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Nonlinear isomorphisms of lattices of Lipschitz functions

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Abstract. The paper contains a number of Banach-Stone type theorems for lattices of uniformly continuous and Lipschitz functions without any linearity assumption. Sample result: two complete metric spaces are Lipschitz homeomorphic if (and only if, of course) the corresponding lattices of Lipschitz functions are isomorphic. Here, a lattice isomorphism is just a bijection preserving the order in both directions, in particular linearity is not assumed.

Introduction

The results presented in this paper could be described as nonlinear Banach-Stone type theorems for lattices of uniformly continuous and Lipschitz functions. Here, by a Banach-Stone theorem we mean the statement that certain (often algebraical) structure of a system of (continuous, real-valued) functions on a topological space X determines some additional (often topological) structure on X. As everyone knows the genuine Banach-Stone theorem says that two compact spaces are homeomorphic provided their corresponding spaces of continuous functions are isometric in the natural supremum norm. See [**5**] for a survey with many historical comments in the linear setting and the references in [**2**] for the nonlinear background.

Let X be a metric space, with distance d. A function $f : X \to \mathbb{R}$ is said to be Lipschitz if

$$\Lambda(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)} < \infty.$$

The set of all Lipschitz functions on X is denoted $\mathrm{Lip}(X)$ and carries several structures: it is a linear space, a lattice and even a Banach lattice. When X has finite diameter it is also a Banach algebra. The nice booklet by Weaver [**11**] contains a lot of information on spaces of Lipschitz functions. In this paper we forget every structure of $\mathrm{Lip}(X)$ but the order and we contemplate it as a lattice. Of course the order in $\mathrm{Lip}(X)$ is the pointwise order inherited from $\mathbb{R}$, with $f \leq g$ meaning $f(x) \leq g(x)$ for all $x \in X$. Let us emphasize that such notions as 'isomorphism', 'homomorphism', and the like refer to the 'default' lattice setting unless otherwise stated.

The main result of the paper is that the lattice structure of $\mathrm{Lip}(X)$ determines the Lipschitz structure of X amongst complete metric spaces of finite diameter: if Y and X are complete metric spaces of finite diameter and there is a lattice isomorphism $T : \mathrm{Lip}(Y) \to \mathrm{Lip}(X)$, then X and Y are Lipschitz homeomorphic (we remark again that a lattice isomorphism is nothing but a bijection preserving the order in both directions, in particular linearity is not assumed). In fact what we shall show is that such a T is implemented by a Lipschitz homeomorphism $\tau : X \to Y$ in the precise way we explain in Theorem 1.

It is worth noting that the corresponding linear result has been obtained only very recently [**9**, Part (d) of the Main Theorem]. See [**4**] for related results.

The somewhat involved proof of this single result occupies most of the paper (Section 1). In Section 2 we give an application to little Lipschitz functions. In Section 3 we prove a non-linear version of a recent result by Garrido and Jaramillo stating that 'unital' uniformly separating lattices determine

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the uniform structure of complete metric spaces. In Section 3 we exhibit an example showing that the hypothesis made in the above results cannot be dropped. This actually follows from standard 'reduction' results for Lipschitz functions, but the uniformly continuous case seems to be new. We close the paper with an esoteric remark on a classical paper by Shirota and some open problems.

Notations and conventions. We use d to denote distance on any metric space. This causes no confusion unless we must consider two different metrics on the same space.

We write $B(x, r)$ for the closed ball of radius r centred at x. The distance between two subsets of X is given by $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$. Given a continuous function $f : X \to \mathbb{R}$ the support of f, abbreviated $\operatorname{supp} f$, is the closure of the set $\{x \in X : f(x) \neq 0\}$.

Finally, given a partially ordered set S, we write S^+ for the subset $\{s \in S : s \geq 0\}$ whenever this makes sense.

1. Lattices of Lipschitz functions

Let us present now the sought after result on Lipschitz lattices.

THEOREM 1. *Let $T : \mathrm{Lip}(Y) \to \mathrm{Lip}(X)$ be an isomorphism, where Y and X are complete metric spaces of finite diameter. Then there is a Lipschitz homeomorphism $\tau : X \to Y$ such that*

$$Tf(x) = t(x, f(\tau(x))) \tag{1}$$

for every $f \in \mathrm{Lip}(Y)$ and all $x \in X$, where $t : X \times \mathbb{R} \to \mathbb{R}$ is given by $t(x, c) = Tc(x)$.

The rather long proof is divided into three parts. First we construct the required map $\tau : X \to Y$ and we show it is a uniform homeomorphism. Then, we use it to get the representation (1). Finally, we use this representation and a category argument to obtain that τ must be Lipschitz in both directions.

1.1. From order to topology. In this part we show how the lattice $\mathrm{Lip}(X)$ determines the topological space X and the uniform structure induced by the distance.

With an eye in the applications to little Lipschitz functions, let us say that a lattice $L(X)$ of uniformly continuous functions on X is uniformly separating if, given subsets A and B of X such that $d(A, B) > 0$ there is $f \in L(X)$ such that $f = 0$ on A and $f = 1$ on B. This notion is borrowed from [**3**]. Only the case $L(X) = \mathrm{Lip}(X)$ is needed to prove Theorem 1.

Throughout this Section $L(X)$ and $L(Y)$ will stand for uniformly separating vector lattices of functions on the metric spaces X and Y, respectively.

To each $f \in L(X)^+$ we associate an open set U_f taking the interior of its support. This is in fact a regular open set (one that agrees with the interior of its closure).

The class of all regular open subsets of X is denoted $R(X)$ and the subclass of those arising as U_f for some $f \in L(X)^+$ is denoted $RL(X)$. These are lattices when ordered by inclusion. Notice that $RL(X)$ contains a base for the topology of X as long as $L(X)$ is uniformly separating.

Our immediate aim is to show that the relations $U_f \subset U_g$ and $\overline{U}_f \subset U_g$ can be expressed within the order structure of $L(X)^+$. To this end, following Shirota [**8**] let us declare $f \subset g$ when for every $h \in L(X)^+$ one has $f \wedge h = 0$ whenever $g \wedge h = 0$. Then, we say that f and g are equivalent if $f \subset g$ and $g \subset f$. Also, we write $f \Subset g$ if, whenever the family (h_α) has an upper bound in $L(X)^+$ and $h_\alpha \subset f$ for all α, there is an upper bound $h \in L(X)^+$ such that $h \subset g$.

LEMMA 1 (Mainly Shirota). *Given $f, g \in L(X)^+$ one has:*

(a) *$f \wedge g = 0$ if and only if $U_f \cap U_g = \varnothing$.*
(b) *$f \subset g$ if and only if $U_f \subset U_g$ if and only if* $\operatorname{supp} f \subset \operatorname{supp} g$.
(c) *If $f \Subset g$, then $d(U_f, U_g^c) > 0$. The converse is true if $L(X)$ is closed under products.*

Proof. (a) is trivial, let us prove (b). By the very definition, we have $f \subset g$ if and only if $g \wedge h = 0$ implies $f \wedge h = 0$. By part (a), this is equivalent to '$U_g \cap U_h = \varnothing$ implies $U_f \cap U_h = \varnothing$', which is clearly equivalent to $U_f \subset U_g$. The last equivalence is obvious.

