# Coarse geometry of James spaces

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# Introduction

Let $f : M \to N$ mapping between metric spaces and $\varphi, \rho : [0, \infty[ \to [0, \infty]$ such that

$$\rho(d(x,y)) \leq d(f(x), f(y)) \leq \omega(d(x,y)) \quad \forall\, x, y \in M$$

- $f$ is *uniform embedding* if $\rho(t) > 0$ for all $t > 0$ and $\lim_{t \to 0+} \omega(t) = 0$
- $f$ is *coarse embedding* if $\lim_{t \to \infty} \rho(t) = \infty$ and $\omega(t) < \infty$ for all $t > 0$

**Theorem (Kalton 2007)**

*$c_0 \not\to X$ coarsely or uniformly if $X$ is reflexive.*

# Kalton's interlaced graphs

- let $\mathbb{M} \subset \mathbb{N}$ infinite. Denote $[\mathbb{M}]^k$ all $k$-subsets $\overline{n} = \{n_1 < \ldots < n_k\} \subset \mathbb{M}$.
- put an edge between $\overline{n} \neq \overline{m} \in [\mathbb{M}]^k$ iff

$$n_1 \leq m_1 \leq n_2 \leq \ldots \leq n_k \leq m_k$$

  or vice versa.
- $d(\overline{n}, \overline{m})$ will be the **shortest path distance**
- Then $diam([\mathbb{M}]^k]) = k$ for all $\mathbb{M} \subset \mathbb{N}$

# Property $\mathcal{Q}$

## Definition (Kalton)

A Banach space $X$ has property $\mathcal{Q}$ if $\exists\, C > 0$ such that $\forall\, k \geq 1$, $\forall\, f : [\mathbb{N}]^k \to X$ Lipschitz $\exists\, \mathbb{M} \subset \mathbb{N}$ infinite

$$diam([\mathbb{M}]^k) < C \cdot Lip(f)$$

*Proof of Kalton's theorem:*

- a Banach space without $\mathcal{Q}$ cannot coarsely embed into a Banach space with $\mathcal{Q}$.
- $X$ reflexive $\Rightarrow$ has $\mathcal{Q}$
- $c_0$ doesn't
- Indeed,
$$Y \text{ has } \mathcal{Q} \Rightarrow [\mathbb{N}]^k \not\to Y \text{ equi-coarsely}$$
- $[\mathbb{N}]^k$ equi-coarsely embed into $c_0$.

# Main result

## Question

$$X \text{ has } \mathcal{Q} \Leftarrow [\mathbb{N}]^k \not\to X \text{ equi-coarsely?}$$

## Answer

No, $J, J^*$ don't have $\mathcal{Q}$ (Kalton) but $[\mathbb{N}]^k \not\to J, J^*$ equi-coarsely.

## Theorem (Lancien-Petitjean-P. 2018)

*Let $1 < p < \infty$ and $X$ be a quasi-reflexive Banach space such that*

- *$X$ admits an equivalent $p$-AUS norm*
- *$X^*$ admits an equivalent $q$-AUS norm ($p + q = pq$)*

*Then $[\mathbb{N}]^k \not\to X, X^*$ equi-coarsely.*

# Comments

- Under the hypotheses of the theorem $[\mathbb{N}]^k \not\hookrightarrow X^{(r)}$ for any $r \geq 1$.
- sufficient condition for $[\mathbb{N}]^k \hookrightarrow X$ is that $X$ admits a spreading model equivalent to the summing basis of $c_0$.
- Freeman-Odell-Sari-Zheng: there exists a **quasi-reflexive** space which admits such a spreading model

# Asymptotic uniform smoothness

## Definition

A norm $\|\cdot\|$ on $X$ is $p$-AUS if $\exists\, C > 0\ \forall\, x \in S_{\|\cdot\|}$ and $\forall\, (x_n) \subset X$ weakly-null we have

$$\limsup \|x + x_n\|^p \leq 1 + C \limsup \|x_n\|^p$$

## Lemma (Lancien-Raja, 2017)

*If $\|\cdot\|$ on $X$ is $p$-AUS then $\exists\, C > 0\ \forall\, x \in S_{\|\cdot\|}$ and $\forall\, (x_n) \subset X^{**}$ $w^*$-null we have*

$$\limsup \|x + x_n\|^p \leq 1 + C \limsup \|x_n\|^p$$

# Asymptotic uniform convexity

## Lemma (Godefroy-Kalton-Lancien 2001)

*A norm $|\cdot|$ on $X$ is $q$-AUS iff the dual norm is $p$-AUC$^*$.*

## Definition

A norm $|\cdot|$ on $X^*$ is $p$-AUC$^*$ if $\exists\ C > 0\ \forall\ x \in S_{|\cdot|}$ and $\forall\ (x_n) \subset X^*$ w$^*$-null we have

$$\liminf |x + x_n|^p \geq 1 + C \liminf |x_n|^p$$

# Proof of the main result

- Let $f_k : [\mathbb{N}]^k \to X$ satisfy $\forall\, k$

  $$\rho(d(\overline{n},\overline{m}))) \leq \|f_k(\overline{n}) - f_k(\overline{m})\| \leq \omega(d(\overline{n},\overline{m})) \quad \forall\, \overline{n},\overline{m} \in [\mathbb{N}]^k$$

