Smooth surjections and surjective restrictions

Jesús A. Jaramillo

Universidad Complutense de Madrid (Madrid, Spain)

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Dedicated to the memory of our friend Bernardo

Based on joint work with:

- Richard M. Aron
- Enrico Le Donne
- Tapio Rajala
- Thomas Ransford

Smooth surjections

- In the geometric nonlinear theory of Banach spaces, *smooth surjections* and their relations with nonlinear quotients play an interesting role.
- If E and F are infinite-dimensional Banach spaces, a first issue is the existence of smooth surjective mappings from E onto F.
- **Theorem** [Bates, 1997]. If E and F are infinite-dimensional separable Banach spaces, there always exists a C^1-smooth surjective map $f : E \to F$.
- Can we obtain a surjection with a higher degree of smoothness?

Existence of smooth surjections

- Bates also gave sufficient conditions for the existence of C^∞-smooth surjections.

- **Theorem** [Bates, 1997]. If E and F are infinite-dimensional separable Banach spaces and E is super-reflexive, there exists a C^∞-smooth surjective mapping $f : E \to F$.

Critical values

- Furthermore, the C^∞-smooth mapping $f : E \to F$ constructed by Bates satisfies that $\text{rank}(Df(x)) \leq 1$, for every $x \in E$.
- In particular, *every* value of f is a *critical value*, that is, the image of a point where the derivative is *not* onto.
- This shows a strong failure of Morse-Sard Theorem in infinite dimension.

Non-existence of smooth surjections

- **Theorem** [Hajék, 1998]. If $E = c_0(\mathbb{N})$ and F is an infinite-dimensional super-reflexive space, there is no C^2-smooth surjective mapping from E onto F.

- Guirao, Hajék and Montesinos (2010) proved that the existence of C^2-smooth surjections from $c_0(\omega_1)$ to $\ell_2(\mathbb{N})$ depends on additional axioms of set theory.

Our aim

- We are interested in the following problem concerning the restrictions of smooth surjections.

- **Main Question (Aron):** Suppose that $f : E \to F$ is a smooth surjective map between Banach spaces, where E is non-separable and F is separable. We ask whether there is a separable subspace $G \subset E$ such that the restriction of f to G remains surjective.

The linear case

- Let $T : E \to F$ be a continuous linear surjection between Banach spaces.
- The classical Bartle-Graves selection theorem asserts that there is a *continuous section* $s : F \to E$ of T. This means that s is continuous and $T(s(y)) = y$ for every $y \in F$.
- Consider the closed subspace G of E generated by $s(F)$. If F is separable, then also G is separable, and it is clear that the restriction of T to G remains surjective.

Smooth nonlinear surjections

- As we will see, the situation in the case of general smooth nonlinear surjections will be different.
- **Remark**. Let E a Banach space and $f : E \to \mathbb{R}$ a continuous surjection. For every $q \in \mathbb{Q}$ there exists some $x_q \in E$ such that $f(x_q) = q$. Then the closed linear space G generated by the x_q's is separable, and $f(G)$ is a connected subset of $\mathbb{R}$ containing $\mathbb{Q}$, so that $f(G) = \mathbb{R}$.

Smooth nonlinear surjections

Theorem [Aron, J. and Ransford, 2013]. Let E be a Banach space with C^m-cellularity $\geq 2^{\aleph_0}$, and let F be a separable Banach space with dimension ≥ 2. Then there exists a C^m-smooth map $f : E \to F$ such that:

1. f is surjective.
2. When restricted to any separable subspace of E, f is not surjective.
3. $\text{rank}(Df(x)) \leq 1$, for every $x \in E$, so in particular every value of f is a critical value.

Cellularity

- Let E be a Banach space, and $1 \leq m \leq \infty$. We say that a subset U of E is a C^m-*cozero set* if

$$U = \{x \in E \,:\, f(x) \neq 0\},$$

where f is a C^m smooth real function on E.

- We say that E is C^m-*smooth* if there is a bounded C^m-cozero set in E.

- Now let κ be a cardinal number. We say that E has C^m-*cellularity* $\geq \kappa$ if there exists a disjoint, locally finite family $\{W_\gamma\}_{\gamma \in \Gamma}$ of C^m-cozero sets of E, with cardinality κ.