(c) Assume $f \Subset g$. For each $x \in U_f$ pick some $h_x : X \to [0,1]$ in $L(X)$ such that $h_x(x) = 1$ and $h_x \subset f$. Of course, the family (h_x) is bounded by 1. Now, if h is an upper bound for (h_x) such that $h \subset g$, then $h \geq 1$ on U_f, $h = 0$ on U_g^c and since h is uniformly continuous we have $d(U_f, U_g^c) > 0$.

Assume $L(X)$ is a ring and $d(U_f, U_g^c) > 0$. Take $u \in L(X)$ such that $u = 0$ off U_g and $u = 1$ on U_f. Now, if $h_\alpha \subset f$ and h is an upper bound for (h_α), then uh is also an upper bound and quite clearly $uh \subset f$. □

Corollary 1. *If $T : L(Y)^+ \to L(X)^+$ is an isomorphism, then the map $\mathfrak{T} : RL(Y) \to RL(X)$ given by $\mathfrak{T}(U_f) = U_{Tf}$ is a well-defined lattice isomorphism.* □

The following results show that isomorphisms of function lattices have a local behaviour.

Lemma 2. *Given $f, g, h \in L(Y)^+$, one has $f \leq g$ on U_h if and only if $f \wedge u \leq g \wedge u$ for every $u \subset h$.*

Therefore if $T : L^+(Y) \to L^+(X)$ is an isomorphism, then, given $f, g \in L^+(Y)$ and $U \in RL(Y)$, one has $f \leq g$ on U if and only $Tf \leq Tg$ on $\mathfrak{T}(U)$, where $\mathfrak{T}$ is as in Corollary 1.

Proof. If $f \leq g$ on U_h and $u \subset h$, then it is straightforward that every function lower than f and u is lower than g, so $f \wedge u \subset g \wedge u$.

As for the converse, it is clear that if $f \wedge ah \leq g \wedge ah$ for every $a \in (0, \infty)$, then $f \leq g$ on U_h. □

Corollary 2. *Let $T : L(Y) \to L(X)$ be an isomorphism. There is a lattice isomorphism $\mathfrak{T} : RL(Y) \to RL(X)$ such that, given $f, g \in L(Y)$ and $U \in RL(Y)$, one has $f \leq g$ on U if and only if $Tf \leq Tg$ on $\mathfrak{T}(U)$. The same is true if we replace '$\leq$' by '$\geq$' or '$=$'.*

Proof. There is no loss of generality in assuming $T0 = 0$. Let $\mathfrak{T}$ be as in Corollary 1. We then have that for $f, g \in L^+(Y)$ and $U \in RL(Y)$ one has $f \leq g$ on U if and only if $Tf \leq Tg$ on $\mathfrak{T}(U)$. It is evident that this property characterizes $\mathfrak{T}(U)$ amongst the members of $RL(X)$.

But 0 plays no special rôle here, so actually we have proved that, given $u \in L(Y)$, there is an isomorphism $\mathfrak{T}_u : RL(Y) \to RL(X)$ such that for $f, g \geq u$ in $L(Y)$ and $U \in RL(Y)$ one has $f \leq g$ on U if and only if $Tf \leq Tg$ on $\mathfrak{T}_u(U)$. As before, this property characterizes $\mathfrak{T}_u(U)$ in $RL(X)$. Now, if $u, v \in L(Y)$, it is easily seen that

$$\mathfrak{T}_u = \mathfrak{T}_{u \wedge v} = \mathfrak{T}_{u \vee v} = \mathfrak{T}_v,$$

so $\mathfrak{T}_u = \mathfrak{T}_0 = \mathfrak{T}$ and the conclusion obtains. □

Before embarking into the proof of the main result, let us remark that $\mathrm{Lip}(X)$ is always uniformly separating. Indeed, if $d(A_0, A_1) > 0$, then the function given by

$$f(x) = \frac{d(x, A_0)}{d(x, A_0) + d(x, A_1)}$$

equals i on A_i, for $i = 0, 1$. Moreover every regular open subset of X arises as U_g for some $g \in \mathrm{Lip}(X)^+$, for if $U \in R(X)$, then $U = U_g$, where $g(x) = d(x, U^c)$.

Proof of Theorem 1. Part I. In this part of the proof we construct the required mapping $\tau : X \to Y$ and we prove it is a uniform homeomorphism. Our reasonings depend on the fact that Lipschitz functions on spaces with finite diameter are bounded and they do not apply to unbounded metrics; see Example 1 below.

So, let $T : \mathrm{Lip}(Y) \to \mathrm{Lip}(X)$ be an isomorphism and let $\mathfrak{T} : R(Y) \to R(X)$ be the lattice isomorphism given by Corollary 2. What we will show is that $\mathfrak{T}$ is induced by a point-mapping $\tau : X \to Y$ in the sense that $\mathfrak{T}(U) = \tau^{-1}(U)$ holds for every $U \in R(Y)$.

Consider the set valued map $\tilde{\tau} : X \to 2^Y$ given by

$$\tilde{\tau}(x) = \bigcap_{x \in \mathfrak{T}(U)} U = \bigcap_{x \in V} \mathfrak{T}^{-1}(V).$$

Let V_n be the open ball of radius $1/n$, centred at x. As $d(V_{n+1}, V_n^c) > 0$, if we write $V_n = U_{h_n}$ for suitably chosen $h_n \in \mathrm{Lip}(X)$ we have $h_{n+1} \Subset h_n$ and thus $T^{-1}h_{n+1} \Subset T^{-1}h_n$ whence if we denote $U_n = \mathfrak{T}^{-1}(V_n) = U_{T^{-1}h_n}$ one has $d(U_{n+1}, U_n^c) > 0$, in particular $\overline{U}_{n+1} \subset U_n$ and

$$\tilde{\tau}(x) = \bigcap_n U_n = \bigcap_n \overline{U}_n.$$

Let us see that $\tilde{\tau}(x)$ is nonempty. For each n, take $y_n \in U_n$ and consider the resulting sequence. Every cluster point of (y_n) is in the closure of every U_n and so in $\tilde{\tau}(x)$. So, if we assume $\tilde{\tau}(x)$ to be empty, then there is $\varepsilon > 0$ and an infinite $M \subset \mathbb{N}$ such that $d(y_n, y_m) \geq \varepsilon$ for $n \neq m$ provided $n, m \in M$. Take a partition $M = M_0 \oplus M_1$ into two infinite subsets and, for $i = 0, 1$, set

$$W_i = \bigcup_{n \in M_i} B(y_n, \varepsilon/3).$$

Clearly $d(W_0, W_1) \geq \varepsilon/3$, so there is a Lipschitz $u : Y \to [0, 1]$ such that $u = 0$ on W_0 and $u = 1$ on W_1. Let f and g be such that $Tf = 0$ and $Tg = 1$. The function $v = f + u \cdot (g - f)$ agrees with f on W_0 and agrees with g on W_1. So, if $w = Tv$, then w takes the values 0 and 1 on any neighbourhood of x, a contradiction.

We see that $\tilde{\tau}(x)$ has exactly one point. If $y \in \tilde{\tau}(x)$, then by the very definition, given $U \in R(Y)$, we have $y \in U$ as long as $\mathfrak{T}(U)$ contains x. Let $S : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ be the inverse of T, $\mathfrak{S} : R(Y) \to R(X)$ the lattice isomorphism associated to S and $\tilde{\sigma} : Y \to 2^X$ the set-valued function associated to $\mathfrak{S}$. Clearly, $\mathfrak{S}$ is nothing but the inverse of $\mathfrak{T}$. Of course, we have proved that $\tilde{\sigma}(y)$ is nonempty. Taking $x' \in \tilde{\sigma}(y)$ we obtain the following implications, for $V \in R(X)$:

$$x \in V \Rightarrow y \in \mathfrak{S}(V) \Rightarrow x' \in V$$

This already implies that $x' = x$ and so, for $y \in \tilde{\tau}(x)$, we have $x \in V$ if and only if $y \in \mathfrak{S}(V)$. But $\mathfrak{S}$ is a lattice isomorphism and so there is at most one y satisfying that condition.

