# From topological vector spaces to topological abelian groups

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# 1 Some duality notions and properties

Let $G_1$, $G_2$ be abelian groups. The set of all group homomorphisms from $G_1$ into $G_2$, $Hom(G_1, G_2)$, with pointwise addition is a group.

The set of all continuous homomorphisms $CHom(G_1, G_2)$ is clearly a subgroup of $Hom(G_1, G_2)$.

The symbols $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ will have the usual meaning.

We identify $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with the multiplicative group of complex numbers with modulus one, endowed with the metric induced by that of $\mathbb{C}$.

For a group $G$, any group homomorphism $\varphi : G \to \mathbb{T}$ is called a *character*.

$Hom(G, \mathbb{T})$ is the algebraic dual of $G$.

The set of all continuous characters $G^\wedge := CHom(G, \mathbb{T})$ will be called the *dual group of G*.

If $G^\wedge$ separates the points of $G$, we will say that $G$ is a *DS-group*.

**(Peter Weyl, Theorem, 1927) Compact groups are DS-groups**

LCA groups are DS-groups

*Real continuous characters* have also been studied. In the class of LCA groups, groups with enough real characters are well known. They were first characterized by Mackey in 1948 as the groups of the form $\mathbb{R}^n \oplus D$

A topological abelian group with enough real characters is a topological group which can be continuously embedded into a locally convex space, and this fact lets these groups enjoy some of the most important properties of real locally convex vector spaces.

One can easily generalize the classical Schauder-Tychonoff fixed point theorem in the following sense:

*If $G$ is a topological abelian group with enough real characters and $K$ is a compact subset of $G$ with $K + K = 2K$, then any continuous map $f : K \to K$ has a fixed point.*

In the dual group $G^\wedge$, we denote by $\sigma(G^\wedge, G)$ the topology of pointwise convergence and by $\tau_{co}$ that of uniform convergence on the compact subsets of $G$.

For short, $G^\wedge_\sigma := (G^\wedge, \sigma(G^\wedge, G))$ and $G^\wedge_c := (G^\wedge, \tau_{co})$.

**If $G$ is compact, $G^\wedge_c$ is discrete. $\mathbb{T}^\wedge_c$ is $\mathbb{Z}$.**

**If $G$ is discrete, $G^\wedge_c$ is compact. $\mathbb{Z}^\wedge_c$ is $\mathbb{T}$.**

**If $G$ is LCA (locally compact abelian), $G^\wedge_c$ is LCA.** $(\mathbb{R}^n)^\wedge_c$ **is** $\mathbb{R}^n$

*The Bohr topology* of $G$, denoted by $\sigma(G, G^\wedge)$, is the weakest topology in $G$ with respect to which all the elements of $G^\wedge$ are continuous. Clearly $\sigma(G, G^\wedge)$ is Hausdorff if and only if $G$ is a DS-group.

This topology coincides with the topology that $G$ inherits from the compact group $(G^\wedge, d)^\wedge_c$ (the Bohr compactification of $G$).

The topological group $(G, \sigma(G, G^\wedge))$ is precompact.

Notice that, if $x \in G$, the mapping $\hat{x} : G^\wedge \to T$ defined by $\varphi \mapsto \varphi(x)$ is a character which is continuous on $G^\wedge_\sigma$ and a fortiori on $G^\wedge_c$.

Obviously the set $\{\hat{x} : x \in G\}$ separates points in $G^\wedge$, therefore $G^\wedge_c$ and $G^\wedge_\sigma$ are DS-groups.

The mapping $\alpha_G : G \to (G_c^\wedge)^\wedge$ defined by $x \mapsto \hat{x}$ is a canonical group homomorphism.

**(Pontryagin van Kampen, theorem) If $G$ is LCA group, then $\alpha_G$ is a topological isomorphism.**

With Pontryagin duality theorem in mind, we claim that the topology $\tau_{co}$ is the most natural for $G^\wedge$. the group $(G_c^\wedge)_c^\wedge$ is called the *bidual group of* $G$.

A topological abelian group $G$ is said to be *Pontryagin reflexive*, if $\alpha_G$ is a topological isomorphism between the groups $G$ and $(G_c^\wedge)_c^\wedge$.

