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Hausdorff-Young type inqualities for vector-valued Dirichlet series

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Joint work with D. Carando and F. Marceca (Universidad de Buenos Aires)

Dirichlet series

$$\sum a_n n^{-s}$$

where $a_n \in X$ (Banach space)

A norm for finite sums (polynomials)

For $1 \leq p < \infty$ define

$$\Big\| \sum_{n=1}^{N} a_n n^{-s} \Big\|_p = \lim_{R \to \infty} \left(\frac{1}{2R} \int_{-R}^{R} \Big\| \sum_{n=1}^{N} a_n \frac{1}{n^{it}} \Big\|_X^p dt \right)^{\frac{1}{p}}$$

- converges
- defines a norm

Hardy space [Bayart]

$$\mathscr{H}_p(X) = \text{completion}$$

Gateway: harmonic analysis

Infinite-dimensional torus

$$\mathbb{T}^\infty = \{(z_n)_n \subset \mathbb{C}\colon |z_n| = 1\}$$

with the normalised Lebesgue measure.

Fourier coefficient

$f \in L_1(\mathbb{T}^\infty, X)$
$\alpha = (\alpha_1, \dots, \alpha_n, 0, 0, \dots)$ with $\alpha_i \in \mathbb{Z}$ for $i = 1, \dots, n$ and $n \in \mathbb{N}$

$$\hat{f}(\alpha) = \int_{\mathbb{T}^\infty} f(z) z^{-\alpha} dz$$

Hardy space

$$H_p(\mathbb{T}^\infty, X) = \left\{ f \in L_p(\mathbb{T}^\infty, X)\colon \exists i,\ \alpha_i < 0 \Rightarrow \hat{f}(\alpha) = 0 \right\}$$

Gateway: harmonic analysis

decomposition as product of primes

Correspondence

$\sum a_n n^{-s} \in \mathscr{H}_p(X)$ if and only if

$$\exists f \in H_p(X) \ \text{s.t.} \ \hat{f}(\alpha) = a_n \ \text{with} \quad n = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$$

Moreover

$$\Big\| \sum a_n n^{-s} \Big\|_{\mathscr{H}_p(X)} = \|f\|_{H_p(\mathbb{T}^\infty, X)}$$

What do we aim at?

Conditions that in some sense relate the $\mathscr{H}_p(X)$-norm with the coefficients

What do we know?

Cotype

X has cotype $2 \leq q \leq \infty$ if there is $C > 1$ so that

$$\Big(\sum_{n=1}^{N} \|x_n\|_X^q \Big)^{\frac{1}{q}} \leq C \int_{\mathbb{T}^N} \Big\| \sum_{n=1}^{N} x_n z_n \Big\|_X dz$$

for every choice $x_1, \ldots, x_N \in X$.

[Carando-Defant-S]

If X has cotype q then for every $\sigma > 1 - \frac{1}{q} = \frac{1}{q'}$ we have

$$\sum_{n=1}^{\infty} \frac{\|a_n\|_X}{n^\sigma} \leq C \Big\| \sum a_n n^{-s} \Big\|_{\mathscr{H}_1(X)}$$

Problem

The inequality is too weak ... can we do better?

Hausdorff-Young inequality – scalar valued

We consider the operator

$$f : \mathbb{T} \to \mathbb{C} \rightsquigarrow \big(\hat{f}(n)\big)_{n \in \mathbb{Z}}$$

Easy

$$L_1(\mathbb{T}) \longrightarrow \ell_\infty(\mathbb{Z}) \text{ bounded}$$

Plancherel

$$L_2(\mathbb{T}) \longrightarrow \ell_2(\mathbb{Z}) \text{ isometry}$$

Interpolating (Haussdorff-Young)

$$L_p(\mathbb{T}) \longrightarrow \ell_{p'}(\mathbb{Z}) \text{ bounded}$$

for $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

Hausdorff-Young inequality – scalar valued

In particular

$$\Big(\sum_{n=0}^{\infty} |\hat{f}(n)|^{p'} \Big)^{\frac{1}{p'}} \leq C \|f\|_p$$

for every $f \in H_p(\mathbb{T})$

This transfers to $\mathbb{T}^\infty$ and gives

$$\Big(\sum_{n=1}^{\infty} |a_n|^{p'} \Big)^{\frac{1}{p'}} \leq C \Big\| \sum a_n n^{-s} \Big\|_{\mathscr{H}_p(\mathbb{C})}$$

for every $1 \leq p \leq 2$.
With a similar idea (taking an operator $\ell_1 \to L_\infty$ and $\ell_2 \to L_2$ and interpolating…) one gets another HY inequality and, from it, deduces

$$\Big\| \sum a_n n^{-s} \Big\|_{\mathscr{H}_p(\mathbb{C})} \leq C \Big(\sum_{n=1}^{\infty} |a_n|^{p'} \Big)^{\frac{1}{p'}}$$

for $2 \leq p \leq \infty$.

