Conditional Risk and Forcing Universes

(Dedicated to the memory of Bernardo Cascales)

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Partially based on joint work with Antonio Avilés (University of Murcia).

Duality theory of Risk Measure

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- The Fenchel transform of ρ is defined to be

$$\rho^{\#}(y) := \sup\{\mathbb{E}[xy] - \rho(x) : x \in \mathcal{X}\} \quad \text{for } y \in \mathcal{X}^{\#}.$$

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Theorem (K. Owari, 2014)

Let $\rho : \mathcal{X} \to \mathbb{R}$ be a convex risk measure. Then ρ is lower semi-continuous w.r.t. $\sigma(\mathcal{X}, \mathcal{X}^\#)$ if and only if ρ is representable, i.e.

$$\rho(x) = \sup\{\mathbb{E}[xy] - \rho^\#(y) \colon y \in \mathcal{X}^\#\} \quad \forall x \in \mathcal{X}.$$

In that case, the following conditions are equivalent:

1. *ρ attains the representation for each $x \in \mathcal{X}$;*
2. *ρ has the Lebesgue property, i.e.*

$$\lim_n x_n = x \text{ a.s.}, \ |x_n| \leq y, \ y \in \mathcal{X} \text{ implies } \lim_n \rho(x_n) = \rho(x);$$

3. *$\rho^\#$ is inf-compact w.r.t. $\sigma(\mathcal{X}^\#, \mathcal{X})$.*

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- Every single module or conditional analogue of a classical theorem needs an adaptation of a classical proof.
- Transfer method between two duality theories:

$$\text{Convex Risk Measures} \Longrightarrow \text{Conditional Risk Measures}.$$

A transfer method from duality theory of convex risk measures to duality theory of conditional risk measures

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"We must ask whether there is any interest in these nonstandard models aside from the independence proof; that is, do they have any mathematical interest? The answer must be yes, but we cannot yet give a really good argument."

Dana Scott, 1969.

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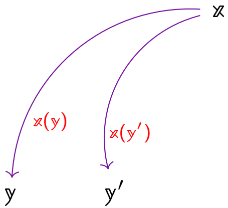

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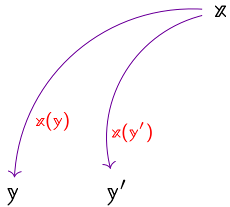

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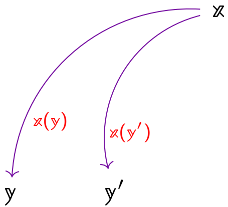

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- A full set-theoretic reasoning is possible.

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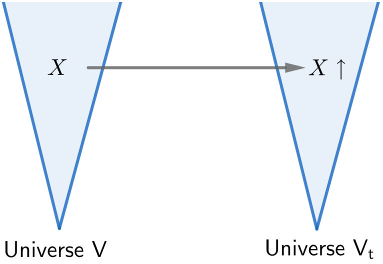

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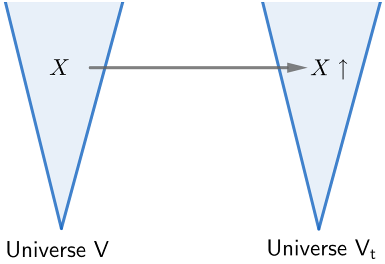

- If we manage to interpret a theorem about $X\uparrow$ as a statement about the original object X, we will have proved a new theorem about X.

