# Extremal structure of the unit ball in Lipschitz-free spaces

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(joint work with Eva Pernecká)

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# Lipschitz functions

Let $(M,d)$, $(M',d')$ be complete metric spaces.
The *Lipschitz constant* of $f : M \to M'$ is

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**Theorem (MacShane, 1934)**

Let $M' \subset M$. Then every $f : M' \to \mathbb{R}$ can be extended to $M$ in such a way that $\|f\|_L$ and $\|f\|_\infty$ are preserved.

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This is not true if $\mathbb{R}$ is replaced by e.g. $\mathbb{C}$.

# Lipschitz spaces

Consider the space of *Lipschitz* functions

$$\mathrm{Lip}(M) = \{f : M \to \mathbb{R} : \|f\|_L < \infty\}.$$

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- Fix a base point $0 \in M$ (we say $M$ is *pointed*)
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Then:

- $\mathrm{Lip}_0(M)$ is a Banach space with norm $\|\cdot\|_L$
- $\mathrm{Lip}_0(M, 0) \cong \mathrm{Lip}_0(M, 0')$ under $f \mapsto f - f(0')$

# Lipschitz-free spaces

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$$\mathcal{F}(M) = \overline{\mathrm{span}}\, \delta(M) \subset \mathrm{Lip}_0(M)^*$$

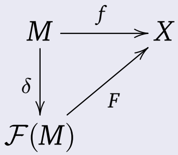
**Theorem (Arens, Eells 1956)**

$$\mathcal{F}(M)^* \cong \mathrm{Lip}_0(M)$$

# Universal property

**Theorem (Weaver 1999)**

Let $M$ be a pointed metric space, $X$ a Banach space, and $f\colon M \to X$ Lipschitz with $f(0) = 0$. Then there is $F \in \mathcal{L}(\mathcal{F}(M), X)$ with $F|_{\delta(M)} = f$, and $\|F\| = \|f\|_L$.



# Universal property

## Theorem (Godefroy, Kalton 2003)

Let $M$ and $N$ be pointed metric spaces, and $f : M \to N$ Lipschitz with $f(0_M) = 0_N$. Then there is $F \in \mathcal{L}(\mathcal{F}(M), \mathcal{F}(N))$ such that $F|_{\delta(M)} = f$ and $\|F\| = \|f\|_L$.

$$\begin{array}{ccc} M & \xrightarrow{\;f\;} & N \\ {\scriptstyle \delta_M}\downarrow & & \downarrow{\scriptstyle \delta_N} \\ \mathcal{F}(M) & \xrightarrow[\;F\;]{} & \mathcal{F}(N) \end{array}$$

# Lipschitz-free subspaces

**Theorem (Kadets 1985)**

If $M' \subset M$, then $\mathcal{F}(M') \subset \mathcal{F}(M)$ isometrically:

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We will assume $0 \in M'$. Otherwise, we mean

$$\mathcal{F}(M') := \mathcal{F}(M' \cup \{0\}).$$

# The intersection property

**Theorem (Aliaga, Pernecká 2018)**

Assume $M$ is bounded and let $K_i \subset M$ be closed. Then

$$\bigcap_i \mathcal{F}(K_i) = \mathcal{F}\left(\bigcap_i K_i\right) .$$

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If $M$ is bounded, $\mathrm{Lip}_0(M)$ is an algebra with pointwise product:

$$\begin{aligned} \|f \cdot g\|_L &\leq \|f\|_L \|g\|_\infty + \|f\|_\infty \|g\|_L \\ &\leq 2\,\mathrm{diam}(M) \|f\|_L \|g\|_L . \end{aligned}$$

# Ideals of Lipschitz functions

Let $\mu \in \mathcal{F}(M)$, $g \in \mathrm{Lip}_0(M)$, and for $f \in \mathrm{Lip}_0(M)$ define

$$(\mu \circ g)(f) = \langle \mu, f \cdot g \rangle \ .$$

Then $\mu \circ g \in \mathcal{F}(M)$.

