0(\mathbb{N}) \to X$ bounded, linear and non-weakly compact. Then T fixes a copy of $c_0(\mathbb{N})$.

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Theorem (Rosenthal'70)

Let X be a B.s. and let $T : c_0(\Gamma) \to X$ bounded and linear such that for some $\epsilon > 0$, $\|T(e_\gamma)\| > \epsilon$, $\forall \gamma \in \Gamma$. Then $\exists \Gamma' \subset \Gamma$ such that $\#\Gamma' = \#\Gamma$ and $T|_{c_0(\Gamma')}$ is an isomorphism.

Papers accepted

- A. L. González and V. Montesinos, **A note on weakly Lindelöf determined Banach spaces**. Czechoslovak Mathematical Journal.
- M. Fabian, A. L. González and V. Zizler, **Flat sets, l_p-generating and fixing $c_0(\Gamma)$ in a nonseparable setting**. Journal of the Australian Mathematical Society.
- M. Fabian, A. L. González, V. Montesinos, **A note on biorthogonal systems in weakly compactly generated Banach spaces**. Annales Academiae Scientiarum Fennicae Mathematica.