Real numbers in the forcing universe V_t

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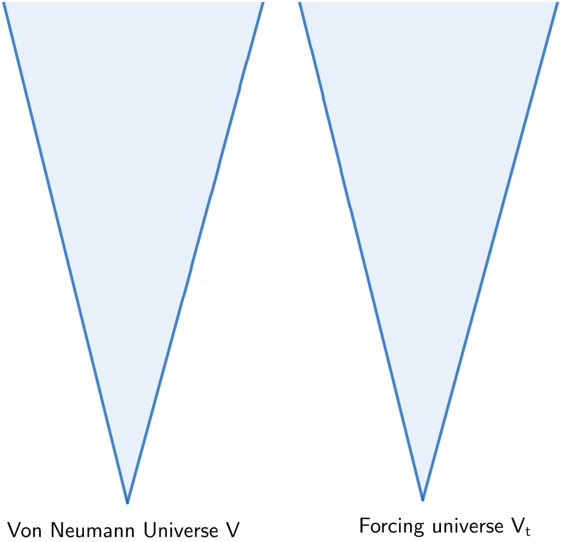

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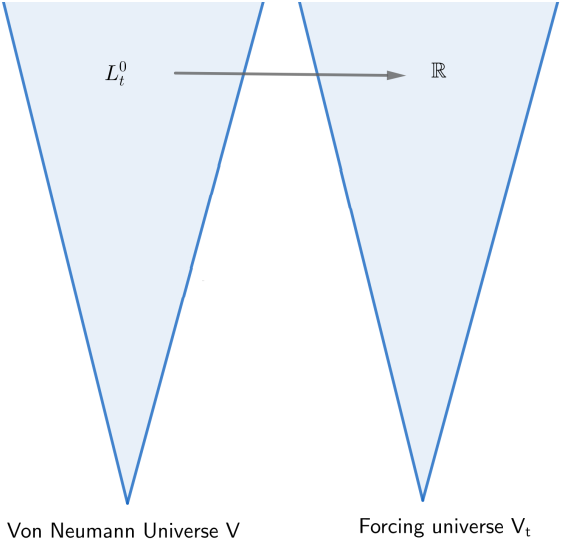

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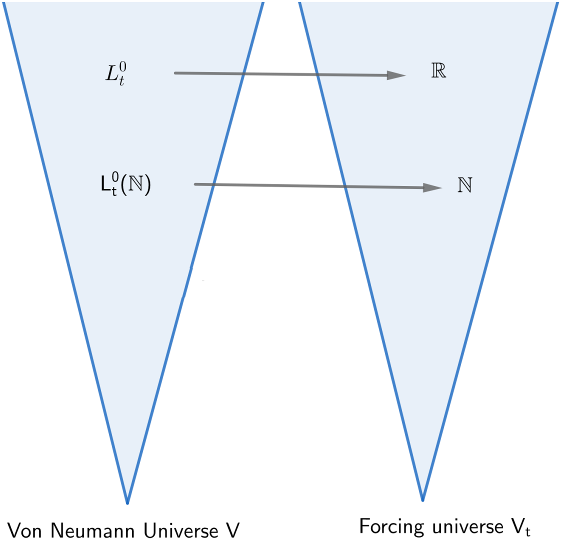

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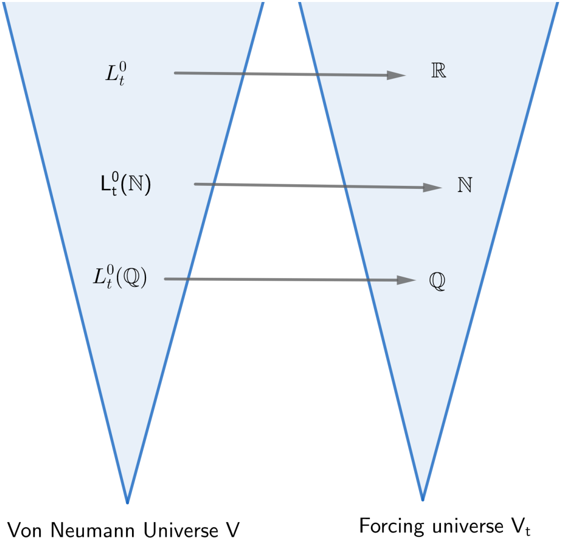

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- We say that ρ is representable if

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The pairing $\langle \mathscr{X}, \mathscr{X}^{\#} \rangle$ allows for the definition of a module analogue of the weak topologies, that we call stable weak topologies and denote by