  with $\rho(t) \to \infty$ as $t \to \infty$ and $\omega(t) < \infty$. (i.e. equi-coarse embedding)
- Fix $k$ large (of the form $k \sim N^{1+p/q}$ for $N$ large).
- Produce a w$^*$-continuous sub-tree $(x_{\overline{n}})_{\overline{n}\in[\mathbb{M}]^{\leq k}}$
  - $x_{\overline{n}} := f(\overline{n})$ if $\overline{n} \in [\mathbb{M}]^k$
  - $x_{\overline{n}} := w^* - \lim_{m\in\mathbb{M}} x_{\overline{n}\frown m}$ if $\overline{n} \in [\mathbb{M}]^j$ with $j < k$.
- Let $z(\overline{n}) := x(\overline{n}) - x(\overline{n}^{\ominus})$
  - so $\sum_{j=1}^{k} z(\overline{n}\restriction_j) = f(\overline{n})$
  - $(z_{\overline{n}})_{\overline{n}\in[\mathbb{M}]^{\leq k}}$ is a w$^*$-null tree
- Notice that $\|z_{\overline{n}}\| \leq \omega(1)$

# Proof of the main result

- Let $|\cdot|$ be the $q$-AUS norm on $X^*$ such that

$$b\|y\| \leq |y| \leq \|y\| \quad \forall\, y \in X^{**}$$

- Pass to a further sub-tree such that

$$\begin{aligned} |f(\overline{n}) - f(\overline{m})|^p &= |\sum_{j=1}^{k} z(\overline{n}\restriction_j) - z(\overline{m}\restriction_j)|^p \\ &\geq const \sum_{j=1}^{k} |z(\overline{n}\restriction_j)|^p + |z(\overline{m}\restriction_j)|^p \end{aligned}$$

  if $n_1 < m_1 < n_2 < \ldots < n_k < m_k$.
- Pass to yet another subtree and get

$$const \cdot \omega(1)^p \geq \sum_{j=1}^{k} |z(\overline{n}\restriction_j)|^p \quad \forall\, \overline{n} \in [\mathbb{M}]^k$$

# Proof of the main result

- Ramsey $\Rightarrow \exists\, K_j \in [0, \omega(1)]$ such that $z(\overline{n} \restriction_j) \sim K_j$ $\forall\, \overline{n} \in [\mathbb{M}]^k,\ \forall\, j \leq k$
- $\Rightarrow$

$$const \cdot \omega(1)^p \geq \sum_{j=1}^{k} K_j^p = \sum_{l=0}^{N^{p/q}} \sum_{j=lN+1}^{(l+1)N} K_j^p$$

$\Rightarrow \exists\, l \leq N^{p/q}$ such that $\sum_{j=lN+1}^{(l+1)N} K_j \leq const \cdot \omega(1)$ provided $k \sim N \cdot N^{p/q}$

- $\Rightarrow \exists\, \overline{n}_0, \overline{m}_0 \in [\mathbb{M}]^{(l+1)N}$ such that $d(\overline{n}_0, \overline{m}_0) = N$ and $2\, const \cdot \omega(1) \geq \|x_{\overline{n}_0} - x_{\overline{m}_0}\|$
- $p$-AUS+quasi-reflexivity $\Rightarrow \exists\, \overline{n}, \overline{m} \in [\mathbb{M}]^k$ such that $d(\overline{n}, \overline{m}) = N$ and

$$3\, const \cdot \omega(1) \geq \|x_{\overline{n}} - x_{\overline{m}}\| = \|f_k(\overline{n}) - f_k(\overline{m})\| \geq \rho(N)$$

- Enough to take $N$ such that $\rho(N) > 3\, const \cdot \omega(1)$. $\square$

# Last slide

**Theorem (Kalton 2007)**

*$c_0 \not\to X$ if $X^{(r)}$ is separable for all r*

**Theorem (Lancien-Petitjean-P. 2018)**

*$\mathcal{JT}, \mathcal{JT}^* \not\to X$ coarsely if $X^{(r)}$ is separable for all r*

- We don't know if $[\mathbb{N}]^k \hookrightarrow \mathcal{JT}, \mathcal{JT}^*$ equi-coarsely.
- Based on the following fact: $\exists$ uncountable $I$ $\forall i \in I$ $\forall k \in \mathbb{N}$ $\exists$ 1-Lipschitz $f_i^k : [\mathbb{N}]^k \to \mathcal{JT}$ such that

$$\lim_{k\to\infty} \inf_{i\neq j} \inf_{\mathbb{M}} \sup_{\overline{n}\in[\mathbb{M}]^k} \|f_i^k(\overline{n}) - f_j^k(\overline{n})\| = \infty$$

Gracias por su atención!

# Banach spaces and optimization: Surprise conference for Robert Deville's 60th birthday! Métabief (France), June 16-21, 2019