This shows that $\tilde{\tau}(x)$ is a singleton for every $x \in X$. That the map $\tau : X \to Y$ sending x into the only element of $\tilde{\tau}(x)$ is continuous is trivial. That this map is a homeomorphism follows by symmetry.

It remains to see that τ is uniformly continuous, that is, $d(x_n, x'_n) \to 0$ in X implies $d(y_n, y'_n) \to 0$ in Y, where $y_n = \tau(x_n), y'_n = \tau(x'_n)$. If we assume the contrary, we get sequences (x_n) and (x'_n) such that $d(x_n, x'_n) \to 0$, while $d(y_n, y'_n)$ is bounded away from zero. Since neither (x_n) nor (x'_n) have convergent subsequences, and passing to a subsequence if necessary, we get $d(A_0, A_1) > 0$ in Y, where $A_0 = \{y_n : n \in \mathbb{N}\}$ and $A_1 = \{y'_n : n \in \mathbb{N}\}$. Take $u \in \mathrm{Lip}(Y)$ such that $u = i$ on a neighbourhood of A_i, where $i = 0, 1$ and proceed as before: assuming $Tf = 0$ and $Tg = 1$, the image of the function $v = f + u \cdot (g - f)$ under T takes the value 0 at every x_n and the value 1 at every x'_n, a contradiction. □

1.2. Functional representation. In this Section we use the map τ to obtain the representation of T appearing in the main result. The key point is the construction of certain Lipschitz functions with suitable oscillation properties we present now.

LEMMA 3. *Let S be a set of real numbers having 0 as a cluster point. There is a Lipschitz function $\varphi : \mathbb{R} \to [0, 1]$ and two infinite subsets M and N of S such that:*

- *$\varphi(t) > t$ for all $t \in M$.*
- *$\varphi = 0$ on a neighbourhood of every $t \in N$.*

Moreover φ can be chosen with Lipschitz constant arbitrarily close to 1.

PROOF. Without loss of generality we may assume 0 is a cluster point of S^+. The action takes place in the plane $\mathbb{R}^2$ and to avoid any risk of confussion, in this proof, we denote by $]a,b[$ the open interval with endpoints a and b. Fix $r>1$. Pick $s_1 \in S\cap]0,1/r[$. Now take $0<s_2<s_1$ in S^+so that the line joining $(s_2,0)$ with (s_1,s_1) has slope at most r, that is:

$$\frac{s_1-0}{s_1-s_2}\leq r.$$

Let $0<s_3<s_2$ so that $]s_3,s_2[\cap S\neq\varnothing$. Next take $s_4<s_3$ in such a way that the line joining (s_4,s_4) with $(s_3,0)$ has slope at most r:

$$\left|\frac{0-s_4}{s_3-s_4}\right|\leq r.$$

Now, replace s_1 by s_4 to obtain s_5 as we did with s_2 and so on.

Let us consider the function ϕ vanishing on the semiaxis $]-\infty,0]$, taking the value s_1 on $[s_1,\infty[$ and whose graph in $]0,s_1[$ is the 'polygonal' defined by the points

$$(s_1,s_1),(s_2,0),(s_3,0),(s_4,s_4),(s_5,0),(s_6,0),(s_7,s_7),\ldots$$

Then $\varphi=r\phi$ is the Lipschitz function we were looking for and, quite clearly, $\Lambda(\varphi)\leq r^2$. $\square$

PROOF OF THEOREM 1. PART II. Let us prove the formula (1), where $\tau: X\to Y$ is the uniform homeomorphism we got in Part I. Plainly, it suffices to prove that, given $f,g\in \mathrm{Lip}(Y)$, one has $Tf(x)=Tg(x)$ if and only if $f(y)=g(y)$, where $y=\tau(x)$. By symmetry, we only need the proof of the 'if' part.

Suppose $f(y)=g(y)$. Replacing f and g by $f\wedge g$ and $f\vee g$ we may assume $f\leq g$. In this case we already know $Tf\leq Tg$ and we must show $Tf(x)=Tg(x)$. This is obvious if $f=g$ on a neighbourhood of y, so in the ensuing argument we assume every neighbourhood of y contains points where $f<g$. In particular y (hence x) is not isolated.

Put $h=g-f$. Then $h(y)=0$ and there is a sequence $y_n\to y$ such that $h(y_n)>0$ for every n. Take $t_n=h(y_n)$ and apply Lemma 3 to the set of these t_n. Let φ the resulting function and define $u=f+\varphi\circ h$. Clearly, every neighbourhood of y contains an open set where $u=f$ and also an open set where $u\geq g$. Therefore, if $v=Tu$, then every neighbourhood of x contains an open set where $v=Tf$ and also an open set where $v\geq Tg$. It follows that $Tf(x)=Tg(x)=v(x)$ and we are done. $\square$

1.3. From order to distance through category. We have arrived to the most delicate point of our main result and we face the proof that τ is Lipschitz. Here we will use in a essential way the fact that Lipschitz lattices are themselves complete metric spaces.

First of all, whenever X has finite diameter we can equip $\mathrm{Lip}(X)$ with the norm $\|f\|=\|f\|_\infty\vee\Lambda(f)$ which makes it into a Banach space. The resulting Banach lattice turns out to be boundedly complete. This simply means that $\bigvee S$ exists as long as the set S is norm bounded in $\mathrm{Lip}(X)$. We hasten to remark that 'norm bounded' implies 'bounded' , but the converse fails.

Also, recall that a lattice homomorphism T is said to be **normal** if it preserves all joints and meets, that is, it satisfies $T(\bigvee S)=\bigvee T(S)$ (respectively, $T(\bigwedge S)=\bigwedge T(S)$) provided $\bigvee S$ (respectively, $\bigwedge S$) exists. Needless to say, lattice isomorphisms are normal.

LEMMA 4. *Let $T:\mathrm{Lip}(Y)\to\mathrm{Lip}(X)$ be a normal homomorphism. Then:*

(a) *If (f_n) is bounded in the norm of $\mathrm{Lip}(Y)$ and converges pointwise to f, then Tf_n converges pointwise to Tf provided (Tf_n) is norm bounded in $\mathrm{Lip}(X)$.*

(b) *T maps an open set of $\mathrm{Lip}(Y)$ into a norm bounded set of $\mathrm{Lip}(X)$.*

Proof. The first part is nearly obvious once one realizes that if (f_n) is bounded in norm, then f_n converges to f pointwise if and only if

$$f = \bigwedge_n \bigvee_{k\geq n} f_k = \bigvee_n \bigwedge_{k\geq n} f_k$$

in the corresponding Lipschitz lattice.