Hausdorff-Young inequality – vector valued?

Still easy

$$L_1(\mathbb{T}, X) \longrightarrow \ell_\infty(\mathbb{Z}, X) \;\text{ bounded}$$

Unfortunately Plancherel does not hold in general ...

Fourier cotype

X has Fourier cotype $2 \leq q \leq \infty$ if there is $C > 1$ so that

$$\Big(\sum_{n=1}^{m} \|x_n\|_X^q\Big)^{\frac{1}{q}} \leq C\Big(\int_{\mathbb{T}} \Big\|\sum_{n=1}^{m} x_n z^n\Big\|_X^{q'} dz\Big)^{\frac{1}{q'}}$$

for every choice $x_1, \ldots, x_m \in X$.

In other words

Given $1 \leq p \leq 2$,

$$L_p(\mathbb{T}, X) \longrightarrow \ell_{p'}(\mathbb{Z}, X)$$

is bounded if and only if X has Fourier cotype p'.

Hausdorff-Young inequality – vector valued?

Theorem

Are equivalent

(a) X has Fourier cotype q (with constant C)

(b) $\left(\sum_{\alpha} \|\hat{f}(\alpha)\|^q\right)^{\frac{1}{q}} \leq C\|f\|_{H_{q'}(\mathbb{T}^\infty, X)}$

(c) $\left(\sum_{n=1}^{\infty} \|a_n\|^q\right)^{\frac{1}{q}} \leq C\left\|\sum a_n n^{-s}\right\|_{\mathscr{H}_{q'}(X)}$

Problem

Having Fourier cotype is too restrictive

Something in between - reformulating what we know

Polynomial in n variables: finite sum

$$P(z) = \sum_{\alpha \in \mathbb{N}_0^n} x_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n} = \sum_{\alpha \in \mathbb{N}_0^n} x_\alpha z^\alpha$$

Degree: $\max\{\alpha_1 + \cdots + \alpha_n\}$.

Cotype *q*

there is $C > 1$ so that for every n and every polynomial P of degree 1 in n variables

$$\Big(\sum_\alpha \|x_\alpha\|_X^q \Big)^{\frac{1}{q}} \leq C \|P\|_1$$

Fourier cotype *q*

there is $C > 1$ so that for every m and every polynomial P of degree m in n variables

$$\Big(\sum_\alpha \|x_\alpha\|_X^q \Big)^{\frac{1}{q}} \leq C \|P\|_q$$

Something in between – polynomial cotype

Polynomial cotype [CDS]

X has polynomial cotype $2 \leq q \leq \infty$ if there is $C > 1$ so that for every m and n and every polynomial of degree m in n variables

$$\Big(\sum_{\alpha} \|x_\alpha\|_X^q \Big)^{\frac{1}{q}} \leq C^m \|P\|_1$$

Fact

Fourier cotype $\Rightarrow$ polynomial cotype $\Rightarrow$ cotype

Something in between – polynomial cotype

Theorem

Are equivalent

(a) X has polynomial cotype q;

(b) there exist $C \geq 1$ and $0 < r < 1$ such that

$$\left(\sum_{n=1}^{\infty} r^{q\,\Omega(n)}\,\|a_n\|^q\right)^{\frac{1}{q}} \leq C\Big\|\sum a_n n^{-s}\Big\|_{\mathscr{H}_1(X)}$$

number of primes in the decomposition, counted with multiplicity

(c) there exist $C \geq 1$ and $0 < r < 1$ such that

$$\left(\sum_{\alpha} r^{q\alpha}\|\widehat{f}(\alpha)\|^q\right)^{\frac{1}{q}} \leq C\|f\|_{H_1(\mathbb{T}^\infty, X)}$$

Something in between – polynomial cotype

Moreover…

If X has non-trivial type the previous are equivalent to

(d) for every $1 < p < \infty$, there exist constants $C \geq 1$ and $0 < r < 1$ such that every function $f \in L_p(\{-1,1\}^\infty, X)$ satisfies

$$\left(\sum_{\substack{A \subset \mathbb{N} \\ A \text{ finite}}} r^{q|A|} \|\widehat{f}(A)\|^q \right)^{1/q} \leq C \|f\|_{L_p(\{-1,1\}^\infty, X)}$$

Spaces with polynomial cotype

Question

Which spaces have polynomial cotype?