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*Proof:* If $\mu = \sum a_n \delta(p_n)$ is finitely supported then

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Otherwise take finitely supported $\mu_n \to \mu$, then

$$\|\mu_m \circ g - \mu_n \circ g\| \leq 2 \, \mathrm{diam}(M) \, \|g\|_L \cdot \|\mu_m - \mu_n\|$$

so $\mu_n \circ g$ converges. The limit is $\mu \circ g$. □

# Ideals of Lipschitz functions

**Proposition**

If $Y$ is an ideal in $\mathrm{Lip}_0(M)$, then $\overline{Y}^{w^*}$ is also an ideal.

Recall: $Y$ is an ideal if $f \in Y$, $g \in \mathrm{Lip}_0(M) \implies f \cdot g \in Y$.

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*Proof:* Let $f \in \overline{Y}^{w^*}$ and $g \in \mathrm{Lip}_0(M)$. Let $U \subset \mathrm{Lip}_0(M)$ be a $w^*$-neighborhood of $f \cdot g$

$$U = \{u \in \mathrm{Lip}_0(M) : |\langle \mu_n, f \cdot g - u \rangle| < \varepsilon \text{ for } n = 1, \ldots, N\}.$$

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Take this $w^*$-neighborhood of $f$:

$$\begin{aligned} V &= \{v \in \mathrm{Lip}_0(M) : |\langle \mu_n \circ g, f - v \rangle| < \varepsilon \text{ for } n = 1, \ldots, N\} \\ &= \{v \in \mathrm{Lip}_0(M) : |\langle \mu_n, (f - v) \cdot g \rangle| < \varepsilon \text{ for } n = 1, \ldots, N\} . \end{aligned}$$

Since $f \in \overline{Y}^{w^*}$, there is $v \in Y \cup V$. Then $u = v \cdot g \in Y \cup U$. $\square$

# Ideals of Lipschitz functions

Let $K \subset M$ be closed. The *kernel of K* is

$$\mathcal{I}(K) = \{f \in \mathrm{Lip}_0(M) : f(x) = 0 \text{ for all } x \in K\} .$$

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**Properties:**

- $\mathcal{F}(K)^\perp = \mathcal{I}(K)$
- $\mathcal{I}(K)_\perp = \mathcal{F}(K)$
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**Theorem (Weaver 1995)**

All $w^*$-closed ideals of $\mathrm{Lip}_0(M)$ are kernels.

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*Proof:* Let $Y = \operatorname{span}\{\mathcal{I}(K_i)\}$. $Y$ is an ideal, so $\overline{Y}^{w^*}$ is an ideal, so $\overline{Y}^{w^*} = \mathcal{I}(H)$ for some $H \subset M$. Thus

$$\bigcap_i \mathcal{F}(K_i) = \bigcap_i \mathcal{I}(K_i)_\perp = \left(\bigcup_i \mathcal{I}(K_i)\right)_\perp = Y_\perp = \left(\overline{Y}^{w^*}\right)_\perp = \mathcal{I}(H)_\perp = \mathcal{F}(H).$$

It is easy to see that $H = \bigcap_i K_i$. $\square$

# Supports in $\mathcal{F}(M)$

Let $\mu \in \mathcal{F}(M)$. The *support of $\mu$* is the smallest closed $K \subset M$ such that

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Equivalently, for any closed $L$, $\mu \in \mathcal{F}(L) \iff K \subset L$.

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We do not know whether all $\mu \in \mathcal{F}(M)$ have a support.

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**Proposition**

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**Theorem (Aliaga, Pernecká 2018)**

If $\mu \in \mathcal{F}(K)$ for a bounded $K \subset M$, then $\mu$ has a support.

# Extreme points

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- $x$ is a *denting point* of $B_X$ iff there are slices of $B_X$ of arbitrarily small diameter containing $x$

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- $x$ is an *exposed point* of $B_X$ iff $\exists f \in B_{X^*}$ such that

$$f(x) = 1 \text{ and } y \in B_X,\ f(y) = 1 \implies y = x$$

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- $x$ is a *strongly exposed point* of $B_X$ iff $\exists f \in B_{X^*}$ such that

$$f(x) = 1 \text{ and } y_n \in B_X,\ f(y_n) \to 1 \implies y_n \to x$$

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$$
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\Downarrow & & \\
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**Problem:** Characterize all of them in Lipschitz-free spaces.

