Small-bound isomorphisms of function spaces

Jiří Spurný

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Banach-Stone theorem

Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$.

Notation

Let K be compact.

- *Let $\mathcal{C}(K, \mathbb{F})$ stand for the space of $\mathbb{F}$-valued continuous functions on K.*
- *Let $\mathcal{M}^1(K)$ denote the space of Radon probability measures on K.*

Theorem (Banach-Stone)

Let K, L be compact spaces. Then K is homeomorphic to L if and only if $\mathcal{C}(K, \mathbb{F})$ is isometric to $\mathcal{C}(L, \mathbb{F})$.

Theorem of Amir and Cambern

Theorem (Amir, Cambern)

Let $T\colon \mathcal{C}(K,\mathbb{F}) \to \mathcal{C}(L,\mathbb{F})$ *be an isomorphism with* $\|T\| \cdot \|T^{-1}\| < 2$*, then* K *is homeomorphic to* L*.*

Theorem (Cohen)

There exist non-homeomorphic compact spaces K*,* L *and an isomorphism* $T\colon \mathcal{C}(K,\mathbb{R}) \to \mathcal{C}(L,\mathbb{R})$ *with* $\|T\| \cdot \|T^{-1}\| = 2$*.*

Compact convex sets

$X \ldots$ a compact convex set in a locally convex (Hausdorff) space.
$\mathcal{A}(X, \mathbb{F}) \ldots$ affine continuous $\mathbb{F}$-valued functions on X

If $\mu \in \mathcal{M}^1(X)$, then ***barycenter*** *$r(\mu)$ satisfies*
$f(r(\mu)) = \int_X f \,\mathrm{d}\mu \,(= \mu(f))$, $f \in \mathcal{A}(X, \mathbb{F})$.
Also, μ ***represents*** *$r(\mu)$. The barycenter exists and it is unique.*

If $\mu = \sum_{i=1}^n a_i \varepsilon_{x_i}$, where $x_i \in X$, $a_i \geq 0$, $\sum_{i=1}^n a_i = 1$, then
$r(\mu) = \sum_{i=1}^n a_i x_i$.

Simplices

Let X be a compact convex set.

Definition (Choquet ordering)

Let $\mu, \nu \in \mathcal{M}^1(X)$. Then $\mu \prec \nu$ if $\int k\, d\mu \leq \int k\, d\nu$ for each convex continuous function k on X.

Theorem (Choquet-Bishop-de-Leeuw)

For each $x \in X$ there exist a $\prec$-maximal measure $\mu \in \mathcal{M}^1(X)$ with $r(\mu) = x$.

Definition (simplex)

The set X is a ***simplex*** *if for each $x \in X$ there exist a unique $\prec$-maximal measure $\mu \in \mathcal{M}^1(X)$ with $r(\mu) = x$.*

Bauer simplices

Let X be a compact convex set and $\mathcal{A}(X, \mathbb{F})$ stand for the space of all affine continuous $\mathbb{F}$-valued functions on X.

Definition (Bauer simplex)

*A simplex X is a **Bauer simplex** if $\operatorname{ext} X$ is closed.*

Theorem

If X is a Bauer simplex, then $\mathcal{A}(X, \mathbb{F}) = \mathcal{C}(\operatorname{ext} X, \mathbb{F})$.

Theorem

If K is a compact, then $\mathcal{C}(K, \mathbb{F}) = \mathcal{A}(\mathcal{M}^1(K), \mathbb{F})$.

Reformulation of isomorphism theorems

Theorem (Banach-Stone)

If X, Y are Bauer simplices and $\mathcal{A}(X, \mathbb{F})$ is isometric to $\mathcal{A}(Y, \mathbb{F})$, then $\operatorname{ext} X$ is homeomorphic to $\operatorname{ext} Y$.

Theorem (Amir, Cambern)

If X, Y are Bauer simplices and there exists an isomorphism $T\colon \mathcal{A}(X, \mathbb{F}) \to \mathcal{A}(Y, \mathbb{F})$ with $\|T\| \cdot \|T^{-1}\| < 2$, then $\operatorname{ext} X$ is homeomorphic to $\operatorname{ext} Y$.