Let us prove (b). We show that, for each real R, the set $\{f : \|Tf\| \leq R\}$ is closed in $\mathrm{Lip}(Y)$. Indeed, if (f_n) converges to f in the Lipschitz norm and $\|Tf_n\| \leq R$ for all n, then (f_n) is norm bounded and pointwise convergent to f, so (Tf_n) is pointwise convergent to Tf, which clearly implies that $\|Tf\| \leq R$. Now, we have

$$\mathrm{Lip}(Y) = \bigcup_{k=1}^{\infty} \{f : \|Tf\| \leq k\}.$$

By Baire's theorem, there is $R \in \mathbb{N}$ such that the (norm closure of) $\{f : \|Tf\| \leq R\}$ has nonempty interior. This completes the proof. □

Let V be a vector lattice and let $g \in V$. Then the map $f \mapsto f + g$ is a lattice automorphism of V. In particular, if $T : \mathrm{Lip}(Y) \to \mathrm{Lip}(X)$ is a homomorphism, $g \in \mathrm{Lip}(Y)$ and $h \in \mathrm{Lip}(X)$, then $f \mapsto h + T(f + g)$ is a homomorphism. It is normal or an automorphism if and only if T is. In particular, if T maps a neighbourhood of g into a norm bounded set, then

$$f \longmapsto T(f - g) - T(-g)$$

maps a ball centered at the origin into a bounded set and sends 0 to 0.

In the following result we use $\lfloor\cdot\rfloor$ for the integer part function.

Lemma 5. *Let d and δ be bounded metrics on X and $T : \mathrm{Lip}(X, \delta) \to \mathrm{Lip}(X, d)$ a homomorphism having the representation $Tf(x) = t(x, f(x))$ for every f and all x. If $\Lambda_d(Tf) \leq R$ for each f in the ball of radius r in $\mathrm{Lip}(X, \delta)$, one has the following:*

(a) *If $a, b \in [-r, r]$ and $|b - a| \leq r\delta(x, y)$, then $|t(x, a) - t(y, b)| \leq R \cdot d(x, y)$.*

(b) *If $0 \leq c \leq \delta(x, y)\lfloor 1/\delta(x, y)\rfloor$, then $|t(x, cr) - t(x, 0)| \leq R \cdot d(x, y)/\delta(x, y)$.*

Proof. (a) There is $f \in \mathrm{Lip}(X, \delta)$ such that $f(x) = a, f(y) = b$ and $\|f\| \leq r$. Now, as $\Lambda_d(Tf) \leq R$, we have

$$|t(x, a) - t(y, b)| = |Tf(x) - Tf(y)| \leq Rd(x, y).$$

(b) We may assume $T0 = 0$. Fix $x, y \in X$ and let N be the least integer such that $N\delta(x, y) > 1$, so that $N - 1 = \lfloor 1/\delta(x, y)\rfloor$. Applying the first part with $a = r\delta(x, y)$ and $b = 0$ we get $t(x, r\delta(x, y)) \leq Rd(x, y)$. And, by symmetry, $t(y, r\delta(x, y)) \leq Rd(x, y)$. Also,

$$|t(x, 2r\delta(x, y)) - t(y, r\delta(x, y))| \leq Rd(x, y) \quad \text{and} \quad |t(y, 2r\delta(x, y)) - t(x, r\delta(x, y))| \leq Rd(x, y),$$

hence $|t(x, 2r\delta(x, y))| \leq 2Rd(x, y)$ and $|t(y, 2r\delta(x, y))| \leq 2Rd(x, y)$. Continuing in this way, we arrive to

$$|t(x, (N - 1)r\delta(x, y))| \leq R(N - 1)d(x, y).$$

Since $(N - 1)\delta(x, y) \leq 1$ the result follows. □

Proof of Theorem 1. Part III. In previous issues of the proof we have seen that $Tf(x) = t(x, f(\tau(x)))$, where $\tau : X \to Y$ is a uniform homeomorphism. Thus, we can transfer the structure of Y to X by defining a new distance through τ taking $\delta(x, x') = d_Y(\tau(x), \tau(x'))$. In this way we may assume in the remainder of the proof that $Y = X$ and τ is the identity on X so that T defines an isomorphism from $\mathrm{Lip}(X, \delta)$ to $\mathrm{Lip}(X, d)$ by the formula $Tf(x) = t(x, f(x))$, where d and δ are uniformly equivalent metrics on X, both bounded and complete.

We must show that the identity is Lipschitz from (X, d) to (X, δ).

Clearly, isomorphisms are normal homomorphisms, so Lemma 4 guarantees that T maps a closed ball of $\mathrm{Lip}(X,\delta)$ into a norm-bounded set of $\mathrm{Lip}(X,d)$. By the remark made after Lemma 4 we may and do assume that $\Lambda_d(Tf) \leq R$ for every $f \in \mathrm{Lip}(X,\delta)$ with $\|f\| \leq r$ and also that $T0 = 0$.

Now, if the identity fails to be Lipschitz from (X,d) to (X,δ), there are sequences (x_n) and (y_n) such that

$$\lim_{n\to\infty} \frac{\delta(x_n,y_n)}{d(x_n,y_n)} = \infty, \tag{2}$$

which already implies $d(x_n,y_n) \to 0$. Since δ is uniformly equivalent to d we also have $\delta(x_n,y_n) \to 0$. Thus, for each $c < 1$, we have $c \leq \delta(x_n,y_n)\lfloor 1/\delta(x_n,y_n)\rfloor$ for n large enough and by Lemma 5(b),

$$t\left(x_n, \frac{r}{2}\right) - t(x_n,0) \leq R\frac{d(x_n,y_n)}{\delta(x_n,y_n)}.$$

If (x_n) has a cluster point in X, say x, then

$$t\left(x, \frac{r}{2}\right) - t(x,0) \leq R \lim_{n\to\infty} \frac{d(x_n,y_n)}{\delta(x_n,y_n)} = 0,$$

a contradiction: in the second part of the proof we established that $t(x,c)$ is strictly increasing in c for each fixed x.

If (x_n) has no cluster point, then neither (y_n) has and there is $\varepsilon > 0$ such that $e(z_n,z_m) \geq \varepsilon$ for $e = d,\delta$, with $z = x,y$ and $n \neq m$. Set $Z = \{x_n, y_n : n \in \mathbb{N}\}$. As bounded Lipschitz functions extend anywhere the 'restriction' of T to Z, given by

$$Tf(z) = t(z, f(z)) \qquad (z \in Z)$$

is an isomorphism of $\mathrm{Lip}(Z,\delta)$ onto $\mathrm{Lip}(Z,d)$ we still call T.

Let ζ denote the involution on Z that permutes x_n and y_n. It is clear that ζ is Lipschitz with respect to d and δ. Thus, we can define a symmetric version of T through

$$Sf = Tf + \zeta^*(T(\zeta^*(f))) = Tf + (T(f\circ\zeta))\circ\zeta.$$

Notice that S maps $\mathrm{Lip}(Z,\delta)$ to $\mathrm{Lip}(Z,d)$. Even if S need not be an isomorphism, it is a homomorphism and, in fact,

$$Sf(z) = t(x_n, f(z)) + t(y_n, f(z))$$

if z is either x_n or y_n. Also, we remark that $S0 = 0$ and $Sf \geq Tf$ for every f. We will construct certain $f \in \mathrm{Lip}(Z,\delta)$ so that $Sf \in \mathrm{Lip}(Z,d)$ forces the ratio $\delta(x_n,y_n)/d(x_n,y_n)$ to be bounded by a constant independent on n, thus contradicting (2).

We remark that the metric structure of Z is so simple that $f : Z \to \mathbb{R}$ is Lipschitz with respect to δ if (and only if) it is bounded and satisfies

$$|f(x_n) - f(y_n)| \leq \Lambda\delta(x_n,y_n)$$

for some Λ independent on n.

