Zhang's inequality for log-concave functions

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(joint with D. Alonso-Gutiérrez and J. Bernués)

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To the memory of Bernardo Cascales

Isoperimetric & Sobolev inequalities

Theorem 1 (Sobolev ineq.)

Let $f : M \to \mathbb{R}$ with $M \subset \mathbb{R}^n$ compact and $f \in \mathcal{C}^1$. Then

$$n|\mathbb{B}_2^n|^{\frac{1}{n}} \left(\int_{\mathbb{R}^n} |f|^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} \leq \int_{\mathbb{R}^n} |\nabla f|$$

"$=$" iff $f = \chi_{\mathbb{B}_2^n}$.

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"$=$" iff $f = \chi_{\mathbb{B}_2^n}$.

Theorem 2 (Isoperimetric ineq.)

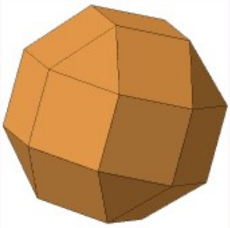
Let $M \subset \mathbb{R}^n$ with "good boundary" and $\overline{M}$ compact. Then

$$n|\mathbb{B}_2^n|^{\frac{1}{n}} \, |M|^{\frac{n-1}{n}} \leq \partial(M).$$

"$=$" iff $M = \mathbb{B}_2^n$.

Definitions

- K is a convex body, i.e., a convex and compact set of $\mathbb{R}^n$.



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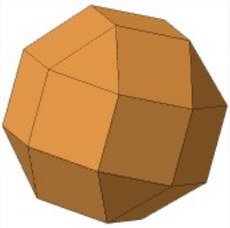
- $\mathcal{K}^n$ is the set of all n-dimensional convex bodies.



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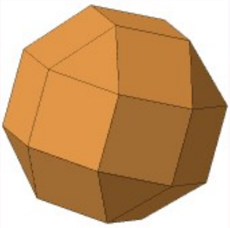
- The polar projection body $\Pi^*(K)$ of $K \in \mathcal{K}^n$ is the unit ball of the norm

$$\|x\|_{\Pi^*(K)} := |x||P_{x^\perp}K|.$$



Petty projection inequality

Theorem 3 (Petty 1971)

Let $K \in \mathcal{K}^n$. Then

$$|K|^{n-1}|\Pi^*(K)| \leq \frac{\pi^{\frac{n}{2}}\Gamma\left(\frac{n+1}{2}\right)^n}{\Gamma\left(\frac{n+2}{2}\right)^n}.$$

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Theorem 4 (Zhang 1991)

Let $K \in \mathcal{K}^n$. Then

$$|K|^{n-1}|\Pi^*(K)| \geq \frac{\binom{2n}{n}}{n^n}.$$

"$=$" iff K is a simplex.

Log-concave integrable functions in $\mathbb{R}^n$

Definition

$f : \mathbb{R}^n \to [0, \infty)$ is log-concave if $f(x) = e^{-u(x)}$ for some $u : \mathbb{R}^n \to (-\infty, \infty]$ convex,

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$\mathcal{F}(\mathbb{R}^n)$ is the set log-concave integrable functions in $\mathbb{R}^n$.

Impact of log-concave functions

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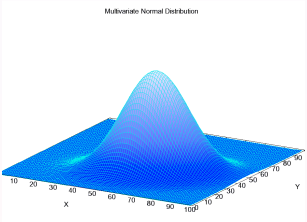

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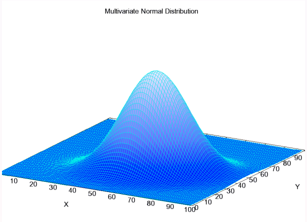

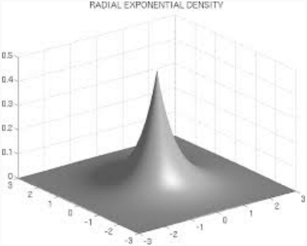

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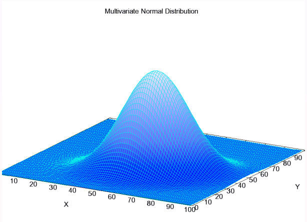
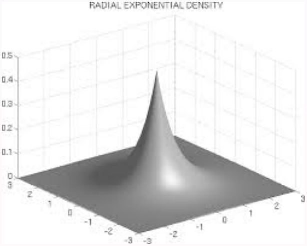
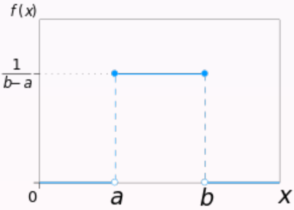
  - ▷ characteristics $\quad a \cdot \chi_K(x),\ K \in \mathcal{K}^n.$







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▷ Brunn-Minkowski inequality

$$|(1-\lambda)K+\lambda C|^{\frac{1}{n}} \geq (1-\lambda)|K|^{\frac{1}{n}}+\lambda|C|^{\frac{1}{n}}$$