We establish now some notations and results for topological real vector spaces, as we did above for abelian groups.

Let $E$ and $F$ be real vector spaces.

Denote by $Lin(E, F)$ the vector space of all linear operators from $E$ into $F$

$Lin(E, \mathbb{R})$ is the algebraic dual of $E$.

If $E$ and $F$ are topological vector spaces, then $CLin(E, F)$ denotes the vector space of all continuous linear operators, and $E^* = CLin(E, \mathbb{R})$ is called the *dual of* $E$.

If $E^*$ separates the points of $E$ we say that $E$ is a *DS-space.*

The weak topologies $\sigma(E, E^*)$ and $\sigma(E^*, E)$ of $E$ and of $E^*$ respectively, are defined as the corresponding pointwise convergence topologies. In $E^*$ we can also consider $\tau_{co}$, the topology of uniform convergence on the compact subsets of $E$.

The spaces $(E^*, \sigma(E^*, E))$ and $(E^*, \tau_{co})$ will be denoted by $E^*_\sigma$ and $E^*_c$ respectively. It is clear that $E^*_\sigma$, and hence $E^*_c$, are DS-spaces.

A topological vector space $E$ is also an additive topological group, and therefore it is possible to consider the group $Hom(E, \mathbb{T})$.

The mapping $p : Lin(E, \mathbb{R}) \to Hom(E, \mathbb{T})$, $p(l) = exp(2\pi i l)$, for all $l \in Lin(E, \mathbb{R})$, is an injective group homomorphism and $p(E^*) = E^\wedge$.

Hewwitt-Ross (23.32 a).

**(Banaszczyck book, (2.3))** $p : E^*_c \to E^\wedge_c$ **is a topological group isomorphism.**

**(Remus- Trigos,1993) The space** $(E, \sigma(E, E^*))$ **and the group** $(E, \sigma(E, E^\wedge))$ **have the same family of compact subsets.**

If $A \subset E$, and $B \subset E^*$ are nonempty sets, we denote by

$$A^\circ = \{x^* \in E^* : |x^*(x)| \leq 1, \forall x \in A\}$$

and by

$$^\circ B = \{x \in E : |x^*(x)| \leq 1, \forall x^* \in B\}.$$

They are closed convex symmetric subsets of $E^*_\sigma$ and of $(E, \sigma(E, E^*))$

According to the bipolar theorem, the converse assertions also hold, i.e. if a subset $A \subset E$ is convex symmetric and $\sigma(E, E^*)$-closed, then $A =^\circ B$ for some subset $B \subset E^*$, and the analogue for subsets of $E^*$.

Let $G$ be a topological abelian group, and let $A \subset G$, $B \subset G^\wedge$ be nonempty subsets. Denote by

$$A^{\triangleright} = \{\varphi \in G^{\wedge} : Re(\varphi(x)) \geq 0, \forall x \in A\}$$

and by

$$B^{\triangleleft} = \{x \in G : Re(\varphi(x)) \geq 0, \forall \varphi \in B\},$$

Clearly, $A^{\triangleright}$ ( $B^{\triangleleft}$) is a closed subset of $X_{\sigma}^{\wedge}$ ( of $X_{\sigma} = (X, \sigma(X, X^{\wedge})))$.

$(A^{\triangleright})^{\triangleleft}$ is called the *quasi-convex hull* of $A$. The set $A$ is said *quasi-convex* when $A = (A^{\triangleright})^{\triangleleft}$.

If $A = \{x\}$, $(A^{\triangleright})^{\triangleleft} = \{x, e_G, x^{-1}\}$

**Aussenhofer If $G$ is a DS group and $A$ is finite, $(A^{\triangleright})^{\triangleleft}$ is finite.**

A topological Abelian group $G$ is *locally quasi–convex* if it has a basis of neighborhoods of zero formed by quasi–convex sets.

Dual groups are locally quasi-convex.

Locally quasi-convex groups were introduce by Vilenkin, 1951.