Let us write $Sf(z) = s(z, f(z))$, where $s(z,c) = t(x_n,c) + t(y_n,c)$ for $z = x_n, y_n$. As T is surjective we can choose $K \in \mathbb{R}$ such that $TK \geq 1$ whence $SK \geq 1$. Fix $n \in \mathbb{N}$ and let N be the least integer such that $N\delta(x_n,y_n) \geq K$. Let z denote either x_n or y_n. We have $s(z, N\delta(x_n,y_n)) - s(z,0) \geq 1$, hence there is $0 \leq m \leq N-1$ for which

$$s(z,(m+1)\delta(x_n,y_n)) - s(z,m\delta(x_n,y_n)) \geq \frac{1}{N} \geq \frac{1}{2(N-1)} \geq \frac{\delta(x_n,y_n)}{2K}. \tag{3}$$

Next, we define $f : Z \to \mathbb{R}$ taking

$$f(x_n) = m\delta(x_n,y_n) \quad \text{and} \quad f(y_n) = (m+1)\delta(x_n,y_n).$$

Clearly, $f \in \mathrm{Lip}(Z, \delta)$ and since $Sf \in \mathrm{Lip}(Z, d)$ we infer from (3) that

$$\frac{\delta(x_n, y_n)}{2K} \leq \Lambda_d(Sf) d(x_n, y_n),$$

in contradiction to (2). This completes the proof. □

2. Little Lipschitz lattices

Now we give an application to little Lipschitz functions. We avoid any pathology by considering in this Section only compact spaces. Let Z be a compact metric space with distance d. Then $\mathrm{lip}(Z)$ consists of those functions in $\mathrm{Lip}(Z)$ satisfying

$$\frac{|f(x) - f(y)|}{d(x, y)} \to 0 \qquad \text{as} \qquad d(x, y) \to 0.$$

It may happen that $\mathrm{lip}(Z)$ contains only the constant functions: $Z = [0, 1]$ is just one example. Thus some additional condition is necessary to get a sensitive space of little Lipschitz functions.

Let us consider the following separation property, introduced by Weaver in [**10**] under a different name. We say that $\mathrm{lip}(Z)$ separates points boundedly if there is a constant $k > 1$ such that for each $x, y \in Z$ there is $f \in \mathrm{lip}(Z)$ satisfying $|f(x) - f(y)| = d(x, y)$ with $\Lambda(f) < k$. It turns out [**11**, Corollary 3.3.5] that if this condition is satisfied for some $k > 1$ then it holds for every $k > 1$.

Theorem 2. *Let Y and X be compact metric spaces such that $\mathrm{lip}(Y)$ and $\mathrm{lip}(X)$ separate points boundedly. Then $\mathrm{lip}(Y)$ and $\mathrm{lip}(X)$ are isomorphic lattices if and only if X and Y are Lipschitz homeomorphic.*

Antonio Jiménez-Vargas and Moisés Villegas-Vallecinos proved the corresponding linear result in [**6**] for Hölder metrics. Recall that if Z is a compact metric space with distance d and $\alpha \in (0, 1)$, then the Hölder space Z^α is just Z with the new distance d^α. It is well-known [**11**, Proposition 3.2.2(b)] that such a $\mathrm{lip}(Z^\alpha)$ separates points boundedly.

The proof in [**6**] can be shortened just invoking duality. Indeed, if $T : \mathrm{lip}(Y) \to \mathrm{lip}(X)$ is a linear bijection preserving the order, then T is continuous (this is proved in [**6**] for Hölder metrics, but the proof goes undisturbed in the general case), and therefore the Banach space double adjoint $T^{**} : \mathrm{lip}(Y)^{**} \to \mathrm{lip}(X)^{**}$ is a bounded linear homeomorphism. On the other hand the separation hypothesis implies $\mathrm{lip}(Y)^{**} = \mathrm{Lip}(Y)$, so T 'extends' to a linear bijection $T^{**} : \mathrm{Lip}(Y) \to \mathrm{Lip}(X)$ that preserves order in both directions (this is easily checked). Hence $T^{**}f = a \cdot (f \circ \tau)$, where $a = T1$ and $\tau : X \to Y$ is a Lipschitz homeomorphism and the same is true for T. Actually this representation is valid for any pair of metric spaces satisfying that the bidual of the little Lipschitz lattice is the big Lipschitz lattice (this can happen even if the involved spaces are not compact; see [**10**]). Needless to say this pattern cannot be followed if T fails to be linear. Instead, we will use the following extension result for little Lipschitz functions on compact spaces [**11**, Theorem 3.2.6(a)]: if $\mathrm{lip}(Z)$ separates points boundedly, then given a closed set Z_0 every $f_0 \in \mathrm{lip}(Z_0)$ with $\Lambda(f_0) < \Lambda$ can be extended to a $f \in \mathrm{lip}(Z)$ with $\Lambda(f) < \Lambda$ and $\|f\|_\infty = \|f_0\|_\infty$.

This clearly implies that $\mathrm{lip}(Z)$ is a uniformly separating vector lattice. Actually it is even a Banach algebra with the norm inherited from $\mathrm{Lip}(Z)$.

Proof of Theorem 2. Let $T : \mathrm{lip}(Y) \to \mathrm{lip}(X)$ be a lattice isomorphism. The first part of the proof goes as in Section 1. However this time $R\,\mathrm{lip}(X)$ need not contain every regular open set and we must replace the neighbourhoods V_n by U_{h_n}, where $h_n \in \mathrm{lip}(X)$ equals 1 on V_{n+1} and vanishes outside V_n.

Moreover [**11**, Proposition 3.1.3], if $\varphi \in \mathrm{Lip}(\mathbb{R})$ and $f \in \mathrm{lip}(Z)$ one has $\varphi \circ f \in \mathrm{lip}(Z)$, so the second part applies verbatim. Thus, we have the following representation

$$Tf(x) = t(x, f(\tau(x))) \qquad (f \in \mathrm{lip}(Y), x \in X) \tag{4}$$

where τ is a homeomorphism and $t(x,c) = Tc(x)$.

Next we claim that T maps an open set of $\mathrm{lip}(Y)$ into a norm bounded subset of $\mathrm{lip}(X)$.

As little Lipschitz lattices do not enjoy the remarkable completeness properties of the big ones the first part of Lemma 4 is useless. However it follows from (4) that $t(x,c)$ is separately continuous in the second variable (a lattice automorphism of $\mathbb{R}$), so T preserves pointwise convergence. For if (f_n) converges pointwise to f in $\mathrm{lip}(Y)$ we have

$$Tf_n(x) = t(x, f_n(\tau(x))) \to t(x, f(\tau(x))) = Tf(x).$$

This shows that the sets $\{f : \|Tf\| \leq R\}$ are all closed in $\mathrm{lip}(Y)$ and by Baire's theorem some of them must have nonempty interior, as we claimed.

Now, using two translations if necessary we may and do assume that for certain $r, R > 0$ one has $\|Tf\| \leq R$ in $\mathrm{lip}(X)$ whenever $\|f\| \leq r$ in $\mathrm{lip}(Y)$. These numbers are fixed for the remainder of the proof. Besides, if we transfer the distance from Y to X through τ, we may consider Y is just X with another (equivalent) distance δ and that $T : \mathrm{lip}(X,\delta) \to \mathrm{lip}(X,d)$ has the form

$$Tf(x) = t(x, f(x)).$$

Now, the crucial estimates are the following:

(a) If $a, b \in [-r, r]$ and $|b-a| < r\delta(x,y)$, then $|t(x,a) - t(y,b)| \leq R \cdot d(x,y)$.
(b) If $0 \leq c < \delta(x,y)\lfloor 1/\delta(x,y) \rfloor$, then $|t(x,cr) - t(x,0)| \leq R \cdot d(x,y)/\delta(x,y)$.