Results of Chu and Cohen

Theorem (Chu-Cohen)

Given compact convex sets X and Y, the sets $\operatorname{ext} X$ and $\operatorname{ext} Y$ are homeomorphic provided there exists an isomorphism $T\colon \mathcal{A}(X,\mathbb{R}) \to \mathcal{A}(Y,\mathbb{R})$ with $\|T\| \cdot \left\|T^{-1}\right\| < 2$ and one of the following conditions hold:

- (i) *X and Y are simplices such that their extreme points are weak peak points;*
- (ii) *X and Y are metrizable and their extreme points are weak peak points;*
- (iii) *$\operatorname{ext} X$ and $\operatorname{ext} Y$ are closed and extreme points of X and Y are split faces.*

Definition

*A point $x \in X$ is a **weak peak point** if given $\varepsilon \in (0,1)$ and an open set $U \subset X$ containing x, there exists a in the unit ball $B_{\mathcal{A}(X,\mathbb{F})}$ of $\mathcal{A}(X,\mathbb{F})$ such that $|a| < \varepsilon$ on $\operatorname{ext} X \setminus U$ and $a(x) > 1 - \varepsilon$.*

Assumption on weak peak points

Theorem (Hess)

For each $\varepsilon \in (0,1)$ *there exist metrizable simplices* X, Y *and an isomorphism* $T\colon \mathcal{A}(X,\mathbb{R}) \to \mathcal{A}(Y,\mathbb{R})$ *with* $\|T\| \cdot \left\|T^{-1}\right\| < 1+\varepsilon$ *such that* $\operatorname{ext} X$ *is not homeomorphic to* $\operatorname{ext} Y$*.*

Theorem (Lazar)

If X, Y *are simplices and* $\mathcal{A}(X,\mathbb{R})$ *is isometric to* $\mathcal{A}(Y,\mathbb{R})$*, then* X *is affinely homeomorphic to* Y*, in particular* $\operatorname{ext} X$ *is homeomorphic to* $\operatorname{ext} Y$*.*

Small bound isomorphisms of spaces of real affine continuous functions

Theorem (Ludvík-S.)

Given compact convex sets X and Y, the sets $\operatorname{ext} X$ and $\operatorname{ext} Y$ are homeomorphic provided there exists an isomorphism $T\colon \mathcal{A}(X,\mathbb{R}) \to \mathcal{A}(Y,\mathbb{R})$ with $\|T\| \cdot \|T^{-1}\| < 2$, extreme points of X and Y are weak peak points and both $\operatorname{ext} X$ and $\operatorname{ext} Y$ are Lindelöf.

Theorem (Dostál-S.)

Given compact convex sets X and Y, the sets $\operatorname{ext} X$ and $\operatorname{ext} Y$ are homeomorphic provided there exists an isomorphism $T\colon \mathcal{A}(X,\mathbb{R}) \to \mathcal{A}(Y,\mathbb{R})$ with $\|T\| \cdot \|T^{-1}\| < 2$ and extreme points of X and Y are weak peak points.

Small bound isomorphisms of spaces of complex affine continuous functions

Theorem (Rondoš-S.)

Given compact convex sets X and Y, the sets $\operatorname{ext} X$ and $\operatorname{ext} Y$ are homeomorphic provided there exists an isomorphism $T\colon \mathcal{A}(X,\mathbb{C}) \to \mathcal{A}(Y,\mathbb{C})$ with $\|T\| \cdot \left\|T^{-1}\right\| < 2$ and extreme points of X and Y are weak peak points.

Complex function spaces

Given a compact Hausdorff space K, we consider a closed subspace $\mathcal{H} \subset \mathcal{C}(K, \mathbb{C})$ which contains constants and separates points of K. By $\mathbf{S}(\mathcal{H})$ we denote the ***state space*** *of $\mathcal{H}$, i.e., the set*

$$\mathbf{S}(\mathcal{H}) = \{s \in \mathcal{H}^*; \|s\| = s(1) = 1\}$$

endowed with the weak* topology. Let $\phi \colon K \to \mathbf{S}(\mathcal{H})$ be the evaluation mapping, then ϕ homeomorphically embeds K into the compact convex set $\mathbf{S}(\mathcal{H})$. The ***Choquet boundary*** $\operatorname{Ch}_{\mathcal{H}} K$ *of $\mathcal{H}$ is defined as*

$$\operatorname{Ch}_{\mathcal{H}} K = \{x \in K; \phi(x) \in \operatorname{ext} \mathbf{S}(\mathcal{H})\}.$$

Then $\operatorname{ext} \mathbf{S}(\mathcal{H}) = \phi(\operatorname{Ch}_{\mathcal{H}} K)$.