**A topological vector space is locally quasi–convex as a group if and only if it is a locally convex space.**

Let $E$ be a topological vector space, and let $p : E^* \longrightarrow E^\wedge$ be the canonical group isomorphism.

For any nonempty balance $A \subset E$, $(A^\triangleright)^\triangleleft = {}^\circ(A^\circ)$

## 2 Compact and equicontinuous sets of $G^\wedge$

**( Noble, 1970) Let $G$ be a topological abelian group and let $U \subset G$ be a neighborhood of the neutral element. Then**

**a) $U^\triangleright$ is an equicontinuous subset of $G^\wedge$.**

**b) $U^\triangleright$ is a compact subset of $G_c^\wedge$.**

**c) The mapping $\alpha_G : G \to (G_c^\wedge)_c^\wedge$ is continuous if and only if any compact subset of $G^\wedge$ is equicontinuous.**

**(Ch., Martin-Peinador, Tarieladze, 1999)Let $G$ be a topological abelian group and let $B \subset G^\wedge$ be a nonempty subset. Then, the following assertions are equivalent:**

**a)** $B$ **is equicontinuous.**

**b) There is a neighborhood** $U$ **of** $e_G$ **such that** $B \subset U^{\triangleright}$**.**

**c)** $B^{\triangleleft}$ **is a neighborhood of the neutral element of** $G$**.**

**(Pettis 1950) Any pointwise convergence sequence of continuous homomorphims from a Baire topological group is equicontinuous.**

**(Equicontinuous principle for groups, Troallic 1996) Let** $G$ **be a Čech-complete group,** $Y$ **a metrizable topological group, and let** $B \subset CHom(X, Y)$ **be any subset which is compact in the topology of pointwise convergence. Then** $B$ **is equicontinuous.**

*The analogous facts, and even stronger, are well known in the context of* *topological vector spaces (Banach-Steinhaus theorem). Namely, if $X$ and $Y$ are topological vector spaces, $X$ is a Baire space and $B \subset CHom(X, Y)$ is bounded in the topology of pointwise convergence, then $B$ is equicontinuous.*

**Corollary Let $G$ be a Čech-complete topological abelian group, and let $B \subset G^\wedge_\sigma$ be a compact subset. Then, $B$ is equicontinuous and consequently it is compact in $G^\wedge_c$.**

**(Glicksberg theorem,1962) Let $G$ be LCA group. Then, any Bohr-compact subset of $G$ is compact in the original topology of $G$.**

Observe that $G^\wedge_c$ is locally compact, therefore Čech-complete, and $G \cong (G^\wedge_c)^\wedge_c$

# 3 The Banach Dieudonné theorem

Let $E$ be a locally convex space and let us denote by $\tau^f$ the topology on $Lin(E, \mathbb{R})$ finest of all those which coincide with $\sigma(E^*, E)$ on every equicontinuous subset of $Lin(E, \mathbb{R})$.

The topology $\tau^f$ was first introduced by Collins, who gave it the name of *equicontinuous weak* topology* or ew* topology. He proved that in general it fails to be a locally convex topology, even if the starting space $E$ is locally convex. Komura (1964) gave an example where the ew* topology fails to be a vector topology. Finally, Valdivia (1974) produced a device to obtain nonregular ew*topologies, [21].

**(Banach Dieudonné Theorem) If $E$ is a metrizable locally convex space $\tau^f$ coincides with the topology of uniform convergence on the precompact subsets of $E$.**

**( Bruguera, Martin-Peinador, Ch. 1999) Let $G$ be an abelian topological group such that the natural mapping $\alpha_G$ is continuous. The following are equivalent:**

**(a) $G^\wedge$ is a k-space with respect to the compact open topology.**

**(b) The compact open topology on $G^\wedge$ is the finest of all those topologies which induce $\sigma(G^\wedge, G)$ on the equicontinuous subsets of $G^\wedge$.**

Proof. Observe that the family of $\tau_{co}$ compact subsets of $G^\wedge$ coincides with that of equicontinuous $\tau_{co}$ closed subsets, and those form a fundamental family of equicontinuous subsets. Thus, statement (b) means that the compact open topology in $G^\wedge$ is the finest of all those which induce $\sigma(G^\wedge, G)$ in the $\tau_{co}$ compact subsets of $G^\wedge$. So, the equivalence between (a) and (b) is proved. □

A natural question now is to find those abelian topological groups $G$ whose dual $G^\wedge$ is a k-space.