This can be proved as we did in Lemma 5, using either the extension result for little Lipschitz functions we quoted before or the separation condition with k close to 1.

After that, the proof is easily completed. Let us see that the formal identity is Lipschitz from (X,d) to (X,δ). Assuming the contrary we find sequences (x_n) and (y_n) such that

$$\lim_{n\to\infty} \frac{d(x_n,y_n)}{\delta(x_n,y_n)} = 0.$$

Since d and δ are bounded, we see that both $d(x_n,y_n)$ and $\delta(x_n,y_n)$ converge to zero. So for every $c < 1$ we have $c < \delta(x_n,y_n)\lfloor 1/\delta(x_n,y_n) \rfloor$ for large n and from the estimate in (b) we get

$$t\left(x_n, \frac{r}{2}\right) - t(x_n, 0) \leq R\frac{d(x_n,y_n)}{\delta(x_n,y_n)}.$$

Let x be a cluster point of (x_n) —recall that X is compact. Then, taking limits in the above inequality, we have $t(x, r/2) = t(x,0)$, a contradiction. □

3. Uniformly separating lattices

In this Section we prove a nonlinear version of a relatively recent result by Maribel Garrido and Jesús Jaramillo on uniformly continuous functions. In the next result the involved lattices are not assumed to be linear. However, it is easily seen that uniformly separating lattices (in the sense of Section 1) must contain the constants 0 and 1 and we can adhere this requirement to the definition.

THEOREM 3. *Let Y and X be complete metric spaces and let $L(Y)$ and $L(X)$ be uniformly separating lattices. Suppose there is a lattice isomorphism $T : L(Y) \to L(X)$ such that $T0 = 0$ and $T1 = 1$. Then Y and X are uniformly homeomorphic.*

The proof follows the lines of Section 1.1, but due to the lack of structure in the involved lattices we need a different approach to get the point mapping $\tau : X \to Y$ out from the lattice isomorphism. The key point is the following general result where a lattice S of open sets of a given topological space X is said to be **basic** if it contains a base of the topology of X.

LEMMA 6. *Let $B(Y)$ and $B(X)$ be basic lattices of open sets for the complete metric spaces Y and X, respectively. If $\mathfrak{T} : B(Y) \to B(X)$ is a lattice isomorphism, then there exist dense subsets X' of X and Y' of Y and a homeomorphism $\tau : X' \to Y'$ such that given $x \in X'$ and $U \in B(Y)$ one has $x \in \mathfrak{T}(U)$ if and only if $\tau(x) \in U$.*

PROOF. Given $(x, y) \in X \times Y$, let us write $x \sim y$ if

$$\bigcap_{y\in U} \mathfrak{T}(U) = \{x\} \quad \text{and} \quad \bigcap_{x\in V} \mathfrak{T}^{-1}(V) = \{y\}$$

First of all notice that if $x \sim y$ and $x \sim y'$, then $y = y'$. Similarly, if $x \sim y$ and $x' \sim y$, then $x = x'$. Let X' be the set of those $x \in X$ for which there exists (a necessarily unique) $y \in Y$ such that $x \sim y$ and Y' the set of those $y \in Y$ such that $x \sim y$ for some $x \in X$. It is pretty obvious that the map $\tau : X' \to Y'$ sending each $x \in X'$ to the only $y \in Y'$ such that $x \sim y$ is a homeomorphism.

It remains to see that Y' is dense in Y. The corresponding statement for X' follows by symmetry.

Let U be a nonempty open subset of Y. We must show that U meets Y'. Take a nonempty $U_1 \in B(Y)$ such that $\overline{U}_1 \subset U$ and $\operatorname{diam} U_1 \le 1$. Choose a nonempty $V_1 \subset \mathfrak{T}(U_1)$, with $\operatorname{diam} V_1 \le 1$. Then choose a nonempty $U_2 \subset \mathfrak{T}^{-1}(V_1)$ with $\overline{U}_2 \subset U_1$ and $\operatorname{diam} U_2 \le 1/2$. Next, take a nonempty $V_2 \subset \mathfrak{T}(U_2)$ such that $\overline{V}_2 \subset V_1$ and $\operatorname{diam} V_2 \le 1/2$. In this way we get sequences (U_n) and (V_n) in $B(Y)$ and $B(X)$, respectively, such that, for each n:

- $\overline{U}_{n+1} \subset U_n$ and $\overline{V}_{n+1} \subset V_n$.
- U_n and V_n have diameter at most $1/n$.
- $\mathfrak{T}(U_{n+1}) \subset V_n \subset \mathfrak{T}(U_n)$.

Now, it is clear that there are $y \in Y$ and $x \in X$ such that

$$\{y\} = \bigcap_n U_n = \bigcap_n \overline{U}_n \quad \text{and} \quad \{x\} = \bigcap_n V_n = \bigcap_n \overline{V}_n.$$

From where it follows that $x \sim y$ and since $y \in U$ we see that Y' is dense in Y. □

PROOF OF THEOREM 3. There is no loss of generality in assuming that every function in $L(Y)$ or $L(X)$ takes values in $[0, 1]$. In any case one can replace $L(Y)$ by

$$L^{[0,1]}(Y) = \{0 \vee (f \wedge 1) : f \in L(Y)\}$$

and similarly with $L(X)$. Also, it is clear that the class of open sets $B(Y) = \{U_f : f \in L(Y)\}$ is a lattice. Moreover, for every $y \in Y$ and every neighbourhood U of y, there is $f \in L(Y)$ vanishing off U and such that $f(y) = 1$, so $B(Y)$ is a basic lattice of (regular) open sets of Y, and similarly for X. Next we define a mapping $\mathfrak{T} : B(Y) \to B(X)$ sending U_f into U_{Tf}. The definition makes sense because Part (b) of Lemma 1 remains true replacing $\operatorname{Lip}(X)$ by $L(X)$. Next, we claim that, for $i = 0, 1$ one has $f = i$ on $U \in S(Y)$ if and only if $Tf = i$ on $\mathfrak{T}(U)$. And this is so because $f = 0$ on U_h is equivalent to $f \wedge h = 0$, while $f = 1$ on U_h is equivalent to $f \wedge u = u$ whenever $u \subset h$.

