Spanish Network of Functional Analysis and Applications

FUNCTIONAL ANALYSIS Network

Espacios de funciones integrables respecto de una medida vectorial. Propiedades y aplicaciones.

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UNIVERSIDAD DE SEVILLA

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People

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Introduction

- (Ω, Σ) measurable space (Σ is a $\sigma-$algebra in Ω).
- $m : \Sigma \longrightarrow X$ (countably additive) vector measure in a (real) Banach space X with dual X^*.
- The semivariation of m. For $A \in \Sigma$,

$$\|m\|(A) := \sup\{|\langle m, x^*\rangle|(A) : x^* \in X^*, \|x^*\| \leq 1\}$$

- $|\langle m, x^*\rangle|$ is the measure variation of the real measure defined by

$$\langle m, x^*\rangle(A) := \langle m(A), x^*\rangle.$$

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- $L^1_w(m)$ is the space of all (equivalence classes of) weakly integrable functions with respect to m. That is, of all real measurable functions $f : \Omega \longrightarrow \mathbb{R}$ such that f is integrable with respect to each $|\langle m, x^* \rangle|$, $x^* \in X^*$. $L^1_w(m)$ becomes a Banach lattice when it is equipped with the natural order m-a.e. and the norm

$$\|f\|_1 := \sup \left\{ \int_\Omega |f| \, d\,|\langle m, x^* \rangle| : \|x^*\| \leq 1 \right\}, \qquad f \in L^1_w(m).$$

For each $A \in \Sigma$ there exists an element $\zeta_f(A) \in X^{**}$ such that

$$\langle x^*, \zeta_f(A) \rangle = \int_A f \, d\langle m, x^* \rangle, \quad \forall x^* \in X^*.$$

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The spaces $L^p(m)$ and $L^p_w(m)$. [6]

$1 < p < \infty$,

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Properties of spaces $L^p(m)$.

$L^p(m), p \geq 1$. [6]

- $L^p(m)$ is an order-continuous Banach lattice with weak order unit.
- In $L^p(m)$ verifies a Dominated Convergence Theorem: Let (f_n) be a sequence of $p-$integrable functions which is pointwise convergent to f and g a $p-$integrable function such that $|f_n| \leq g$ for each n ($m-$a.e.). Then $f \in L^p(m)$ and (f_n) converges to f in norm
- If E is a $p-$convex order-continuous Banach lattice with a weak order unit, then E is lattice isomorphic to some $L^p(m)$.

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- [4, Th. 4] Let E be any $p-$convex Banach lattice with the $\sigma-$Fatou property and possessing a weak unit which belongs to $E_a := \{x \in E : |x| \geq u_n \downarrow 0 \text{implies} \|u_n\| \downarrow 0\}$ (the elements of E with $\sigma-$order continuous norm). Then there exists a vector measure m such that E is Banach lattice isomorphic to $L^p_w(m)$.

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Properties of spaces $L^p(m)$ and $L^p_w(m)$. Reflexivity.

Theorem. [6, Cor. 3.10]

For every $p > 1$, the following conditions are equivalent:

(a) $L^p_w(m)$ is reflexive.

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(d) $L^p_w(m) = L^p(m)$ as Banach lattices.

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Remarks.

- $L^1(m)$ and $L^1_w(m)$ may be reflexive ($\Longrightarrow$ they are equal).
- $L^p(m)$ and $L^p_w(m)$, $p > 1$, may be non reflexive ($\Longleftrightarrow L^p(m) \neq L^p_w(m)$).

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Properties of spaces $L^p(m)$ and $L^p_w(m)$. Inclusions.

Proposition. [6]

For $p > q \geq 1$ we have $L^p_w(m) \subset L^q(m)$ and this inclusion is a $L-$weakly compact operator. In particular, it is a weakly compact operator.

Remarks.

- The unit ball of $L^p_w(m)$ is weakly relatively compact in each $L^q(m)$ where it is included.
- $H \subset L^p(m)$ bounded $\Rightarrow$ $q-$uniformly integrable, $1 \leq q < p$,

$$\lim_{\|m\|(A)\to 0} \|f\chi_A\|_q = 0 \text{ uniformly in } H.$$

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The integration operator.