# Elementary molecules

**Theorem (Weaver 1995)**

Every preserved extreme point of $B_{\mathcal{F}(M)}$ is a molecule $u_{pq}$.

An *elementary molecule* is $u_{pq} = \dfrac{\delta(p) - \delta(q)}{d(p,q)} \in S_{\mathcal{F}(M)}$.

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**Properties:**

- $\|f\|_L = \sup \left\{ \langle u_{pq}, f \rangle : p, q \in M \right\}$ since

$$\langle u_{pq}, f \rangle = \frac{f(p) - f(q)}{d(p,q)}$$

- $B_{\mathcal{F}(M)} = \overline{\operatorname{conv}} \left\{ u_{pq} : p, q \in M \right\}$

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**Theorem (de Leeuw 1961)**

If there is $f \in \mathrm{Lip}_0(M)$ that peaks at $p, q$, then $u_{pq}$ is a preserved extreme point of $B_{\mathcal{F}(M)}$.

# Recent advances

**Theorem (García-Lirola, Procházka, Rueda 2017)**

TFAE:

(i) $u_{pq}$ is a strongly exposed point of $B_{\mathcal{F}(M)}$.

(ii) There is $f \in \mathrm{Lip}_0(M)$ that peaks at $p, q$.

(iii) $p, q$ do not have property (Z), that is:

$$(\neg Z) \quad \text{There is } C > 0 \text{ such that, for all } x \neq p, q$$
$$d(p,x) + d(x,q) - d(p,q) \geq C \min\{d(p,x), d(q,x)\}$$

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**Theorem (García-Lirola, Petitjean, Procházka, Rueda 2017)**

Every preserved extreme point of $B_{\mathcal{F}(M)}$ is denting.

# Recent advances

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# Necessary conditions

Let $p, q \in M$. The *metric segment* between $p$ and $q$ is

$$[p, q] = \{x \in M : d(p, x) + d(x, q) = d(p, q)\}.$$

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If $u_{pq}$ is an extreme point of $B_{\mathcal{F}(M)}$, then $[p,q] = \{p,q\}$.

*Proof:* If $x \in [p,q]$ then $u_{pq} \in [u_{px}, u_{xq}]$:

$$\begin{aligned} u_{pq} = \frac{\delta(p)-\delta(q)}{d(p,q)} &= \frac{\delta(p)-\delta(x)}{d(p,q)} + \frac{\delta(x)-\delta(q)}{d(p,q)} \\ &= \frac{d(p,x)}{d(p,q)} u_{px} + \frac{d(x,q)}{d(p,q)} u_{xq}. \qquad \square \end{aligned}$$

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Let $p, q \in M$. The *extended metric segment* between $p$ and $q$ is

$$\beta[p,q] := \{\xi \in \beta M : d(p,\xi) + d(\xi,q) = d(p,q)\}.$$

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**Proposition (Aliaga, Guirao 2017)**

If $u_{pq}$ is preserved extreme in $B_{\mathcal{F}(M)}$, then $\beta[p,q] = \{p,q\}$.

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**Proposition (Aliaga, Guirao 2017)**

If $u_{pq}$ is preserved extreme in $B_{\mathcal{F}(M)}$, then $\beta[p,q] = \{p,q\}$.

*Proof:* If $\xi \in \beta[p,q]$ then $\delta(\xi) \in \mathrm{Lip}_0(M)^* = \mathcal{F}(M)^{**}$ and

$$u_{pq} = \frac{d(p,\xi)}{d(p,q)} u_{p\xi} + \frac{d(\xi,q)}{d(p,q)} u_{\xi q}$$

where $u_{p\xi} = \dfrac{\delta(p) - \delta(\xi)}{d(p,\xi)}$, $u_{\xi q} = \dfrac{\delta(\xi) - \delta(q)}{d(\xi,q)} \in S_{\mathcal{F}(M)^{**}}$. □

# Necessary conditions...

| | molecule $u_{pq}$ | non-molecule |
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| extreme | $[p,q] = \{p,q\}$ | |
| exposed | $[p,q] = \{p,q\}$ | |
| preserved extreme | $\beta[p,q] = \{p,q\}$ | impossible |
| denting | $\beta[p,q] = \{p,q\}$ | impossible |
| strongly exposed | $p, q$ do not have property (Z) | impossible |

# ...are also sufficient conditions

**Theorem (Aliaga, Guirao 2017)**

If $\beta[p,q] = \{p,q\}$ then $u_{pq}$ is a preserved extreme point of $B_{\mathcal{F}(M)}$.

