Locally uniformly rotund renormings of the spaces of continuous functions on Fedorchuk compacts

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joint work with

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Let us recall that a Banach space E (or the norm in E) is said to be *locally uniformly rotund* (*LUR* for short) if

$\lim_n \|x_n - x\| = 0$ whenever

$\lim_n \|(x_n + x)/2\| = \lim_n \|x_n\| = \|x\|.$

The *LUR* renorming techniques for a Banach space are based in two different approaches.

In the first one, for enough convex functions on the Banach space are constructed to apply Deville's master lemma, to get an equivalent LUR norm. Originally this method use the powerful tool of projectional resolutions of the identity.

The second one is based to a characterization of those Banach spaces that admit a *LUR* renorming by means of a linear topological condition. Namely

Theorem 1.(A.Moltó, J.Orihuela, S.T., '97) A normed space E is *LUR* renormable if and only if for every $\varepsilon > 0$ we can write in such a way

$$E = \bigcup_{n \in N} E_{n,\varepsilon}$$

that for every $x \in E$ there exist $n \in \boldsymbol{N}$ and open half space H with $x \in H \cap E_{n,\varepsilon}$, $diam(H \cap E_{n,\varepsilon}) < \varepsilon$.

It turns out that it is rather difficult to apply former theorem in concrete cases. This motivates us to build up some transfer technique designed to transfer a good convexity property from a normed space to another combining with topological properties of metric space of covering type. This is done in A.Moltó, J.Orihuela, M. Valdivia, S.T., Non-linear transfer technique, Lect. Notes Math., 1951, Springer, Berlin, 2009

We present a particular result in this direction which we use latter. This is a nonlinear (convex) version for *LUR* renorming of Banach spaces with strong Markushevich basis.

Theorem 2. Let E be a normed space and let F be a norming for E subspace of E^*. Assume that there is a locally bounded map Φ from E into $c_0(\Gamma)$ for some set Γ such that:

(i) for every finite set $A \subset \Gamma$ is specified a separable subspace Z_A of E with properties:

(a) $Z_A \subset Z_B$ whenever $A \subset B \subset \Gamma$;

(b) $x \in \overline{span \cup_{n \in N} Z_{K_n}}^{\|.\|}$, whenever $x \in E$ and $\{K_n : n \in N\}$ is an increasing sequence (i.e. $K_m \subset K_n$ if $m<n$) of finite subsets of Γ;

(ii) for every $\gamma \in \Gamma$ the real function $\delta_\gamma \circ \Phi$ on E is non-negative, convex and $\sigma(E, F)$ - lower semi - continuous, where δ_γ is the Dirac measure on Γ at γ.

Then E admits an equivalent $\sigma(E,F)$ - lower semi-continuous LUR norm.

Example 1. Assume that E has a srtrong Markushevich basis $\{e_\gamma : \gamma \in \Gamma\}$ with conjugate system $\{e^*_\gamma : \gamma \in \Gamma\}$, that is $e^*_\beta(e_\gamma) = \delta_{\beta\gamma}$, and for every $x \in E$ we have

$$x \in \overline{span} \{e_\gamma : e^*_\gamma(x) \neq 0, \gamma \in \Gamma\}^{-\|.\|}$$

Let

$$Z_A = \text{span}\{e_\gamma : \gamma \in A\}$$

and define $\Phi : E \to c_0(\Gamma)$ by formula

$$\Phi x(\gamma) = |e^*_\gamma(x)|.$$

From the definition of strong Markushevich basis for every $x \in E$ we have

$$x \in \overline{span} \{e_\gamma : \Phi x(\gamma) = |e^*_\gamma(x)| \neq 0, \gamma \in \Gamma\}^{--\|.\|}$$

So Φ satisfies the hypothesis of the former theorem.

Lot of papers are devoted to find different classes of compact space X for which $C(X)$ admits an equivalent pointwise lower semicontinuous *LUR* norm. Moving on now to topological properties, it was proved consecutively that $C(X)$ has such norm if X is *Eberlein, Talagrand, Gul'ko* compact space. It turns out that if X hails from one of these three classes of compact spaces then it can be treated as a subset of a pointwise compact cube $[0, 1]^{\Gamma}$, in such a way that, given $t \in X$, its coordinates $t(\gamma)$, $\gamma \in \Gamma$, behave according to certain rules.

Every compact space X from the classes above shares the property that X may be embedded in $[0, 1]^{\Gamma}$, such that given any $t \in X$, the *support* *supp* $t = \{ \gamma \in \Gamma : t(\gamma) \neq 0 \}$ is countable. In general, a space satisfying this property is called *Corson* compact. The space X is called *Valdivia* compact if it is as above, but, in this case, only a pointwise dense subset of points of $X \subseteq [0, 1]^{\Gamma}$ are required to have countable support. These classes have long been relevant to renorming theory. In all this cases $C(X)$ admits projectional resolutions of the identity and therefore admits a pointwise lower semicontinuous *LUR* norm.