Large cellularity

- If Γ is a set with cardinality $\geq \kappa$, for $1 \leq p < \infty$ the space $\ell_p(\Gamma)$ has C^∞-cellularity $\geq \kappa$.

- If Γ is a set with cardinality $\geq \kappa$, the space $c_0(\Gamma)$ also has C^∞-cellularity $\geq \kappa$.

- For any infinite set I, the space $\ell_\infty(\mathrm{I})$ has C^∞-cellularity $\geq 2^{card(\mathrm{I})}$.

Cellularity

Lemma. Let E be a Banach space, and consider a cardinal $\kappa > \aleph_0$. The following conditions are equivalent:

- E has C^m-cellularity $\geq \kappa$.
- There exist a set Γ with cardinality κ and a C^m-smooth mapping $\phi : E \to c_0(\Gamma)$, whose range contains the unit vector basis $\{e_\gamma\}_{\gamma\in\Gamma}$ of $c_0(\Gamma)$.
- There exist a C^m-smooth Banach space Z and a C^m-smooth mapping $\phi : E \to Z$, whose range has density character $\geq \kappa$.

Smooth surjections

- The previous Theorem applies, in particular, to smooth surjections from $\ell_p(2^{\aleph_0})$ to $\mathbb{R}^2$.
- We wonder what happens with smooth surjections from $\ell_p(\omega_1)$ to $\mathbb{R}^2$.
- **Theorem** [Hajék and Johannis, 2018]. Let E be a non-separable super-reflexive Banach space and let F be a separable Banach space with dimension ≥ 2. Then there exists a C^∞-smooth surjection $f : E \to F$, such that the restriction of f to any separable subspace of E fails to be surjective.

Uniformly open maps

- A map $f : X \to Y$ between metric spaces is said to be *uniformly open* (or *co-uniformly continuous*) if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $x \in X$:

$$f\,(B(x, \varepsilon)) \supset B(f(x), \delta).$$

- We say that f is a *co-Lipschitz* map if there is a constant $C > 0$ such that $\delta = \varepsilon / C$.

- We say that f is a *Lipschitz quotient* if it is both Lipschitz and co-Lipschitz.

Uniformly open surjections

- **Theorem** [Aron, J. and Le Donne, 2017]. Let $f : X \to Y$ be a continuous, uniformly open surjection between metric spaces, where X is complete. Then there is a subset $Z \subset X$ such that $\mathrm{dens}(Z) = \mathrm{dens}(Y)$ and the restriction $f|_Z$ remains surjective.

- The *density character* $\mathrm{dens}(X)$ is the smallest cardinality of a dense subset of X.

Regular smooth surjections

- **Corollary.** Let $f : E \to F$ be a C^1-smooth surjection between Banach spaces. If the set of critical values of f has cardinality $\leq \mathrm{dens}(F)$, there is a subspace G of E such that $\mathrm{dens}(G) = \mathrm{dens}(F)$ and the restriction $f|_G$ remains surjective. In particular this applies when the set of critical values of f is countable.

- **Example** [Aron, J. and Le Donne, 2017]. There exists a C^∞-smooth surjection $f : \ell_2(2^{\aleph_0}) \to \mathbb{R}^2$ such that:
  1. When restricted to any separable subspace of $\ell_2(2^{\aleph_0})$, f is not surjective.
  2. The set of critical values of f has zero-measure in $\mathbb{R}^2$.

Uniformly open surjections: refinement

- **Theorem** [Kania and Rmoutil, 2017]. Let $f : X \to Y$ be a continuous, uniformly open surjection between metric spaces, where X is complete. Then there is a subset $Z \subset X$ with $\mathrm{dens}(Z) = \mathrm{dens}(Y)$, such that the restriction $f|_Z$ remains uniformly open and surjective.

Open surjections

- **Theorem** [J., Le Donne and Rajala, 2019]. Let $f : X \to Y$ be a continuous, open surjection between metric spaces, where X is complete. Then there is a subset $Z \subset X$ with $\mathrm{dens}(Z) = \mathrm{dens}(Y)$, such that the restriction $f|_Z$ remains surjective.

THANK YOU!