*(Bourbaki EVT IV) If $E$ is a metrizable locally convex vector space, $E_c^*$ is a k-space.*

The proof only uses the group structure and the metrizability of the space $E$.

**(Ch. 1998). If G is an abelian metrizable group, $G^\wedge$ is a k-space.**

**The same assertion holds for the dual of an almost metrizable group.**

**Aussenhofer (1999).**

An abelian topological group $G$ is *almost metrizable* if and only if it contains a compact subgroup $K$ such that $G/K$ is metrizable.

**A topological group is Čech complete if and only if it is almost metrizable and complete. LCA groups are Čech complete.**

let us suppose that $H$ is a dense subgroup of $G$. A continuous character of $H$ can be uniquely extended to a continuous character of $G$. Hence, if $i : H \to G$ denotes the embedding, the dual mapping $i^\wedge : G_c^\wedge \to H_c^\wedge$, $\aleph \to \aleph|_H$ is an isomorphism. Trivially, it is continuous.

**(Ch. Aussenhofer, 1998-1999)Let $G$ be a metrizable topological abelian group. Assume that $H$ is a dense subgroup of $G$. Then,**

$i^\wedge : G_c^\wedge \to H_c^\wedge$ **is a topological isomorphism.**

Proof. Since $H_c^\wedge$ and $G_c^\wedge$ are k-spaces, we prove that both dual groups $H_c^\wedge$ and $G_c^\wedge$ have the same compact sets.

Take a compact subset $K$ of $H_c^\wedge$. Evidently $K$ is closed in $G_c^\wedge$. Since $H$ is metrizable, K is equicontinuous. So, there exists some neighborhood $U$ of $o \in G$ such that $K \subset (U \cap H)^\circ$.

Let $W$ be a neighborhood of $o \in G$ such that $W + W \subset U$. Notice that $W \subset \overline{U \cap H}$. (If $x \in W$, we can take a net $(x_\alpha)_{\alpha \in \mathcal{A}} \subset H$ such that $x_\alpha \in x + W \subset U$, for all $\alpha \geq \alpha_o$; hence, $x \in \overline{U \cap H}$.) Observe that $(U \cap H)^\circ = (\overline{U \cap H})^\circ \subset W^\circ$; which in turn implies that $K \subset W^\circ$ and therefore $K$ is compact in $G^\wedge$. This shows that $H^\wedge$ and $G^\wedge$ have the same compact sets.

**If G is an almost metrizable abelian topological group, then**

**a) The finest of all those topologies which induce $\sigma(G^\wedge, G)$ on the equicontinuous subsets of $G^\wedge$ coincides with the compact open topology in $G^\wedge$ and therefore it is a group topology.**

**b) $G_c^\wedge$ satisfies Glicksberg theorem.**

**c) $G_c^\wedge$ is a k-space.**

Applying the above results we obtain the following statement in the framework of topological vector spaces.

*The bidual group of a locally bounded space whose dual separates points is its Banach envelope.*

Let $E$ be a locally bounded topological vector space whose dual space $E^*$

separates points of $E$. Fix a balanced, bounded neighborhood $B$ of $o$ in $E$ and denote by $\| \quad \|$ the gauge of $B$. Let us recall that the *Banach envelope* of $E$ is the completion of the normed space $(E, \| \quad \|_c)$, where $\| \quad \|_c$ is the gauge of $co(B)$ ( the convex hull of $B$). The Banach envelope of $E$ will be denoted by $B(E)$. The functional $\| \quad \|_c$ is a norm on $X$, the identity map $I_d : (X, \| \quad \|) \to (X, \| \quad \|_c)$ is continuous and $X$ is a dense subspace of $B(X)$ (Kalton book p.27).

