Mixed-frequency VAR models with Markov-switching dynamics

Maximo Camacho[*]
Universidad de Murcia

Abstract: This paper extends the Markov-switching vector autoregressive models to accommodate both the typical lack of synchronicity that characterizes the real-time daily flow of macroeconomic information and economic indicators sampled at different frequencies. The results of the empirical application suggest that the model is able to capture the features of the NBER business cycle chronology very accurately.

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[*] M. Camacho, Departamento de Métodos Cuantitativos para la Economía y la Empresa. Facultad de Economía y Empresa. Campus de Espinardo, 30100 Universidad de Murcia, Spain. Phone: +34 968 367982 Fax: +34 968 367905. E-mail: mcamacho@um.es

1. Introduction

Early proposed by Sims (1980), the Vector Autoregression (VAR) specification is one of the most successful, flexible and easy to use models for the analysis of multivariate time series. Since the influential work of Hamilton (1989) many authors have used Markov-switching extensions of these models to capture the business cycle regime shifts typically observed in economic data. Some examples of Markov-Switching VAR (MS-VAR) models are Krolzig (1997), Camacho and Perez Quiros (2002) and Paap, Sergers and van Dijk (2009).

Although economic data are rarely collected at the same instances in time, these standard MS-VAR applications are restricted to use economic indicators that must be available at the same frequency. In addition, these applications also rely on the unrealistic assumption that the real-time data flow of all the variables involved in the empirical analyses occurs at the same time.

To overcome these drawbacks, this paper develops a model that extends to a Markov-switching context the Mixed Frequency VAR (MF-VAR) model proposed by Mariano and Murasawa (2011). I call this model Markov-Switching Mixed Frequency VAR (MS-MF-VAR). The extension offers the interesting additional information of converting the business cycle signals provided by several quarterly and monthly economic indicators into recession probabilities.

I apply the method to US coincident indicators to compute inferences of the US business cycles. The component indicators, used by Mariano and Murasawa (2011) in a linear MF-VAR model to construct a monthly index of the economic activity, are quarterly real GDP and the four monthly coincident indicators that make up the coincident indicator currently released by the Conference Board. Using these indicators in a Markov-switching MF-VAR model, the inferred monthly probabilities of recession suggest that the nonlinear extension of the model is capable of identifying the US business cycles with very high accuracy.

2. MF-VAR models with Markov-switching

2.1. Specification

Let X_t, be a vector of N economic indicators that may include N_1 quarterly indicators, $X_{t,1}$, and N_2 monthly indicators, $X_{t,2}$. Let me assume that ln X_t, contains a unit root. To start with, let me assume that all the indicators are observed every month, although I will relax this assumption soon.

If the sample mean of the within quarter activity can be well approximated by the geometric mean, Mariano and Murasawa (2003) show that the quarter-over-quarter growth rate of quarterly indicators computed at each month of the sample, $Y_{t,1}^q$, can be expressed as the averaged sum of previous month-over-month growth rates

$$Y_{t,1}^q = \frac{1}{3}Y_{t,1}^m + \frac{2}{3}Y_{t-1,1}^m + Y_{t-2,1}^m + \frac{2}{3}Y_{t-3,1}^m + \frac{1}{3}Y_{t-4,1}^m . \quad (1)$$

The month-over-month growths of the economic indicators, $Y_t^m = \left(Y_{t,1}^m{}', Y_{t,2}^m{}'\right)'$, are then assumed to follow a Markov-switching VAR(p) process. Therefore, the constant terms, the autoregressive coefficients and the covariance matrix are driven by an unobservable two-state variable s_t

$$Y_t^m = \eta_{s_t} + \phi_{1,s_t} Y_{t-1}^m + \ldots + \phi_{p,s_t} Y_{t-p}^m + \varepsilon_t, \ldots\ldots\ldots \qquad \ldots\ldots\ldots\ldots(2)$$

where $\varepsilon_t \sim iN\left(0, \Sigma_{s_t}\right)$. The state variable is assumed to evolve according to an irreducible 2-state Markov chain whose transition probabilities are defined by

$$p\left(s_t = j / s_{t-1} = i, s_{t-2} = h, \ldots, \varphi_{t-1}\right) = p\left(s_t = j / s_{t-1} = i\right) = p_{ij}, \qquad (3)$$

where i,j=0,1, and φ_t refers to the information set up to period t. In short, this model endogenously permits the model parameters to switch as the date and regime changes.