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  For any $\eta \in L^0_t$, $\{\rho \leq \eta\}$ is closed w.r.t. $\sigma_s(\mathscr{X}, \mathscr{X}^{\#})$;

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$$\lim_n x_n = x \text{ a.s.}, \ |x_n| \leq z, \ z \in \mathscr{X} \text{ implies } \lim_n \rho(x_n) = \rho(x) \text{ a.s.}.$$

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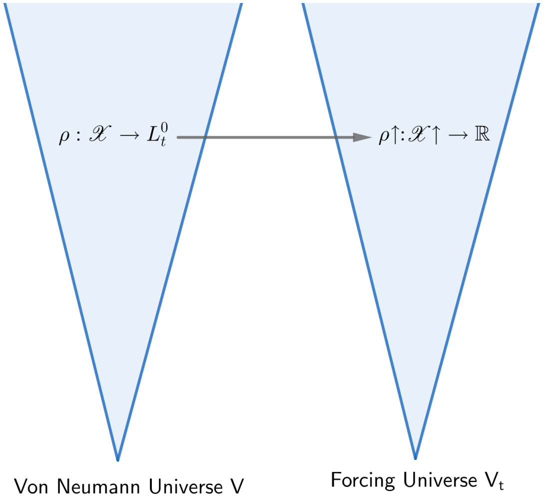

Robust representation of conditional risk measures

Recall the general version of the Jouini-Schachermayer-Touzi theorem:

Theorem

Let $\rho : \mathcal{X} \to \mathbb{R}$ be a convex risk measure. Then ρ is lower semi-continuous w.r.t. $\sigma(\mathcal{X}, \mathcal{X}^{\#})$ if and only if ρ is representable, i.e.

$$\rho(x) = \sup\{\mathbb{E}[xy] - \rho^{\#}(y) \colon y \in \mathcal{X}^{\#}\} \quad \forall x \in \mathcal{X}.$$

In that case, the following conditions are equivalent:

1. *ρ attains the representation for each $x \in \mathcal{X}$;*
2. *ρ has the Lebesgue property, i.e.*

$$\lim_n x_n = x \text{ a.s.}, \ |x_n| \leq y, \ y \in \mathcal{X} \text{ implies } \lim_n \rho(x_n) = \rho(x);$$

3. *$\rho^{\#}$ is inf-compact w.r.t. $\sigma(\mathcal{X}^{\#}, \mathcal{X})$.*

Robust representation of conditional risk measures

Thanks to the transfer principle we derive the following robust representation theorem:

Theorem

Let $\rho : \mathscr{X} \to L^0_t$ be a conditional risk measure. Then ρ is $\mathcal{F}_t$-lower semi-continuous w.r.t. $\sigma_s(\mathscr{X}, \mathscr{X}^\#)$ if and only if ρ admits a representation

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Examples of model spaces

- L^∞ type modules:

$$L_{t,T}^{\infty} := \left\{ x \in L_T^0 \colon \exists \eta \in L_t^0 \text{ such that } |x| \leq \eta \right\}.$$

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- Orlicz type modules: Suppose that $\phi : [0,\infty) \to [0,\infty]$ is a Young function

$$L^\phi_{t,T} := \left\{x \in L^0_T \colon \exists \varepsilon \in L^0_t,\ \varepsilon > 0 \text{ a.s.},\ E[\phi(\varepsilon^{-1}|x|)|\mathcal{F}_t] < \infty \text{ a.s.}\right\}.$$

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- Orlicz-heart type modules: Suppose that $\phi : [0, \infty) \to [0, \infty]$ is a Young function

$$H^\phi_{t,T} := \left\{ x \in L^0_T \colon \forall \varepsilon \in L^0_t,\ \varepsilon > 0 \text{ a.s., } E[\phi(\varepsilon^{-1}|x|)|\mathcal{F}_t] < \infty \text{ a.s.} \right\}.$$

References

References

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Thank you for your attention!