Known [CDS]

- Fourier cotype
- local unconditional structure + cotype

In particular

- $\mathscr{L}_p$-spaces have polynomial cotype q for $q = \max(2, p)$
- for $2 \leq p \leq \infty$, Schatten classes $\mathscr{S}_p$ have polynomial cotype p

Spaces with polynomial cotype – an interesting feature

Recall polynomial cotype

For every m and n and every polynomial of degree m, $P(z) = \sum_{\alpha\in\mathbb{N}_0^n} x_\alpha z^\alpha$ in n variables

$$\Big(\sum_{\alpha} \|x_\alpha\|_X^q\Big)^{\frac{1}{q}} \leq C^m \|P\|_1 \qquad (\star)$$

Tetrahedral polynomial

All variables appear with power at most 1, that is

$$P(z) = \sum_{\alpha\in\{0,1\}^n} x_\alpha z^\alpha$$

Theorem

X has polynomial cotype q if and only if $(\star)$ holds for every tetrahedral polynomial of degree m in n variables

Spaces with polynomial cotype

Other properties that imply polynomial cotype

- Walsh cotype $q \Rightarrow$ polynomial cotype q
- non-trivial type + cotype 2 $\Rightarrow$ polynomial cotype 2
- Gaussian Approximation Property [Casazza-Nielsen] (+ cotype q) $\Rightarrow$ polynomial cotype q
    - l.u.st. + cotype q
    - type 2 + cotype q
    - Gordon-Lewis property + cotype q
- q-uniform PL-convex $\Rightarrow$ polynomial cotype q

Spaces with polynomial cotype

Uniform PL-convexity [Davis-Garling-Tomczak Jaegermann]

X is q-uniformly PL-convex (for $q \geq 2$) if there exists $\lambda > 0$ such that

$$\|x\|^q + \lambda\|y\|^q \leq \int_{\mathbb{T}} \|x + zy\|^q dz,$$

for all $x, y \in X$.

Spaces that are *q*-uniformly PL-convex

- q-uniformly convex
- any non-commutative L_1-space (for $q = 2$) [Haagerup]
- spaces with ARNP (for $q = 2$) [Haagerup-Pisier]
- Schatten classes $\mathscr{S}_p$ (for $q = \max(p, 2)$)

In particular

Schatten classes have polynomial cotype $q = \max(p, 2)$

Spaces with polynomial cotype

What is left?

We do not know of a Banach space that does not fall into one of the previous classes

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Candidate

$$L_1(\mathbb{T})/H_1(\mathbb{T})$$

is not uniformly PL-convex, but we do not know if it has GAP (or polynomial cotype).

Can we have reversed inequalities?

Recall the 'second' Hausdorff-Young inequality

$$\|f\|_{p'} \leq C\Big(\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^p\Big)^{\frac{1}{p}}$$

for $1 \leq p \leq 2$, that led to

$$\Big\| \sum a_n n^{-s} \Big\|_{\mathscr{H}_{p'}(\mathbb{C})} \leq C\Big(\sum_{n=1}^{\infty} |a_n|^p\Big)^{\frac{1}{p}}$$

Polynomial type

A Banach space X has polynomial type $1 \leq p \leq 2$ if there is $C > 1$ so that for every m and n and every polynomial of degree m in n variables

$$\|P\|_1 \leq C^m\Big(\sum_{\alpha} \|x_\alpha\|_X^p\Big)^{\frac{1}{p}}$$

Polynomial type

Theorem

Are equivalent

(a) X has polynomial type p

(b) for some $1 \leq q < \infty$ there exist $R, C \geq 1$ such that

$$\Big\| \sum a_n n^{-s} \Big\|_{\mathscr{H}_q(X)} \leq C \Big(\sum_{n=1}^{\infty} R^{p\Omega(n)} \|a_n\|^p \Big)^{\frac{1}{p}}$$

(c) for every $1 \leq q < \infty$ there exist $R, C \geq 1$ such that

$$\Big\| \sum a_n n^{-s} \Big\|_{\mathscr{H}_q(X)} \leq C \Big(\sum_{n=1}^{\infty} R^{p\Omega(n)} \|a_n\|^p \Big)^{\frac{1}{p}}$$

and ...

analogous inequalities for $H_q(\mathbb{T}^\infty, X)$ and $L_q(\{-1,1\}^\infty, X)$.

Polynomial type

Relation with cotype

- X polynomial type $p \Rightarrow X^*$ polynomial cotype p'
- X polynomial cotype q + non-trivial type $\Rightarrow X^*$ polynomial type q'

Spaces having polynomial type

- type 2 $\Rightarrow$ polynomial type 2
- Gordon-Lewis property + type $p \Rightarrow$ polynomial type p
- p-uniform smooth $\Rightarrow$ polynomial type p

For example…

$\mathscr{L}_p$-spaces and Schatten classes $\mathscr{S}_p$ have polynomial type $\min(2, p)$

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