The MF-VAR specification this model can be easily stated in state space representation and estimated by using the Kalman filter. Let me assume that $p \leq 5$, for instance p=1.[1] Calling $Y_t = \left(Y_{t,1}^q{}', Y_{t,2}^m{}'\right)'$, the measurement equation, $Y_t = H_{s_t}\beta_t + E_t$, where $E_t \sim iN\left(0, R\right)$, can be defined as

$$Y_t = \left[\begin{pmatrix} \frac{1}{3}I_{N_1} & 0 \\ 0 & I_{N_2} \end{pmatrix}, \begin{pmatrix} \frac{2}{3}I_{N_1} & 0 \\ 0 & I_{N_2} \end{pmatrix}, \begin{pmatrix} I_{N_1} & 0 \\ 0 & I_{N_2} \end{pmatrix}, \begin{pmatrix} \frac{2}{3}I_{N_1} & 0 \\ 0 & I_{N_2} \end{pmatrix}, \begin{pmatrix} \frac{1}{3}I_{N_1} & 0 \\ 0 & I_{N_2} \end{pmatrix}\right] \begin{pmatrix} Y_t^m \\ Y_{t-1}^m \\ Y_{t-2}^m \\ Y_{t-3}^m \\ Y_{t-4}^m \end{pmatrix}, \qquad (4)$$

where E_t=0 and R=0. The transition equation, $\beta_t = \mu_{s_t} + F_{s_t}\beta_{t-1} + V_t$, with $V_t \sim iN\left(0, Q_{s_t}\right)$, can be stated as

$$\begin{pmatrix} Y_t^m \\ Y_{t-1}^m \\ Y_{t-2}^m \\ Y_{t-3}^m \\ Y_{t-4}^m \end{pmatrix} = \begin{pmatrix} \eta_{s_t} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \phi_{1,s_t} & 0 & 0 & 0 & 0 \\ I_N & 0 & 0 & 0 & 0 \\ 0 & I_N & 0 & 0 & 0 \\ 0 & 0 & I_N & 0 & 0 \\ 0 & 0 & 0 & I_N & 0 \end{pmatrix} \begin{pmatrix} Y_{t-1}^m \\ Y_{t-2}^m \\ Y_{t-3}^m \\ Y_{t-4}^m \\ Y_{t-5}^m \end{pmatrix} + \begin{pmatrix} \varepsilon_t \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad (5)$$

where

[1] Modifying the expressions for higher lag orders is straightforward.

$$Q_{s_t} = \begin{pmatrix} \Sigma_{s_t} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (6)$$

2.2. Estimation and signal extraction

The Kalman filter can be used to estimate model's parameters and to infer unobserved components. Starting the algorithm with initial values $\beta_{0|0}$ and $P_{0|0}$, the prediction equations when $s_{t-1}=i$ and $s_t=j$ are

$$\beta_{t|t-1}^{(i,j)} = \mu_j + F_j \beta_{t-1|t-1}^{i}, \quad (7)$$

$$P_{t|t-1}^{(i,j)} = F_j P_{t-1|t-1}^{i} F_j' + Q_j, \quad (8)$$

where $\beta_{t|t-1}^{(i,j)}$ is an inference on β_t conditional on information up to t-1 given the states $s_{t-1}=i$ and $s_t=j$, $P_{t|t-1}^{(i,j)}$ is its covariance matrix, $\beta_{t-1|t-1}^{i}$ is an inference on β_{t-1} based on information up to t-1 given the state $s_{t-1}=i$ and $P_{t-1|t-1}^{i}$ is its covariance matrix.