Now we apply Lemma 6 to get a homeomorphism $\tau : X' \to Y'$ between dense subspaces in such a way that given $U \in S(Y)$ and $x \in X'$ one has $\tau(x) \in U$ if and only if $x \in \mathfrak{T}(U)$. We are going to see that τ is uniformly continuous on X'. Assuming the contrary we have sequences (x_n) and (x'_n) in X' such that $d(x_n, x'_n) \to 0$, while $d(y_n, y'_n)$ does not converge to zero, where $y_n = \tau(x_n), y'_n = \tau(x'_n)$. As

(y_n) and (y'_n) cannot converge to the same limit, there is an infinite set $M \subset \mathbb{N}$ and $\delta > 0$ such that $d(y_n, y'_m) > \delta$ for all $n, m \in M$. Set

$$A_0 = \bigcup_{n\in M} B(y_n, \delta/3) \quad \text{and} \quad A_1 = \bigcup_{n\in M} B(y'_n, \delta/3),$$

then $d(A_0, A_1) \geq \delta/3$ and there is $f \in L(Y)$ such that $f = i$ on A_i, for $i = 0, 1$. Therefore, for $n \in M$, we have $Tf(x_n) = 0$ and $Tf(x'_n) = 1$, a contradiction which completes the proof. $\square$

COROLLARY 3. *Let Y and X be complete metric spaces. Suppose $L(Y)$ and $L(X)$ are uniformly separating vector lattices of bounded functions that are isomorphic as mere lattices. Then Y and X are uniformly homeomorphic.*

PROOF. Let $T : L(Y) \to L(Y)$ be a lattice isomorphism. We may assume without loss of generality that $T0 = 0$. Put $u = 1_Y \vee T^{-1}1_X$ and $v = Tu = 1_X \vee T1_Y$. Next notice that since $u, v \geq 1$ Theorem 3 remains true if we replace the condition $T1 = 1$ by $Tu = v$ provided $L(Y)$ has the property that if A and B are subsets of Y such that $d(A, B) > 0$ there is $f \in L(Y)$ such that $f = 0$ on A and $f = u$ on B and $L(X)$ has the analogous property with respect to v.

To check the relevant condition for $L(Y)$, take sets A and B such that $d(A, B) > 0$. Take some $h \in L(Y)$ such that $h = 0$ on A and $h = 1$ on B. If $M \geq 0$ is any constant satisfying $M \geq u$, then $f = u \wedge Mh$ does what we need. $\square$

As a byproduct of the proof we have the following explicit description of the isomorphisms of lattices of regular open sets of complete metric spaces. Notice that regular open sets play a major rôle in lattice theory; see [**1**].

PROPOSITION 1. *Let Y and X be complete metric spaces with dense subspaces Y' and X'. Suppose $\tau : X' \to Y'$ is a homeomorphism. Then the mapping $\mathfrak{T} : R(Y) \to R(X)$ given by*

$$\mathfrak{T}(U) = \overset{\circ}{\overline{\tau^{-1}(U \cap Y')}} \tag{5}$$

is a lattice isomorphism. And, conversely, every lattice isomorphism arises in this way.

PROOF. The first part follows from the fact that $A \mapsto A' = A \cap Y'$ defines an isomorphism between $R(Y)$ and $R(Y')$ whose inverse is obtained sending $B \in R(Y')$ to the interior of the closure of B in Y, and similarly for X. Thus if $\tau : X' \to Y'$ is a homeomorphism between dense subspaces, then the map $\mathfrak{T}$ defined in (5) is just the composition

$$R(Y) \longrightarrow R(Y') \xrightarrow{\tau^{-1}} R(X') \longrightarrow R(X).$$

To prove this let us introduce the following notation. Given $B \subset Y'$, we write $\mathrm{cl}_{Y'}(B)$ for the closure of B in Y' and $\mathrm{int}_{Y'}(B)$ for the interior of B in Y'. As before, the bar and the circle stand, respectively, for the closure and the interior in the whole space Y. Now, we have:

- If A is open in Y, then $\mathrm{cl}_{Y'}(A \cap Y') = \overline{A} \cap Y'$.
- If F is closed in Y, then $\mathrm{int}_{Y'}(F \cap Y') = \overset{\circ}{F} \cap Y'$.

We check the first point only: the second one easily follows by duality taking complements. That $\mathrm{cl}_{Y'}(A \cap Y') \subset \overline{A} \cap Y'$ is trivial. The reversed inclusion is as follows. If $y \in \overline{A} \cap Y'$, there is a sequence (y_n) in A converging to y. As Y' is dense in A for each n there is $y'_n \in A \cap Y'$ such that $d(y_n, y'_n) < 1/n$. It follows that (y'_n) converges to y, which belongs to $\mathrm{cl}_{Y'}(A \cap Y')$, as required.

Let now A be open in Y and put $A' = A \cap Y'$. We have

$$\mathrm{int}_{Y'}\,\mathrm{cl}_{Y'}(A') = \mathrm{int}_{Y'}\,\mathrm{cl}_{Y'}(A \cap Y') = \mathrm{int}_{Y'}(\overline{A} \cap Y') = \overset{\circ}{\overline{A}} \cap Y'.$$

It follows that A' is regular if A is. Next, if $A, B \in R(Y)$ are such that $A \cap Y' = B \cap Y'$, then $A = B$. Indeed we have $\overline{A} = \overline{B}$ and so $A = B$.

It remains to see that each $C \in R(Y')$ can be obtained as the intersection of Y' with some member of $R(Y)$: but is is easily seen that taking the interior of the closure of C in Y suffices. This ends the proof of the first statement.

To prove the converse, let $\mathfrak{T} : R(Y) \to R(X)$ be an isomorphism and let $\tau : X' \to Y'$ be as in Lemma 6. It is pretty obvious from the definition of $x \sim y$ and the first part of the proof that given $U \in R(Y)$ one has $\mathfrak{T}(U) \cap X' = \tau^{-1}(U \cap Y')$, which implies (5). □

4. Counterexamples

The following example shows that the hypothesis on the diameters cannot be removed in Theorem 1. It also shows at once that the 'unital' character of the isomorphism is necessary in Theorem 3 and that Corollary 3 fails for lattices of unbounded functions. Please notice that linearity would not help!

EXAMPLE 1. Two non-uniformly homeomorphic complete metric spaces Y and X such that the lattices $\mathrm{Lip}(Y)$ and $\mathrm{Lip}(X)$ are (even linearly) isomorphic.

PROOF. Set $X = \{n^2 + i \in \mathbb{R} : n \in \mathbb{N}, n \geq 2, i = 0, 1\}$ and $Y = \{n + i/n \in \mathbb{R} : n \in \mathbb{N}, n \geq 2, i = 0, 1\}$. We equip these spaces with the restriction of usual distance in $\mathbb{R}$. Consider the map sending each $f \in \mathrm{Lip}(Y)$ into the function

$$Tf(n^2 + i) = nf(n + i/n).$$

We claim that T defines an (obviously linear) isomorphism between $\mathrm{Lip}(Y)$ and $\mathrm{Lip}(X)$. We take advantage of the fact that the Lipschitz constant of functions defined either on Y or on X can be computed using only 'adjacent' points, so

$$\Lambda_X(g) = \sup_{n\geq 2}\left\{|g(n^2+1) - g(n^2)|, \frac{|g((n+1)^2) - g(n^2+1)|}{2n}\right\},$$

while

$$\Lambda_Y(f) = \sup_{n\geq 2}\left\{n|f(n+1/n) - f(n)|, \frac{n}{n-1}|f(n+1) - f(n+1/n)|\right\}.$$