Selfadjoint complex function spaces

Definition

Given a function space $\mathcal{H}$, we say that $x \in K$ is a ***weak peak point*** *if given $\varepsilon \in (0,1)$ and an open set $U \subset K$ containing x, there exists $f \in B_{\mathcal{H}}$ such that $|f| < \varepsilon$ on $\operatorname{Ch}_{\mathcal{H}} K \setminus U$ and $f(x) > 1 - \varepsilon$.*

Theorem (Rondoš-S.)

For $i = 1, 2$, let K_i be a compact space and $\mathcal{H}_i$ be a selfadjoint closed subspace of $\mathcal{C}(K_i, \mathbb{C})$ which contains constants and separates points of K_i. Let each point of $\operatorname{Ch}_{\mathcal{H}_i} K_i$ be a weak peak point.
If there exists an isomorphism $T : \mathcal{H}_1 \to \mathcal{H}_2$ satisfying $\|T\| \cdot \|T^{-1}\| < 2$, then $\operatorname{Ch}_{\mathcal{H}_1} K_1$ is homeomorphic to $\operatorname{Ch}_{\mathcal{H}_2} K_2$.

General function spaces on locally compact space

Let K be a locally compact Hausdorff space and $\mathcal{H} \subset \mathcal{C}_0(K, \mathbb{F})$ be a closed subspace. Then $x \in K$ is in the ***Choquet boundary*** $\operatorname{Ch}_{\mathcal{H}} K$ ***of*** *$\mathcal{H}$ if the point evaluation functional $\phi(x)$ defined as $\phi(x)(h) = h(x)$, $h \in \mathcal{H}$, is an extreme point of $B_{\mathcal{H}^*}$. Again, a point $x \in K_i$ is a* ***weak peak point*** *if for a given $\varepsilon \in (0, 1)$ and a neighborhood U of x there exists a function $h \in B_{\mathcal{H}}$ such that $h(x) > 1 - \varepsilon$ and $|h| < \varepsilon$ on $\operatorname{Ch}_{\mathcal{H}} K \setminus U$.*

Theorem (Rondoš-S.)

For $i = 1, 2$, let $\mathcal{H}_i$ be a closed subspace of $\mathcal{C}_0(K_i, \mathbb{F})$ for some locally compact space K_i. Assume that each point of $\operatorname{Ch}_{\mathcal{H}_i} K_i$ is a weak peak point and let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be an isomorphism satisfying $\|T\| \cdot \|T^{-1}\| < 2$. Then $\operatorname{Ch}_{\mathcal{H}_1} K_1$ is homeomorphic to $\operatorname{Ch}_{\mathcal{H}_2} K_2$.

One-sided versions

Theorem (Jarosz)

If K_1, K_2 are locally compact spaces, $A \subset \mathcal{C}_0(K_1, \mathbb{C})$ is an extremely regular closed subspace and a not necessarily surjective isomorphism $T \colon A \to \mathcal{C}_0(K_2, \mathbb{C})$ satisfies $\|T\| \cdot \|T^{-1}\| < 2$, then K_1 is a continuous image of a subset of K_2.

Theorem (Rondoš-S.)

For $i = 1, 2$, let $\mathcal{H}_i$ be a closed subspace of $\mathcal{C}_0(K_i, \mathbb{F})$ for some locally compact space K_i. Assume that each point of $\mathrm{Ch}_{\mathcal{H}_1} K_1$ is a weak peak point and let $T \colon \mathcal{H}_1 \to \mathcal{H}_2$ be an into isomorphism satisfying $\|T\| \cdot \|T^{-1}\| < 2$. Then there exists a set $L \subset \mathrm{Ch}_{\mathcal{H}_2} K_2$ and a continuous surjective mapping $\varphi \colon L \to \mathrm{Ch}_{\mathcal{H}_1} K_1$.

Cardinality of the Choquet boundaries

Theorem (Cengiz)

Locally compact spaces K_1, K_2 have the same cardinality provided $\mathcal{C}_0(K_1, \mathbb{F})$ is isomorphic to $\mathcal{C}_0(K_2, \mathbb{F})$.

Theorem (Rondoš-S.)

For $i = 1, 2$, let $\mathcal{H}_i$ be a closed subspace of $\mathcal{C}_0(K_i, \mathbb{F})$ for some locally compact space K_i. Assume that each point of $\mathrm{Ch}_{\mathcal{H}_i} K_i$ is a weak peak point and let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be an isomorphism. Then the cardinality of $\mathrm{Ch}_{\mathcal{H}_1} K_1$ is equal to the cardinality of $\mathrm{Ch}_{\mathcal{H}_2} K_2$.

Thank you for your attention.