These expressions can be used to compute prediction errors and their covariance matrix

$$v_{t|t-1}^{(i,j)} = Y_t - H_j \beta_{t|t-1}^{(i,j)}, \quad (9)$$

$$\Omega_{t|t-1}^{(i,j)} = H_j P_{t|t-1}^{(i,j)} H_j', \quad (10)$$

which can be used to evaluate the log likelihood function

$$l_t = \sum_{j=1}^{2} \sum_{i=1}^{2} -\frac{1}{2} \left[\ln\left(2\pi \left|\Omega_{t|t-1}^{(i,j)}\right|\right) + v_{t|t-1}^{(i,j)}{}' \left(\Omega_{t|t-1}^{(i,j)}\right)^{-1} v_{t|t-1}^{(i,j)} \right]. \quad (11)$$

At each iteration, one can use the nonlinear filter proposed by Hamilton (1989), to compute the joint probabilities $p\left(s_t = j, s_{t-1} = i / \varphi_t\right)$ and the filtered probabilities $p\left(s_t = j / \varphi_t\right)$.

Once the observed variables are realized at the end of time t, in each iteration the state vector and its covariance matrix are updated as follows

$$\beta_{t|t}^{(i,j)} = \beta_{t|t-1}^{(i,j)} + P_{t|t-1}^{(i,j)} H_j' \left(\Omega_{t|t-1}^{(i,j)}\right)^{-1} v_{t|t-1}^{(i,j)} \quad (12)$$

$$P_{t|t}^{(i,j)} = P_{t|t-1}^{(i,j)} - P_{t|t-1}^{(i,j)} H_j' \left(\Omega_{t|t-1}^{(i,j)}\right)^{-1} H_j P_{t|t-1}^{(i,j)}. \quad (13)$$

As noted by Kim (1994), the above Kalman filter produces 2-fold increases in the total number of cases to consider. Thus, if the filter does not reduce the number of terms at each time t, it becomes computationally unfeasible. In line with this author, I propose to collapse the Kalman filter to a single posterior at each t and substitute (12) and (13) by

$$\beta_{t|t}^{j} = \frac{\sum_{i=1}^{2} p\left(s_t = j, s_{t-1} = i / \varphi_t\right) \beta_{t|t}^{(i,j)}}{p\left(s_t = j / \varphi_t\right)}, \quad (14)$$

$$P_{t|t}^{j} = \frac{\sum_{i=1}^{2} p\left(s_t = j, s_{t-1} = i/\varphi_t\right)\left[P_{t|t}^{(i,j)} + \left(\beta_{t|t}^{j} - \beta_{t|t}^{(i,j)}\right)\left(\beta_{t|t}^{j} - \beta_{t|t}^{(i,j)}\right)'\right]}{p\left(s_t = j/\varphi_t\right)}, \quad (15)$$

respectively.

Once parameters are estimated, the smoothed probabilities, which are based on all the information available in the sample, can be estimated iteratively from the filtered probabilities as follows

$$p\left(s_t = j/\varphi_T\right) = \sum_{k=1}^{2} \frac{p\left(s_{t+1} = k/\varphi_T\right) p\left(s_t = j/\varphi_t\right) p\left(s_{t+1} = k/s_t = j\right)}{p\left(s_{t+1} = k/\varphi_t\right)}. \quad (16)$$

The smoothed probabilities are very useful to evaluate the in-sample accuracy of a Markov-switching model to capture the business cycles.

2.3. Dealing with missing observations

So far, we have assumed that all the variables included in the model are always available at monthly frequencies for all time periods. However, this assumption is quite unrealistic when using the model to compute real-time inferences of the business cycles for two reasons. The first reason has to do with mixing quarterly and monthly frequencies. Since quarterly data is only observed in the third month of the respective quarter, quarterly indicators exhibit two missing observations in the first two months of each quarter. The second reason has to do with the flow of real-time data, which implies that the publication lag of the indicators is different. Therefore, the data vintages typically exhibit missing data at the end of the sample of the indicators with larger publication delays.