**Theorem (Aliaga, Pernecká 2018)**

If $[p,q] = \{p,q\}$ then $u_{pq}$ is an extreme point of $B_{\mathcal{F}(M)}$.

# Tools for the proof

Let $\widetilde{M} = \{(p,q) \in M \times M : p \neq q\}$.
The *de Leeuw transform* is the mapping $\Phi\colon \operatorname{Lip}(M) \to C(\widetilde{M})$

$$\Phi f(p,q) = \frac{f(p) - f(q)}{d(p,q)} = \langle u_{pq}, f \rangle .$$

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**Properties:**

- $\Phi$ is an isometric embedding $\operatorname{Lip}_0(M) \to C_b(\widetilde{M}) \cong C(\beta\widetilde{M})$
- $\Phi^*\colon \mathcal{M}(\beta\widetilde{M}) \to \operatorname{Lip}_0(M)^*$ is onto
- $\Phi^*\delta_{(p,q)} = u_{pq}$

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Let $\widetilde{M} = \{(p,q) \in M \times M : p \neq q\}$.
The *de Leeuw transform* is the mapping $\Phi\colon \operatorname{Lip}(M) \to C(\widetilde{M})$

$$\Phi f(p,q) = \frac{f(p) - f(q)}{d(p,q)} = \langle u_{pq}, f \rangle .$$

We define the coordinate projections in $\beta\widetilde{M}$ by extending

$$\mathrm{id}\colon \widetilde{M} \to M \times M \quad \Longrightarrow \quad \pi : \beta\widetilde{M} \to \beta M \times \beta M$$

# Tools for the proof

Consider the set

$$\mathcal{E} := \Big\{\psi \in B_{\mathrm{Lip}_0(M)^*} : \text{if } f \in B_{\mathrm{Lip}_0(M)} \text{ and } \Phi f(p,q) = 1, \\ \text{then also } \langle f, \psi\rangle = 1\Big\}.$$

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Suppose $u_{pq} = \frac{1}{2}(\psi_1 + \psi_2)$ for $\psi_1, \psi_2 \in B_{\mathrm{Lip}_0(M)^*}$.
If $f \in B_{\mathrm{Lip}_0(M)}$ and $\Phi f(p,q) = 1$ then

$$1 = \langle f, u_{pq} \rangle = \frac{1}{2} \left( \langle f, \psi_1 \rangle + \langle f, \psi_2 \rangle \right) \leq 1$$

so $\langle f, \psi_1 \rangle = \langle f, \psi_2 \rangle = 1$. Thus $\psi_1, \psi_2 \in \mathcal{E}$.

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It suffices to show:

- If $\beta[p,q] = \{p,q\}$ then $\mathcal{E} = \{u_{pq}\}$.
- If $[p,q] = \{p,q\}$ then $\mathcal{E} \cap \mathcal{F}(M) = \{u_{pq}\}$.

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Consider also the set

$$\mathcal{D} := \{\zeta \in \beta\widetilde{M} : \text{if } f \in B_{\mathrm{Lip}_0(M)} \text{ and } \Phi f(p,q) = 1, \\ \text{then also } |\Phi f(\zeta)| = 1\}.$$

(Any $f$ that attains its Lipschitz constant between $p$ and $q$ also attains its Lipschitz constant at $\zeta$.)

# The concentration lemma

**Lemma**

Let $U \subset \beta\widetilde{M}$ be open and such that $\mathcal{D} \subset U$. Then there is a constant $C_U$ such that

$$|\mu|\left(\beta\widetilde{M} \setminus U\right) \leq C_U \cdot (\|\mu\| - 1)$$

for any $\mu \in \mathcal{M}(\beta\widetilde{M})$ such that $\Phi^*\mu \in \mathcal{E}$.