We mention that in $C(X)$ spaces we have a canonical map to $c_0(\Gamma)$. Indeed if $X \subseteq [0,1]^\Gamma$ the uniform continuity of every $h \in C(X)$ allows us to define the oscillation map

$\Omega : C(X) \to c_0(\Gamma)$

by formula

$\Omega(h(\gamma))=sup\{h(t)-h(s)\colon t, s \in X, (t-s)1_{\Gamma\setminus\{\gamma\}} = 0\}$,

where $h \in C(X)$. It is easy to see that Ω is a bounded map satisfying condition (*ii*) of Theorem 2, i.e. $\delta_\gamma \circ \Omega$ on E is non-negative, convex and pointwise lower semi-continuous, where δ_γ is the Dirac measure on Γ at γ.

In [MOTV] is shown that Ω satisfies a condition similar to condition (*i*) of Theorem 1 when X is Helly compact of monotone functions on [0,1].

The aim of this talk is to give a new class of compact spaces for which the corresponding space of continuos functions is *LUR* renormable. The following definition goes back to V.V. Fedorchuk.

Definition1. Continuous map $f : X \to Y$ is said to be fully closed if for every disjoint closed subsets A and B of X the set $f(A) \cap f(B)$ is finite.

Our main result is next

Theorem 3. Let X be a compact space admitting a fully closed map f onto a metric compact Y such that the fibers $f^{-1}(y)$ are metrizable for every $y \in Y$. Then $C(X)$ admits an equivalent pointwise lower semi-continuous *LUR* norm.

The above class of compact spaces is particular case of Fedorchuk compacts.

Example 2. Denote with L the lexicographic square. The projection of L onto the first factor is fully closed and all its fibers are homeomorphic to the closed interval.

G.Alexandrov'88 showed that $C(L)$ is LUR renormable. Latter R. Haydon, J. Jayne, I. Namioka and C. A. Rogers' 00 proved that if $Q = [0, 1]^{\Gamma}$ is lexicographical cube then $C(Q)$ is *LUR* renormable if and only if Γ is a countable ordinal.

Let $f: X \to Y$ be a continuous map of compacts.
Given $y \in Y$ define

$$osc_{f^{-1}(y)}(h) = sup_{s,t \in f^{-1}(y)} \{h(t) - h(s)\}$$

for $h \in C(X)$. Clearly $osc_{f^{-1}(y)}(\cdot)$ is a pointwise lower semi-continuous semi-norm in $C(X)$.

We introduce fiberwise oscillation map

$\Omega_f : C(X) \to l_\infty(Y)$
by formula
$\Omega_f(h)(y) = osc_{f^{-1}(y)}(h)$,

$\forall y \in Y$, where $h \in C(X)$.

Clearly the map Ω_f is bounded. Since

$\delta_y \circ \Omega_f(h) = osc_{f^{-1}(y)}(h)$

we get that Ω_f satisfies condition (*ii*) of Theorem 2, that is $\forall y \in Y$ the function $\delta_y \circ \Omega_f(h)$ is non negative, convex function of h and $osc_{f^{-1}(y)}(\cdot)$ is a pointwise lower semi-continuous semi-norm on $C(X)$.

Proposition1. The continuos map $f : X \rightarrow Y$ of compacts is fully closed if and only if Ω_f maps $C(X)$ into $c_0(Y)$.

Proof. Assume that the set

$H_{h,\varepsilon} = \{y \in Y : diam(h(f^{-1}(y))) \geq \varepsilon\}$

is infinite for some $h \in C(X)$ and $\varepsilon > 0$. We show that f is not fully closed.

Denote with Λ the space of all non empty subsets of closed interval $[-\|h\|_\infty, \|h\|_\infty]$ endowed with Hausdorff metric d. It is well known that (Λ, d) is compact.

We can consider $\{h(f^{-1}(y)) : y \in H_{h,\varepsilon}\}$ as infinite family of closed subsets of the interval $[-\|h\|_\infty, \|h\|_\infty]$, i.e. $\{h(f^{-1}(y)) : y \in H_{h,\varepsilon}\} \subseteq \Lambda$.
Since Λ is metric compact there exists a sequence $\{y_n : n \in N\}$ of different points of $H_{h,\varepsilon}$ and subset M of the interval $[-\|h\|_\infty, \|h\|_\infty]$ such that

$$lim_n d(h(f^{-1}(y_n)), M))=0.$$

It is obvious that $diam(M) \geq \varepsilon > 0$. Pick two different real numbers $u,v \in M$. There exist closed disjoint intervals I and J such that its interiors contain u and v respectively. Clearly for all enough large n

$$h(f^{-1}(y_n)) \cap I \neq \emptyset , \qquad h(f^{-1}(y_n)) \cap J \neq \emptyset$$

Set $A=h^{-1}(I)$ and $B=h^{-1}(J)$. Since I and J are closed disjoint we get that A and B are closed disjoint sets in X too.