As described in Mariano and Murasawa (2003), the system of equations remains valid with missing data after a subtle transformation. These authors propose replacing the missing observations with random draws r_t, whose distribution cannot depend on the parameter space that characterizes the Kalman filter.[2] Then, the measurement equation are transformed conveniently in order to allow the Kalman filter to skip the missing observations when updating.

Let Y_{it} be the i-th element of the vector Y_t and R_{ii} be its variance. Let His_t be the i-th row of the matrix Hs_t, which has $5N$ columns, and let 0_{5N} be a row vector of $5N$ zeroes. The measurement equation can be replaced by the following expressions

$$Y_{it}^{+} = \begin{cases} Y_{it} & \text{if } Y_{it} \text{ is observable} \\ r_t & \text{otherwise} \end{cases}, \quad (17)$$

$$H_{is_t}^{+} = \begin{cases} H_{is_t} & \text{if } Y_{it} \text{ is observable} \\ 0_{5N} & \text{otherwise} \end{cases}, \quad (18)$$

$$E_{it}^{+} = \begin{cases} E_{it} & \text{if } Y_{it} \text{ is observable} \\ r_t & \text{otherwise} \end{cases}, \quad (19)$$

[2] We assume that $r_t \sim N\left(0, \sigma_r^2\right)$ for convenience but replacements by constants would also be valid.

$$R_{is_t}^{+}=\begin{cases} R_{iis_t} & \text{if } Y_{it} \text{ is observable} \\ \sigma_r^2 & \text{otherwise} \end{cases}, \tag{20}$$

and $\mu_{s_t}^{+}$, which is a column vector with the drifts $\eta_{s_t}^{+}$ in the first 5 cells and $4N$ zeroes elsewhere, where

$$\eta_{is_t}^{+}=\begin{cases} \eta_{is_t} & \text{if } Y_{it} \text{ is observable} \\ 0 & \text{otherwise} \end{cases}. \tag{21}$$

This substitution leads to a time-varying state space model with no missing observations so the Kalman filter can be directly applied to Y_{it}^{+}, $\mu_{s_t}^{+}$, $H_{is_t}^{+}$, E_{it}^{+}, $R_{is_t}^{+}$.

3. Application

The purpose of this section is to show how the regime-switching MF-VAR described in the previous section works in empirical applications. Toward this end, I extend the empirical analysis of Mariano and Murasawa (2010), who construct a coincident index of the US economic activity from a linear MF-VAR, to additionally compute inferences of the US business cycles from a MS-MF-VAR. Following these authors, the component indicators are quarterly growth rates of real GDP and the four monthly coincident indicators that make up the Coincident Indicator released by the Conference Board: Employees on non-agricultural payrolls, Personal income less transfer payments, Index of industrial production and Manufacturing and trade sales, all of them in monthly growth rates.

The data vintage was downloaded on July, 18th 2013 and the sample period is from January 1960 to June 2012. It is worth noting that many missing data appear in this data vintage. According to the release calendar of the indicators used in the analysis, several outliers appear at the end of the time series. In particular, the latest figures of the indicators are March 2013 for GDP, March 2013 for sales, May 2013 for income and June 2013 for industrial production and employment. In addition, the quarterly GDP is observable every third period only.

Following Mariano and Murasawa (2010), I select a lag length p=1. Following Hamilton (1989) and Chauvet (1998) who consider that the shifts do not depend on the dynamics of the autoregressive process or the covariance matrices, only the drifts are allowed to switch. Figure 1 compares the monthly estimates of the US economic activity that are obtained from the linear MF-VAR model and from my Markov-switching extension. Both indicators are in consonance with the NBER-referenced business cycles, which are plotted as shaded areas. The positive growth rates are sometimes interrupted by broad changes of direction that seem to mark quite well the US recessions.