$(E, \| \quad \|)$ and $(E, \| \quad \|_c)$ have the same dual vector space, hence their dual groups are algebraically isomorphic. If $K$ is a compact set in $(E, \| \quad \|)^\wedge_c$, there exists a natural number $n$ such that $K \subset (\frac{1}{n}B)^\circ = (\frac{1}{n}co(B))^\circ$. Therefore $K$ is also compact in $(E, \| \quad \|_c)^\wedge_c$. Hence $(E, \| \quad \|)^\wedge_c$ and $(E, \| \quad \|_c)^\wedge_c$ have the same compact subsets. So, they are isomorphic. On the other hand, $(E, \| \quad \|_c)^\wedge_c$ and

$B(E)_c^\wedge$ are topologically isomorphic. In other words, the bidual group of a locally bounded space is its Banach envelope.

## 4 Nuclear groups

The class of nuclear group forms a Hausdorff variety (i.e. it is closed under forming subgroups, Hausdorff quotient groups and arbitrary products) and contains all LCA groups and all nuclear vector spaces. (It is shown that it is strictly larger than the Hausdorff variety generated by all LCA groups).

A locally convex vector space $E$ is named *nuclear vector space* if for every symmetric and convex neigbourhood $U$ of 0 there exists a symmetric and convex neighbourhood $W$ of 0 such that $d_k(W, U) \leq 1/k$ for all $k \in \mathbb{N}$

*A normed space is a nuclear vector space if and only if it is finie-dimensional*

*The space* $\mathbb{R}^{(I)}$ *with the locally convex sum topology is a nuclear space, if and only if* $I$ *is countable.*

*Locally convex vector spaces endowed with their weak topology are nuclear spaces.*

*Every bounded subset of a nuclear vector space is totally bounded*

*A metrizable locally convex space* $E$ *is nuclear if and only if the set of all summable families in* $E$ *coincides with the set of all absolutely summable families in* $E$*.*

**(Banaszczyk, 1986): Additive subgroups of nuclear spaces are weakly**

**closed.**

**(Banaszczyk, 1984) If a metrizable locally convex non nuclear it contains a non trivial discrete additive subgroup which is weakly dense.**

A vector space $E$ endowed with a Hausdorff group topology $\tau$ which has a neighbourhood basis of 0 consisting of symmetric and convex sets is named a *locally convex vector group.*

If, in addition, for every symetric and convex neighbourhood $W$ of 0 such that $d_k(W, U) \leq 1/k$ for all $k \in \mathbb{N}$, it is called a *nuclear vector group.*

**Banaszczyk, 1990 Every nuclear group is topologically isomorphic to a subgroup of a Hausdorff quotient group of a nuclear vector group.**

**(Aussenhofer Banaszczyk) Properties of nuclear groups**

Nuclear groups are locally quasi-convex.

Quotients, products, countable direct sums and completions of nuclear groups are Schwartz groups.

Subgroups of nuclear groups are nuclear groups.

Any group which is locally isomorphic with a nuclear group is a nuclear group.

Every nuclear group can be embedded into a product of metrizable nuclear groups.

Čech complete nuclear groups are Pontryagin reflexive.

**(Banaszczyk, Martín-Peinador) Every nuclear group satisfies Glicksberg theorem.**

Example of a group which does not satisfies Glicksberg Theorem. The sequence of standard unit vectors $(e_n)$ in the sequence space $l_2$ converges weakly to 0.

**(Galindo, Hernandez). Let X be a completely regular space. Then the free abelian group A(X) satisfies Gliksberg theorem.**

Observe that the $A(l_2) \to l_2$ is a projection. In general Hausdorff quotient groups of groups which satisfies Glicksberg Theorem need not have this property.

## 5 Schwartz groups.

- $E$ real vector space, $U \subset E$ balanced and absorbing.

  When is $\{\frac{1}{n}U : n \in \mathbb{N}\}$ a basis of neighborhoods of zero for a vector space topology $\mathcal{T}_U$ on $E$?