We have

$f(A)=\{y\in Y: \exists\, x\in f^{-1}(y) \; with \; h(x)\in I\}$,

$f(B)==\{y\in Y: \exists\, x\in f^{-1}(y) \; with \; h(x)\in J\}$.

So

$y_n \in f(A) \cap f(B)$

for all enough large n.

Hence f is not fully closed map.

Assume now that f is not fully closed.Then there exist disjoint closed subsets A, B in X such that the intersection $D = f(A) \cap f(B)$ is infinite. There exists $g \in C(X)$ such that $g1_A = 1$ and $g1_B = 0$. So for every $y \in D$ we have $\Omega_f(g)(y) = 1$. Hence $H_{g,1}$ contains infinite set D. ■

Let A be an arbitrary subset of Y. Consider a partition of a compact X whose nontrivial elements are sets $f^{-1}(y)$ for $y \in Y \setminus A$. Let Y_A be the quotient space corresponding to this partition (with respect to f), that is

$$Y_A = \{f^{-1}(y): y \in Y \setminus A\} \cup \{\{x\}: f(x) \in A\}.$$

Let $f_A : X \to Y_A$ be a quotient map, corresponding to this partition, that is

$$f_A(x) = \begin{cases} f^{-1}(f(x)) & \text{if } f(x) \in Y \setminus A \\ \{x\} & \text{otherwise} \end{cases}$$

In Y_A we consider standard factor topology i.e. $U \subset Y_A$ is open iff $f_A^{-1}(U)$ is open in X.

Fact. If Y is metrizable with metrizable fibers and A is countable then Y_A is metrizable too.

Given $A \subset Y$ define
$Z_A = \{h \in C(X) : supp(\Omega_f(h)) \subseteq A\}$.
From the definition of Ω_f it follows that Z_A is a closed subspace of $C(X)$. We investigate Z_A using that $f : X \to Y$ is fully closed map of compacts.

Lemma 1. The spaces Z_A and $C(Y_A)$ are isomorphically isometric. The linear operator $T_A : Z_A \to C(Y_A)$ defined by formula

$$T_A h = h \circ f_A^{-1}$$

give the isometry , i.e.

$$\|T_A h\|_\infty = \|h\|_\infty .$$

Corollary. If Y is metrizable with metrizable fibers and A is countable then Z_A is separable.

Proposition 2. Let $f : X \rightarrow Y$ be a fully closed map of compact X onto a metric compact Y with metrizable fibers $f^{-1}(y)$. Let K be a finite subset of a countable set $A \subset Y$. Then for every $h \in \mathrm{Z}_A$ we have

$$\mathrm{dist}(h, \mathrm{Z}_K) = \inf\{\| h - g \|_\infty : g \in \mathrm{Z}_K\} \leq \| \Omega_f(h) 1_{Y \setminus K} \|_\infty .$$

Proposition 3. The map of the fiberwise oscillation $\Omega_f : C(X) \to c_0(Y)$ satisfies the conditions (a) and (b) of (i) Theorem 1 for any fully closed map $f : X \to Y$ of compact X onto a metric compact Y with metrizable fibers $f^{-1}(y)$, $y \in$Y, that is :

(i) for every finite set $A \subset \Gamma$ is specified a separable subspace Z_A of E with properties:

(a) $Z_A \subset Z_B$ whenever $A \subset B \subset \Gamma$;

(b) $h \in \overline{span \bigcup_{n \in N} Z_{K_n}}^{\|.\|}$

whenever $x \in E$ and $\{K_n : n \in N\}$ is an increasing sequence (i.e. $K_m \subset K_n$ if $m<n$) of finite subsets of Γ with $supp(\Omega_f(h)) \subset \bigcup_{n \in N} K_n$.

Proof. Let us remember that we set
$Z_A = \{h \in C(X) : supp(\Omega_f(h)) \subseteq A\}$.
If A is finite or countable we showed that Z_A is separable. Evidently condition (a) holds. Pick
$h \in C(X)$ and set
$K_m = \{y \in Y : \Omega_f(h)(y) \geq 1/m\}$
Since $\Omega_f(h) \in c_0(Y)$ we get that all K_m are finite.
From former proposition we get $dist(h, Z_{Km}) < 1/m$.■