In spite of the similar performance of the linear and nonlinear approaches to construct monthly indicators of the US economic activity, the latter approach offers the interesting additional information of converting the business cycle signals provided by the economic indicators into recession probabilities. To assert the accuracy of the model to account for the business cycles, Figure 2 plots the values of the smoothed recession probabilities of state s_t=1. According to the reasonable matching between the quarters of high probabilities of state 1 and the NBER recessions, it is easy to interpret state 1 as recession and the series plotted in this figure as probabilities of being in recession. The probabilities are close to either zero or one, suggesting that the model is capturing well the underlying pattern of the dichotomous shifts between expansions and recessions.

In spite of the high correlation between the probabilities of recession and the NBER referenced recessions, the question is whether or not my new extension outperforms existing models. The natural competitor is the MS-VAR model of Krolzig (1997), Camacho and Perez Quiros (2002) and Paap, Sergers and van Dijk (2009). In addition, I check the relative performance of the model with the Markov-Switching Dynamic Factor (MS-DF) model proposed by Kim and Yoo (1995) and Chauvet (1998).[3] To quantify the relative ability of these models to detect the actual state of the business cycle, I compute Quadratic Probability Score (QPS), which is the mean squared deviation of the different probabilities of recession from a NBER-referenced recessionary dummy.

Table 1 shows that MF-MS-VAR outperforms MS-VAR and MS-DF, although the improvements with respect to MS-VAR are larger. The *p*-values of the null of different accuracy (Diebold and Mariano, 1995) show that the improvements are statistically significant in the case of MF-MS-VAR versus MS-VAR, but one cannot reject the null of equal predictive accuracy of MF-MS-VAR and MS-DF at any reasonable significance level. However, this analysis is in-sample and omits the effect of the asynchronous releases that characterizes the real-time flow of macroeconomic information.

To perform a more realistic assessment of the actual empirical reliability of MF-MS-VAR, I develop a pseudo real-time analysis as suggested by, among others, Giannone, Reichlin and Small (2008). Towards this end, I use the latest available dataset to construct data vintages that are recursively updated monthly in the middle of each month. The first data vintage of our experiment refers to September 15, 1976 and the last data vintage refers to July 15, 2013.

Since the publications dates of the indicators exhibit relatively stable calendars, the successive vintages can easily replicate the publication lags that characterize real-time analyses. It implies that the pseudo real-time probabilities inferred from MS-VAR and MS-DF must be computed as two-month-ahead forecasts since sales exhibit publication delays of about two months.[4] According to Table 1, MF-MS-VAR outperforms both MS-VAR and MS-DF since it exhibits lower QPS than these models. According to the Diebold-Mariano tests, the gains are statistically significant in both cases, although at significance levels greater than 0.05 in the latest case.

4. Conclusion

Nowadays, there has been a great deal of interest in modeling the business cycle features of multivariate time series. The MS-VAR models used in the empirical analyses are only of limited usefulness in practice since they are restricted to use economic indicators that must be sampled at the same frequencies and that cannot exhibit publication delays. The MF-VAR models are restricted to linear analysis and do not provide recession probabilities. To overcome these drawbacks, this paper extends the MS-VAR models to deal with the problem of mixed sampling frequencies and to account for asynchronously published economic indicators and the MF-VAR models to account for regime switching business cycle asymmetries.

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[3] In contrast to these authors, I do not impose a single-index factor structure in the model.

[4] All the model use parameters estimated from the latest available sample. Re-estimating the models is unfeasible since it would imply estimating large numbers of parameters with short samples.

