Plasticity of the unit ball and related problems

A survey of joint results with B. Cascales, C. Angosto, J. Orihuela, E.J. Wingler, and O. Zavarzina, 2011 – 2018

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Definition of plasticity

Let U be a metric space. A map $F : U \to U$ is called non-expansive, if $\rho(F(x), F(y)) \leq \rho(x, y)$ for all $x, y \in U$. In 2006 S.A. Naimpally, Z. Piotrowski and E.J. Wingler introduced and studied the following property of metric spaces.

Definition

A metric space U is called an expand-contract plastic space (or simply, a plastic space) if every non-expansive bijection from U onto itself is an isometry.

In other words, a bijective map $F : U \to U$ of a plastic space either preserves all the distances, or there are both pairs of elements with increasing distances and pairs of elements with decreasing distances.

Namioka's proof of plasticity for compact spaces

Ellis Theorem (1958). Let K be a compact, $S \subset C(K, K)$ be a semigroup for the composition, and let $\Sigma \subset K^K$ be the pointwise closure of S. The following are equivalent:

1. Each member of Σ is one to one.
2. Σ is a group with $id\colon K \to K$ being the identity element of the group.

Namioka's argument. Let K be a metric compact, $F\colon K \to K$ be a non-contractive bijective map. Consider the semigroup $S = \{F^n : n \in \mathbb{N}\} \subset C(K, K)$ and the corresponding Σ. Each member of S is non-contractive, so each member of Σ is non-contractive and hence is one to one. By Ellis theorem Σ is a group, so $F^{-1} \in \Sigma$. Consequently, F^{-1} is non-contractive, which means that F is an isometry.

More examples

There are many other plastic spaces outside of compact ones. Say, every precompact space is plastic, and there are examples of not precompact (and even unbounded) plastic spaces. In an infinite-dimensional Banach space X there can be very "good" subsets that are not plastic spaces. For example, consider

$$U = \left\{ x = (x_n) \in H : \sum_{k=-\infty}^{0} |x_k|^2 + \frac{1}{4} \sum_{k=1}^{\infty} |x_k|^2 \le 1 \right\} \subset \ell_2(\mathbb{Z}).$$

Define the (linear) weighted left shift operator T as follows: $Te_n = e_{n-1}$ for $n \neq 1$ and $Te_1 = \frac{1}{2} e_0$. This operator maps U to U bijectively, is non-expansive and is not an isometry, so U is not plastic, but it is a closed convex and bounded set.

The main problem

Remark that the set U in this example is a solid ellipsoid in a Hilbert space H, so in many senses it does not differ much from the unit ball of H. Nevertheless, in the sense of plasticity the unit ball is quite different from this U. Namely, in recent paper by Cascales, Kadets, Orihuela and Wingler it is shown that the unit ball of every strictly convex Banach space is plastic.

It is an open question, whether the same result remains true without the strict convexity assumption.

We also don't know the answer to an analogous question whether every bijective non-expansive map $F\colon B_X \to B_Y$ between balls of two different spaces should be an isometry.

Mankiewicz's theorem

The following result by P. Mankiewicz (1972) explains better what is plasticity in the case of the unit ball.

Theorem. If $A \subset X$ and $B \subset Y$ are convex with non-empty interior, then every bijective isometry $F : A \to B$ can be extended to a bijective affine isometry $\tilde{F} : X \to Y$. □

Taking into account that in the case of A, B being unit balls every isometry maps 0 to 0, this result implies that every bijective isometry $F : B_X \to B_Y$ is the restriction of a linear isometry from X onto Y.

The key lemma

Lemma. Let $F : B_X \to B_Y$ be a bijective non-expansive map such that $F(S_X) = S_Y$. Let $V \subset S_X$ be the subset of all those $v \in S_X$ that $F(av) = aF(v)$ for all $a \in [-1, 1]$. Denote $A = \{tx : x \in V, t \in [-1, 1]\}$, then $F|_A$ is a bijective isometry between A and $F(A)$.

The idea of the proof. Consider the directional derivative of the function $x \mapsto \|x\|_X$ at $u \in S_X$ in the direction $v \in X$:

$$u^*(v) = \lim_{a \to 0^+} \frac{1}{a} \left(\|u + av\|_X - \|u\|_X \right).$$

If for some $u \in S_X$ and $v \in A$ we have $u^*(-v) = -u^*(v)$, then a few lines of inequalities gives us $(F(u))^*(F(v)) = u^*(v)$.