Theorem

For $p > 1$,

a) *The integration map* $I_m : f \in L^p_w(m) \longrightarrow I_m(f) := \int f \, dm \in X$ *is weakly compact.*

b) *The integration map* $I_m : f \in L^p_w(m) \longrightarrow I_m(f) := \int f \, dm \in X$ *is compact if and only if the range of m is a relatively compact subset of X.*

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Theorem (S. Okada, W. Ricker, L. Rodriguez-Piazza, [9]) The integration map $I_m : f \in L^1(m) \longrightarrow I_m(f) := \int f\, dm \in X$ is compact if and only if m verifies

(i) m has finite variation.

(ii) m has a Radon-Nikodým derivative $G = \frac{dm}{d|m|} \in L^1(|m|, X)$ with respect to $|m|$.

(iii) m the function G has $|m|-$essentially relatively compact range in X.

In this case $L^1(m) = L^1(|m|)$ and $I_m(f) = \int fG\, d|m|$ for all $f \in L^1(m)$.

Related Spaces. [5].

Definition.

Given a norming set $\Lambda \subset X^*$ let us consider the space $L^{1,s}_{\Lambda}(\nu)$ of all $f \in L^0(m)$ such that

a) $f \in L^1(\langle m, x^* \rangle)$ for every $x^* \in \Lambda$.

b) for each $A \in \Sigma$ there is $\xi_{f,\Lambda}(A) \in X$ such that

$$\langle \xi_{f,\Lambda}(A), x^* \rangle = \int_A f d\langle m, x^* \rangle \quad \text{for every } x^* \in \Lambda.$$

Related Spaces

Proposition

Given a norming set $\Lambda \subset X^*$,

a) $L^1(m) \subset L^{1,s}_\Lambda(m) \subset L^1_w(m)$.

b) $\left(L^{1,s}_\Lambda(m), \| \ \|_1\right)$ *is a Banach lattice.*

Remark

- *Topological conditions on* Λ *in order to obtain*

$$L^{1,s}_\Lambda(m) = L^1(m) \text{ or } L^{1,s}_\Lambda(m) = L^1_w(m).$$

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Multiplication operators. [3].

•Let us consider multiplication operators

$$M_g : f \in \mathcal{F} \longrightarrow M_g(f) := gf \in \mathcal{G} \tag{1}$$

between function spaces $\mathcal{F}$ and $\mathcal{G}$ where each space is an $L^p(m)$ or $L^p_w(m)$ space.

•**Remark.** If $p, q > 1$ are conjugate exponents

$$\begin{aligned} L^1(m) &= L^p(m) \cdot L^q(m) = \\ &= L^p_w(m) \cdot L^q(m) = L^p(m) \cdot L^q_w(m). \\ L^1_w(m) &= L^p_w(m) \cdot L^q_w(m). \end{aligned}$$

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Multiplication operators from $L^p(m)$ into $L^1(m)$, $p > 1$.

Theorem

Let $p, q > 1$ be conjugate exponents and $g \in L^0(m)$.

(1) $g \in L^q_w(m) \iff M_g : L^p(m) \longrightarrow L^1(m)$ *is continuous.*

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Multiplication operators from $L^p(m)$ into $L^p(m)$, $p \geq 1$.

Theorem

Let $p \geq 1, g \in L^0(m)$ and m_δ the restriction of m to $G_\delta = \{\omega \in \Omega : |g(\omega)| \geq \delta\}$ for each $\delta > 0$.

(1) $M_g : L^p(m) \longrightarrow L^p(m)$ *is continuous* $\Longleftrightarrow g \in L^\infty(m)$.

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- *The condition $L^p(m_\delta) = L^p_w(m_\delta)$ for each $\delta > 0$ does not imply that $L^p(m_G) = L^p_w(m_G)$.*
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$M_g : L^p(m) \longrightarrow L^p(m)$ weakly compact

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Let $p \geq 1, g$ and m_δ as before.

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(2) *When m is an atomless vector measure with $\sigma-$finite variation, $M_g : L^1(m) \longrightarrow L^1(m)$ is weakly compact if and only if $g = 0$ $m-$a.e.*

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Let $p \geq 1, g$ and m_δ as before. $M_g : L^p_w(m) \longrightarrow L^p_w(m)$ is compact $\iff g \in L^\infty(m)$ and $\dim(L^p(m_\delta)) < \infty$ for each $\delta > 0$.