Answer: When $U$ is *pseudoconvex* (i. e. there exists $n \in \mathbb{N}$ such that $\frac{1}{n}U + \frac{1}{n}U \subset U$)

- $G$ abelian group, $0 \in U \subset G$, $U$ symmetric

Define for every $n \in \mathbb{N}$

$$U_{(n)} = \{x \in G : x \in U,\ 2x \in U, \dots, nx \in U\}$$

(If $G$ is a vector space and $U$ is balanced, then $U_{(n)} = \frac{1}{n}U$)

When is $\{U_{(n)} : n \in \mathbb{N}\}$ a basis of neighborhoods of zero for a group topology $\mathcal{T}_U$ on $G$?

Answer: When there exists $n \in \mathbb{N}$ such that $U_{(n)} + U_{(n)} \subset U$

If $U$ is a quasi-convex subset of $G$ then the family $\{U_{(n)} : n \in \mathbb{N}\}$ is a basis of neighborhoods of 0 for a locally quasi-convex group topology $\tau_U$ on $G$. We write $G_U$ for the Hausdorff group topology asociated to $(G, \tau_U)$. If $U \subseteq U$, $\varphi_{VU} : (G, \tau_V) \to (G, \tau_U)$ are the linking homomorphism.

(Grothendieck, 1954) A Hausdorff locally convex space $E$ is said to be a *Schwartz space* if for every convex and balanced $U \in \mathcal{N}_0(E)$ there exists $V \in \mathcal{N}_0(E)$ which is $\mathcal{T}_U$–precompact.

(Rolewicz, 1961, in metrizable case) A Hausdorff topological vector space $E$ is said to be a Schwartz space if for every $U \in \mathcal{N}_0(E)$ there exists $V \in \mathcal{N}_0(E)$ such that

$$\forall n \in \mathbb{N} \quad \exists F \text{ finite subset of } E \text{ such that } V \subset F + \tfrac{1}{n}U$$

(Aussenhofer, Dominguez, Tarieladze, Ch.) A Hausdorff topological Abelian group $G$ is said to be *a Schwartz group* if for every $U \in \mathcal{N}_0(G)$ there exists $V \in \mathcal{N}_0(G)$ such that

(*) $\forall n \in \mathbb{N} \quad \exists F$ finite subset of $G$ such that $V \subset F + U_{(n)}$

If $U$ generates a group topology (e. g. if $U$ is quasi-convex), (*) is equivalent with: $V$ is $\mathcal{T}_U$–precompact.

**Every Schwartz space is a Schwartz group.**

**Every bounded subset of a locally quasi–convex Schwartz group is precompact.**

Bounded sets in topological groups (Hecjman): A subset $B$ of a topological abelian group $G$ is bounded if for every $U \in \mathcal{N}_0(G)$ there exist a finite $F \subset G$

and $n \in \mathbb{N}$ such that

$$B \subset F + U + \overset{n}{\ldots} + U.$$

**Permanence and structural properties of Schwartz groups**

Quotients, products, countable direct sums and completions of Schwartz groups are Schwartz groups.

Subgroups of locally quasi–convex Schwartz groups are Schwartz groups.

Any group which is locally isomorphic with a locally quasi–convex Schwartz group is a Schwartz group.

Every locally quasi–convex Schwartz group can be embedded into a product of metrizable Schwartz groups.

**Characterizations of Schwartz groups**

**Let $G$ be a locally quasi–convex topological abelian group. $G$ is a Schwartz group if and only if for every quasi–convex $U \in \mathcal{N}_0(G)$ there exists $V \in \mathcal{N}_0(G)$ such that $V \subset U$and the linking homomorphism $\varphi_{VU}$ is precompact (i.e. $\varphi_{VU}(W)$ is precompact for some neighborhood of $0$ in $G_V$).**

**Nuclear groups are Schwartz groups**

**Free locally convex, over hemicompact k-spaces are Schwartz vector spaces**

**Topological abelian groups which are hemicompact k-spaces are Schwartz groups.**

Most important example: duals of metrizable groups.

Proof: Let $A(G)$ (resp. $L(G)$) be the free topological Abelian group (resp. the free locally convex space) over the topological space $G$. $G$ is a quotient of $A(G)$, which is a topological subgroup of $L(G)$. $L(G)$ is a Schwartz space.

**(Aussenhofer, 2008) Locally quasi-convex Schwartz groups satisty Glicksber Theorem.**