The idea of demonstration: continuation

Fix arbitrary $y_1, y_2 \in A$. Let $E = \mathrm{span}\{y_1, y_2\}$, and let $W \subset S_E$ be the set of smooth points of S_E (which is dense in S_E). All the functionals x^*, where $x \in W$, are linear on E, so $x^*(-y_i) = -x^*(y_i)$, for $i = 1, 2$. Now

$$\begin{aligned} \|y_1 - y_2\|_X &= \sup\{x^*(y_1 - y_2) : x \in W\} \\ &= \sup\{x^*(y_1) - x^*(y_2) : x \in W\} \\ &= \sup\{(F(x))^*(F(y_1)) - (F(x))^*(F(y_2)) : x \in W\} \\ &\leq \|F(y_1) - F(y_2)\|_Y. \end{aligned}$$

So $\|F(y_1) - F(y_2)\| = \|y_1 - y_2\|$. $\square$

Elementary properties

Let $F : B_X \to B_Y$ be a non-expansive bijection. Then, the following holds true.

1. $F(0) = 0$.
2. $F^{-1}(S_Y) \subset S_X$.
3. If X is strictly convex then Y is strictly convex.
4. If $F(x) \in \mathrm{ext}(B_Y)$, then $F(ax) = aF(x)$ for all $a \in [-1, 1]$.
5. If Y is strictly convex then $F(S_X) = S_Y$.

Why?

1. 0_X has distance at most 1 from all elements of B_X, so $F(0_X)$ has distance at most 1 from all elements of B_Y, i.e. $F(0_X) = 0_Y$.
2. Every element of the open ball of X is of distance smaller than 1 to 0, so it cannot be mapped to an element of S_Y.
3. Let Y be not strictly convex, then for some $y \in S_Y$ there are infinitely many elements $z \in S_Y$ such that $\|y + z\| = 2$. Then

$$\|F^{-1}(y)+(-F^{-1}(-z))\| = \|F^{-1}(y)-F^{-1}(-z)\| \geq \|y-(-z)\| = 2.$$

4. Say, for $0 < a < 1$ we have $\|ax\| = a$, $\|ax - x\| = 1 - a$, consequently $\|F(ax)\| = a$, $\|F(ax) - F(x)\| = 1 - a$. But for $F(x) \in \mathrm{ext}(B_Y)$ the only element $z \in Y$ with the property $\|z\| = a$, $\|z - F(x)\| = 1 - a$ is the element $aF(x)$.

The list of results in chronological order

Let $F : B_X \to B_Y$ be a non-expansive bijection. Then each of the following conditions imply that F is an isometry.

1. $X = Y$ are strictly convex (Cascales , Kadets, Orihuela, Wingler, 2016).
2. $X = Y = \ell_1$ (Kadets, Zavarzina, 2016).
3. Y is strictly convex, or $Y = \ell_1$, or Y is finite-dimensional (Zavarzina, 2018).
4. Y is ℓ_1-sum of strictly convex Banach spaces (Kadets, Zavarzina, 2018).
5. X is strictly convex (Angosto, Kadets, Zavarzina, 2018).
6. S_Y is a union of finite-dimensional polyhedral faces (Angosto, Kadets, Zavarzina, 2018).

Problems

1. Which of the spaces ℓ_∞, $C(K)$, c_0, L_1 have plastic unit balls?
2. Does RNP or at least reflexivity imply the plasticity of the unit ball?
3. If a space is isomorphic to a Hilbert space, should its ball be plastic?
4. Does the existence of a non-expansive bijection $F : B_X \to B_Y$ imply that X and Y are isomorphic?

One more problem

What is the characterization of bounded closed convex plastic subsets of an infinite-dimensional Hilbert space?

A step in this direction was recently made by O. Zavarzina (preprint, 2018).

Let $a(n) > 0$, $n \in \mathbb{N}$. Suppose $\inf_n a(n) > 0$ and $\sup_n a(n) < +\infty$ and consider an ellipsoid with these semiaxes in a Hilbert H:

$$E = \left\{ x = \sum_{n \in \mathbb{N}} x_n e_n \in H : \sum_{n \in \mathbb{N}} \left| \frac{x_n}{a(n)} \right|^2 \leq 1 \right\},$$

where e_n are elements of a given orthonormal basis.

Zavarzina's theorem about ellipsoids

Let ellipsoid E be as above, then the following conditions are equivalent:
(1) every linear operator $T\colon H \to H$ that maps E to E bijectively is an isometry;

(2) Every subset B of the set $a(\mathbb{N})$ of semi-axes of E that consists of more than one element possesses at least one of the following properties:

1. B has a maximum of finite multiplicity;
2. B has a minimum of finite multiplicity.