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Particular case: For every $\psi \in \mathcal{E}$ there is $\mu \in \mathcal{M}(\beta\widetilde{M})$ such that

$$\Phi^*\mu = \psi \quad \text{and} \quad \|\mu\| = \|\psi\| = 1.$$

By the Lemma, this $\mu$ is supported on $\mathcal{D}$.

# The key construction

**Lemma**

If $\zeta \in \mathcal{D}$ then $\pi(\zeta) \in \beta[p,q] \times \beta[p,q]$.

# The key construction

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*Sketch of proof:* Let $\zeta \in \beta\widetilde{M}$, $\pi(\zeta) = (\xi, \eta)$ where $\xi, \eta \in \beta M$.
Suppose $\xi \notin \beta[p,q]$ or $\eta \notin \beta[p,q]$.

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Suppose $\xi \notin \beta[p,q]$ or $\eta \notin \beta[p,q]$.

Construct nets $x_i \to \xi$, $y_i \to \eta$ in $M$ and $f \in \mathrm{Lip}(\{p,q,x_i,y_i\})$ as

$$f(t) = \begin{cases} d(t,q) & \text{if } t = p \\ (1-\varepsilon)\cdot d(t,q) & \text{if } t = q \text{ or } x_i \text{ or } y_i \end{cases}.$$

If done right, $\|f\|_L = 1$.

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If done right, $\|f\|_L = 1$.

Extend $f$ to $M$, then $\Phi f(p,q) = 1$ and $|\Phi f(\zeta)| \leq 1 - \varepsilon$.
So $\zeta \notin \mathcal{D}$. □

# The preserved extreme case

**Theorem (Aliaga, Guirao 2017)**

If $\beta[p,q] = \{p,q\}$ then $u_{pq}$ is a preserved extreme point of $B_{\mathcal{F}(M)}$.

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*Proof:* Let $\psi \in \mathcal{E}$. Take $\mu \in \mathcal{M}(\beta\widetilde{M})$ with $\|\mu\| = 1$ and $\Phi^*\mu = \psi$. Then $\mu$ is supported on

$$\mathcal{D} \subset \{(p,q),(q,p)\} \cup \pi^{-1}(\{(p,p),(q,q)\}).$$

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Consider the action of $\mu$ on $f \in \mathrm{Lip}_0(M)$ such that

- $f$ is constant in neighborhoods of $p$ and $q$
- $f(p) \neq f(q)$

to see that $\mu$ is supported on $(p,q)$ and $(q,p)$. Thus $\psi = u_{pq}$. $\square$

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$\mathcal{E} \cap \mathcal{F}(M) \subset \mathcal{F}([p,q])$.

If $[p,q] = \{p,q\}$, it is immediate that $\mathcal{E} \cap \mathcal{F}(M) = \{u_{pq}\}$.

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**Theorem (Aliaga, Pernecká 2018)**

$\mathcal{E} \cap \mathcal{F}(M) \subset \mathcal{F}([p,q])$.

*Proof:* Let $m \in \mathcal{E} \cap \mathcal{F}(M)$. For $\varepsilon > 0$ define

$$[p,q]_\varepsilon = \{x \in M : d(p,x) + d(x,q) < d(p,q) + \varepsilon\}$$
$$\beta[p,q]_\varepsilon = \{\xi \in \beta M : d(p,\xi) + d(\xi,q) < d(p,q) + \varepsilon\}$$
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We will show that $m \in \mathcal{F}([p,q]_\varepsilon)$, so

$$m \in \bigcap_{\varepsilon>0} \mathcal{F}(\overline{[p,q]_\varepsilon}) = \mathcal{F}\left(\bigcap_{\varepsilon>0} \overline{[p,q]_\varepsilon}\right) = \mathcal{F}([p,q]).$$

# The extreme case

**Theorem (Aliaga, Pernecká 2018)**

$\mathcal{E} \cap \mathcal{F}(M) \subset \mathcal{F}([p,q])$.