Furthermore, in this case, $m(\Sigma_G) := \{m(A) : A \in \Sigma, A \subseteq G\}$,

$$\left\{ \frac{1}{(\|m\|(A))^{\frac{1}{p}}} \int_A g\, dm : A \in \Sigma^+ \right\} \text{ and } \left\{ \int_A g\, dm : A \in \Sigma \right\}$$

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Complex Interpolation. First Method of Calderón. [2]

Let (X_0, X_1) be a Banach couple of complex Banach spaces: Denote by $\mathcal{F}(X_0, X_1)$ the space of all functions $f : \overline{\Omega} \longrightarrow X_0 + X_1$, ($\Omega = \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$) having the following properties:

- f is continuous and bounded in the norm of $X_0 + X_1$.
- f is analytic in Ω in the norm of $X_0 + X_1$
- $t \in \mathbb{R} \longrightarrow f(it) \in X_0$ is continuous and bounded.
- $t \in \mathbb{R} \longrightarrow f(1 + it) \in X_1$ is continuous and bounded.

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- For all $t_1, t_2 \in \mathbb{R}$, $g(j + it_1) - g(j + it_2) \in X_j, j = 0, 1$ and

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Complex Interpolation of $L^p(m)$ spaces.

Given the Banach interpolation couple (X_0, X_1) where $X_0 = L^{p_0}(m)$ or $L^{p_0}_w(m)$ and $X_1 = L^{p_1}(m)$ or $L^{p_1}_w(m)$, $1 \leq p_0 \neq p_1$ we obtain $[X_0, X_1]_{[\theta]}$ and $[X_0, X_1]^{[\theta]}$, through their relationship with

- the Calderón-Lozanovskii's construction: $X_0^{1-\theta} X_1^{\theta}$, (over Banach lattices of measurable functions)

$$\left\{x : |x| \leq \lambda |x_0|^{1-\theta} |x_1|^{\theta}, x_0 \in B_1(X_0), x_1 \in B_1(X_1)\right\}$$

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- the Gustavsson-Peetre's method: $\langle X_0, X_1, \theta\rangle, 0 < \theta < 1$. The space of elements $x \in X_0 + X_1$ such that there exists a sequence $(x_k)_{\in\mathbb{Z}}$ of elements $x_k \in X_0 \cap X_1$ such that
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Relations

- For a Banach couple (see [8, Th. 5 and Section 7])

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- For a Banach couple of lattices of measurable functions (norm one inclusions)

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- If X_0 or X_1 has order continuous norm,

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- If $X_0^{1-\theta} X_1^{\theta}$ has the Fatou property,

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- If X_0 or X_1 has order continuous norm,

$$[X_0, X_1]_{[\theta]} = X_0^{1-\theta} X_1^{\theta} \text{ (isometry)}.$$

- If $X_0^{1-\theta} X_1^{\theta}$ has the Fatou property,

$$[X_0, X_1]^{[\theta]} = X_0^{1-\theta} X_1^{\theta} \text{ (isometry)}.$$

Relations

- For a Banach couple (see [8, Th. 5 and Section 7])

$$\langle X_0, X_1, \theta\rangle \subset [X_0, X_1]^{[\theta]}.$$

- For a Banach couple of lattices of measurable functions (norm one inclusions)

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Complex Interpolation of $L^p(m)-$spaces. [7].

Theorem

Let $1 \le p_0 \neq p_1 < \infty, 0 < \theta < 1$ *be and* $p(\theta)$ *defined by* $\dfrac{1}{p(\theta)} = \dfrac{1-\theta}{p_0} + \dfrac{\theta}{p_1}$. *Then:*

$$(1) \quad [L^{p_0}(m), L^{p_1}(m)]_{[\theta]} = \left[L^{p_0}_w(m), L^{p_1}(m)\right]_{[\theta]} = \left[L^{p_0}_w(m), L^{p_1}_w(m)\right]_{[\theta]} = L^{p(\theta)}(m).$$

$$(2) \quad [L^{p_0}(m), L^{p_1}(m)]^{[\theta]} = \left[L^{p_0}_w(m), L^{p_1}(m)\right]^{[\theta]} = \left[L^{p_0}_w(m), L^{p_1}_w(m)\right]^{[\theta]} = L^{p(\theta)}_w(m).$$

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