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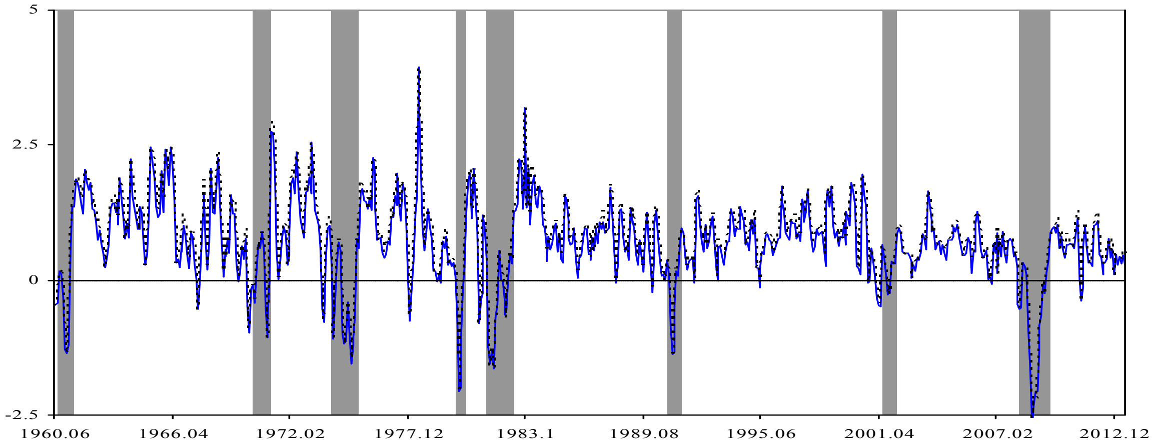
Table 1. Analysis of the relative performance

	In-sample 1960.06-2013.06	Pseudo real-time 1976.08-2013.06
	Quadratic probability score	
MS-MF-VAR	0.08	0.06
MS-VAR	0.18	0.18
MS-DF	0.09	0.08
	p-value of no different accuracy test	
MS-MF-VAR vs MS-VAR	0.00	0.00
MS-MF-VAR vs MS-DF	0.36	0.05

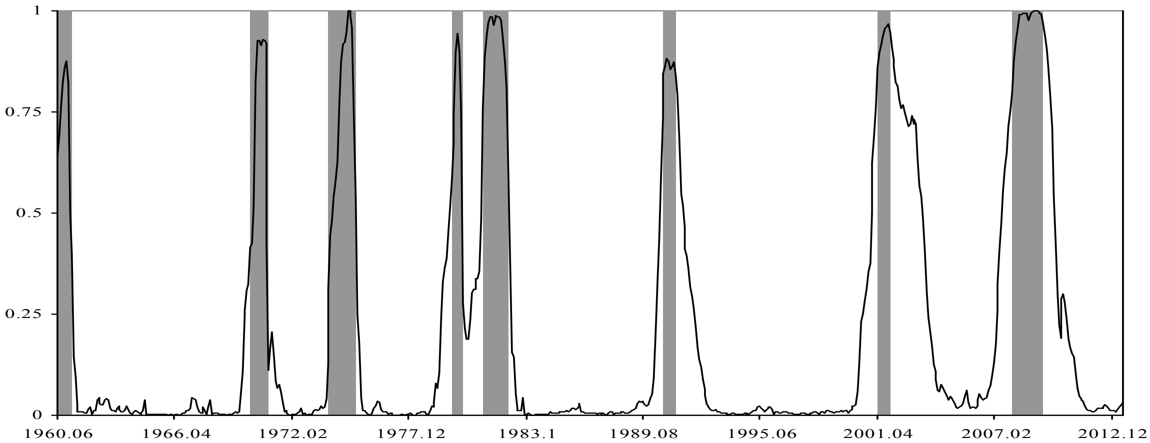
Notes. Quadratic probability score measures the mean squared deviation of the different types of inferences from a recessionary dummy that is constructed from the NBER business cycles. The test of no different accuracy is the Diebold and Mariano (1995) test.

Figure 1. Monthly indicators of quarterly growth rate



Notes. Dotted (straight) line refers to the monthly index of quarterly GDP growth rate from Markov-switching (lineal) mixed-frequency VAR model. Shaded areas correspond to recessions as documented by the NBER. The sample goes from 1960.06 to 2013.06.

Figure 2. Smoothed probabilities of state 1



Notes. Shaded areas correspond to recessions as documented by the NBER. The sample goes from 1960.06 to 2013.06.