But,

$$|Tf(n^2+1) - Tf(n^2)| = n|f(n+1/n) - f(n)| \leq n\Lambda_Y(f)/n = \Lambda_Y(f)$$

and

$$\begin{aligned}
\frac{|Tf((n+1)^2) - Tf(n^2+1)|}{2n} &= \frac{|(n+1)f(n+1) - nf(n+1/n)|}{2n} \\
&= \frac{|(n+1)f(n+1) - (n+1)f(n+1/n) + f(n+1/n)|}{2n} \\
&\leq \frac{n+1}{2n}|f(n+1) - f(n+1/n)| + \frac{|f(n+1/n)|}{2n} \\
&\leq \frac{n+1}{2n} \cdot \Lambda_Y(f) \cdot \frac{n-1}{n} + \frac{|f(n+1/n) - f(2) + f(2)|}{2n} \\
&\leq \frac{3}{4}\Lambda_Y(f) + \frac{\Lambda_Y(f)}{2n}(n+1/n-2) + \frac{|f(2)|}{2n} \\
&\leq \frac{5}{4}\Lambda_Y(f) + |f(2)|,
\end{aligned}$$

so $\Lambda_X(Tf) \le \frac{5}{4}\Lambda_Y(f) + |f(2)|$ and T maps $\mathrm{Lip}(Y)$ into $\mathrm{Lip}(X)$. To see T is surjective let us show that for each $g \in \mathrm{Lip}(X)$ the function $f : Y \to \mathbb{R}$ given by

$$f(n + i/n) = \frac{g(n^2 + i)}{n}$$

is Lipschitz. Obviously one then has $Tf = g$. We have

$$\frac{|f(n + 1/n) - f(n)|}{1/n} = |g(n^2 + 1) - g(n^2)| \le \Lambda_X(g).$$

Also,

$$\begin{aligned}
\frac{n}{n-1}|f(n+1) - f(n+1/n)| &= \frac{n}{n-1}\left|\frac{g((n+1)^2)}{n+1} - \frac{g(n^2+1)}{n}\right| \\
&= \frac{|ng((n+1)^2) - (n+1)g(n^2+1)}{(n+1)(n-1)} \\
&\le \frac{n}{(n+1)(n-1)} \cdot \Lambda_X(g) \cdot 2n + \frac{|g(n^2+1)|}{(n+1)(n-1)} \\
&\le \frac{8}{3}\Lambda_X(g) + \frac{|g(n^2+1) - g(4) + g(4)|}{(n+1)(n-1)} \\
&\le \frac{11}{3}\Lambda_X(g) + \frac{|g(4)|}{3}.
\end{aligned}$$

Whence $\Lambda_Y(f) \le \frac{11}{3}\Lambda_X(g) + |g(4)|/3$, which completes the proof. □

Garrido and Jaramillo proved in [**4**, Theorem 3.10] that two complete metric spaces are Lipschitz homeomorphic if and only if there is a linear and **unital** lattice isomorphism between the corresponding spaces of Lipschitz functions. The above example shows that 'unital' is needed here. And the next one that neither 'linear' can be omitted.

Example 2. Let $\mathbb{N}_1$ denote the set of integers with the discrete metric instead of the usual metric. Obviously $\mathbb{N}_1$ is not Lipschitz homeomorphic with $\mathbb{N}$. However, there is a lattice isomorphism $T : \mathrm{Lip}(\mathbb{N}_1) \to \mathrm{Lip}(\mathbb{N})$ such that $T0 = 0$ and $T1 = 1$.

Proof. Notice $\mathrm{Lip}(\mathbb{N}_1)$ is nothing but the space of bounded sequences. Put

$$Tf(n) = \begin{cases} f(n) & \text{if } |f(n)| \le 1 \\ nf(n) & \text{otherwise} \end{cases}$$

It is easily verified that T defines an isomorphism of $\mathrm{Lip}(\mathbb{N}_1)$ onto $\mathrm{Lip}(\mathbb{N})$. □

5. Concluding remarks

The paper [**8**] contains the statement that two complete metric spaces are uniformly homeomorphic if the corresponding lattices of uniformly continuous functions are isomorphic [**8**, Theorem 6].

While it is apparent that Shirota's proof works for bounded functions (we refer the interested reader to [**2**] for a contemporary proof), a serious gap occurs in the 'unbounded' case. It is worth noticing that Nagata had already proved a closely related for bounded uniformly continuous functions which are uniformly continuous outside a finite set [**7**, Theorem 2].

Perhaps the following explanations are in order. Let $U(X)$ denote the lattice of all uniformly continuous functions on X and $U^*(X)$ the sublattice of bounded functions in $U(X)$. Consider Shirota's relations '$\subset$' and '$\Subset$' we used in Section 1 in $U^*(X)$ and $U(X)$. As before, $f \subset g$ is equivalent to $U_f \subset U_g$ both in $U^*(X)^+$ and in $U(X)^+$, but the meaning of $f \Subset g$ depends on the 'ambient' lattice.

Indeed, one has $f \Subset g \Leftrightarrow d(U_f, U_g^c) > 0$ in $U^*(X)$, by Lemma 1(c). However, the implication ($\Leftarrow$) may fail in $U(X)$. To see this, take $X = \mathbb{R}$ with the usual distance and the sets:

$$V = \bigcup_n (n - 1/8, n + 1/8) \qquad \text{and} \qquad W = \bigcup_n (n - 1/4, n + 1/4).$$

Clearly, $d(V, W^c) = 1/8$. Define f and g taking $f(x) = d(x, V^c)$ and $g(x) = d(x, W^c)$, so that $V = U_f$ and $W = U_g$. Let us see that the relation $f \Subset g$ does not hold in $U(\mathbb{R})$. Indeed, for $n \in \mathbb{N}$, let h_n be piecewise linear function defined by the conditions $h_n(n) = n, h_n(n \pm \frac{1}{8}) = 0$. Then $h_n \subset f$ for all n and the sequence (h_n) is bounded by $|\cdot|$. However no uniformly continuous function $h \subset g$ can be an upper bound for (h_n).

So, let us close the paper with the following.

Problem. Let Y and X be complete metric spaces such that the lattices $U(Y)$ and $U(X)$ are isomorphic. Must X and Y be uniformly homeomorphic? What if $U(Y)$ and $U(X)$ are linearly isomorphic?

It is apparent that the problem reduces to find a condition equivalent to '$d(U_f, U_g^c) > 0$' (or to '$d(U_f, U_g) = 0$') within the order structure of $U(X)^+$. This could be an impossible task: Example 1 shows that these conditions cannot be expressed within the order structure of $\mathrm{Lip}(X)$ if X has infinite diameter.

References

[1] G. Birkhoff, Lattice theory. Corrected reprint of the 1967 third edition. American Mathematical Society Colloquium Publications, 25. American Mathematical Society, Providence, R.I., 1979

[2] F. Cabello Sánchez, Homomorphisms on lattices of continuous functions, Positivity (to appear). Available at `http://kolmogorov.unex.es/~fcabello`

[3] M.I. Garrido and J.Á. Jaramillo, A Banach-Stone theorem for uniformly continuous functions. Monatsh. Math. 131 (2000), no. 3, 189–192

[4] — and —, Homomorphisms on function lattices, Monatsh. Math. 141 (2004) 127–146

[5] — and —, Variations on the Banach-Stone theorem, Extracta. Math. 17 (2002) no. 3, 351–385

[6] A. Jiménez-Vargas and M. Villegas-Vallecinos, Order isomorphisms of little Lipschitz algebras, Houston J. Math. (to appear)

[7] J. Nagata, On lattices of functions on toplogical spaces and of functions on uniform spaces, Osaka Math. J. 1 (2) (1949) 166–181.

[8] T. Shirota, A generalization of a theorem of I. Kaplansky. Osaka Math. J. 4 (1952) 121–132.

[9] N. Weaver, Lattices of Lipschitz functions, Pacific J. Math. 164 (1) (1994) 179–193

[10] —, Duality for locally compact Lipschitz spaces, Rocky Mountain J. Math. 26 (1996) 337–353

[11] —, Lipschitz Algebras, World Scientific, Singapore 1999.

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