Fix $\delta > 0$ and express $m$ as

$$m = \sum_{n=1}^{\infty} a_n u_{x_n y_n} \quad \text{where} \quad \sum_{n=1}^{\infty} |a_n| < 1 + \delta.$$

Let

$$\mu = \sum_{n=1}^{\infty} a_n \delta_{(x_n, y_n)} \in \mathcal{M}(\beta \widetilde{M}).$$

Then $\Phi^* \mu = m$ and $\|\mu\| < 1 + \delta$.

# The extreme case

**Theorem (Aliaga, Pernecká 2018)**

$\mathcal{E} \cap \mathcal{F}(M) \subset \mathcal{F}([p,q])$.

Let $\mu'(E) = \mu(E \cap U_\varepsilon)$. Then $\Phi^*\mu' = m'$ where

$$m' = \sum_{x_n, y_n \in [p,q]_\varepsilon} a_n u_{x_n y_n} \in \mathcal{F}([p,q]_\varepsilon)$$

and

$$\begin{aligned} \|m - m'\| = \|\Phi^*(\mu - \mu')\| \leq \|\mu - \mu'\| &= |\mu| (\beta\widetilde{M} \setminus U_\varepsilon) \\ &\leq C_{U_\varepsilon} \cdot (\|\mu\| - 1) < \delta C_{U_\varepsilon}. \end{aligned}$$

Thus $m$ is $(\delta C_{U_\varepsilon})$-close to $\mathcal{F}([p,q]_\varepsilon)$. $\square$

# What about exposed points?

| | molecule $u_{pq}$ | non-molecule |
|---|---|---|
| extreme | $[p,q]=\{p,q\}$ | |
| exposed | | |
| preserved extreme | $\beta[p,q]=\{p,q\}$ | impossible |
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# What about exposed points?

**Theorem (García-Lirola 2018 / Petitjean, Procházka 2018)**

If $[p,q] = \{p,q\}$ then $u_{pq}$ is an exposed point of $B_{\mathcal{F}(M)}$.

The exposing functional is

$$f_{pq}(x) = \frac{d(p,q)}{2}\,\frac{d(x,q)-d(x,p)}{d(x,q)+d(x,p)} - \text{constant}$$

# Current knowledge

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**Main question**

Let $M$ be a complete pointed metric space.
Are all extreme points of $B_{\mathcal{F}(M)}$ molecules?

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Known to be true when:

- $M$ is compact and $\mathcal{F}(M) = \mathrm{lip}_0(M)^*$ (Weaver 1999)
  - $M$ is compact Hölder
  - $M$ is compact and countable (Dalet 2015)
  - $M$ is compact and ultrametric (Dalet 2015)

Recall that $\mathrm{lip}_0(M) = \left\{ f \in \mathrm{Lip}_0(M) : \Phi f \in C_0(\widetilde{M}) \right\}$.

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    - $M$ is compact Hölder
    - $M$ is compact and countable (Dalet 2015)
    - $M$ is compact and ultrametric (Dalet 2015)
- $M$ has a natural predual that is a subspace of $\mathrm{lip}_0(M)$
  (Garcia-Lirola, Petitjean, Procházka, Rueda 2017)

Recall that $\mathrm{lip}_0(M) = \left\{ f \in \mathrm{Lip}_0(M) : \Phi f \in C_0(\widetilde{M}) \right\}$.

**Main question**

Let $M$ be a complete pointed metric space.
Are all extreme points of $B_{\mathcal{F}(M)}$ molecules?

Equivalently,

(1) Are all extreme points of $B_{\mathcal{F}(M)}$ exposed?
(2) Are all exposed points of $B_{\mathcal{F}(M)}$ molecules?

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Let $M$ be a complete pointed metric space.
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Let $M$ be a complete pointed metric space.
Does every $\mu \in \mathcal{F}(M)$ admit a support?

Equivalently,

Let $K_i \subset M$ be closed. Is it always true that

$$\bigcap_i \mathcal{F}(K_i) = \mathcal{F}\left(\bigcap_i K_i\right) \quad ?$$

# Thank you for your attention!

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