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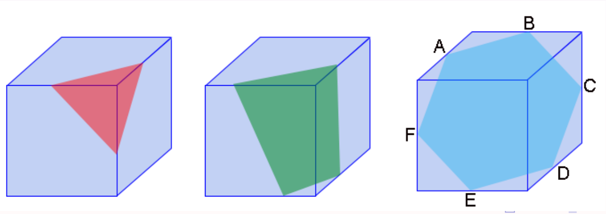
▷ marginal densities of convex compact sets $K \in \mathcal{K}^n$

$$f : H \to \mathbb{R}, \quad f(x) := |K \cap (x+H^{\perp})|$$

where H is an i-dimensional plane, is $\frac{1}{n-i}$-concave.



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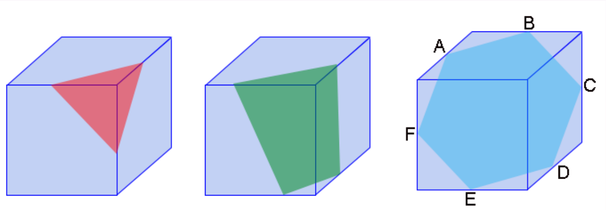
▷ Smallest common concavity in marginals on $\mathbb{R}^n$ for all $n \geq 1$ are the 0-concave integrable functions, i.e., log-concave integrable ones.



Polar projection body of f

For every $f \in \mathcal{F}(\mathbb{R}^n)$, let $\Pi^*(f)$ the polar projection body of f be the unit ball of the norm

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where $P_{x^\perp} f(y) = \max_{s \in \mathbb{R}} f\left(y + s\frac{x}{|x|}\right)$.

Functional affine inequalities

Theorem 5 (Zhang 1999)

Let $f \in \mathcal{F}(\mathbb{R}^n) \cap \mathcal{C}^1$. Then

$$\|f\|_{\frac{n}{n-1}} |\Pi^*(f)|^{\frac{1}{n}} \leq \frac{\pi^{\frac{1}{2}} \Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{n+2}{2}\right)}.$$

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"=" iff $f = \chi_E$ with E an ellipsoid. More generally

$$\|f\|_{\frac{n}{n-1}} \left(\int_{\mathbb{S}^{n-1}} \|\nabla_u f\|_1^{-n} du \right)^{\frac{1}{n}} \leq \frac{n^{\frac{1}{n}} \pi^{\frac{1}{2}} \Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{n+2}{2}\right)}.$$

Functional affine inequalities

Theorem 6 (Alonso-Gutiérrez, Bernués, G.M. +2018)

Let $f \in \mathcal{F}(\mathbb{R}^n)$. Then

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \min\{f(y), f(x)\} dydx \leq 2^n n! \|f\|_\infty \|f\|_1^{n+1} |\Pi^*(f)|.$$

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Observation

If $f(x) = e^{-\|x\|_K}$ with $K \in \mathcal{K}^n$ then Thm. 6 becomes Thm. 4, i.e.

$$\frac{\binom{2n}{n}}{n^n} \leq |K|^{n-1}|\Pi^*(K)|.$$

Proof of Theorem 6

Definition

Let $f \in \mathcal{F}(\mathbb{R}^n)$. Then

$$K_t(f) := \{x \in \mathbb{R}^n : f(x) \geq e^{-t}\|f\|_\infty\} \quad \forall\, t \geq 0.$$

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Lemma 1

Let $f \in \mathcal{F}(\mathbb{R}^n)$. The covariogram $g : \mathbb{R}^n \to \mathbb{R}$ of f

$$g(x) := \int_0^\infty e^{-t}|K_t(f) \cap (x + K_t(f))|dt$$
$$= \int_{\mathbb{R}^n} \min\left\{\frac{f(y)}{\|f\|_\infty}, \frac{f(y-x)}{\|f\|_\infty}\right\} dy$$

is even and $g \in \mathcal{F}(\mathbb{R}^n)$.

Proof of Theorem 6

Lemma 2

Let $f \in \mathcal{F}(\mathbb{R}^n)$ and g its covariogram. For every $0 < \lambda_0 < 1$ then

$$2\|f\|_1 \Pi^*(f) = \bigcap_{0<\lambda<\lambda_0} \frac{K_{-\log(1-\lambda)}(g)}{\lambda}.$$

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$$= \frac{\|x\|_{\Pi^*(f)}}{2\|f\|_\infty}.$$

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then $x \in K_{-\log(1-\lambda)}(g)/\lambda$ and thus

$$2\|f\|_1 \Pi^*(f) \subset \frac{K_{-\log(1-\lambda)}(g)}{\lambda}$$

for every $0 < \lambda < 1$. $\square$

Proof of Theorem 6

Definition (Ball 1988)

Let $f \in \mathcal{F}(\mathbb{R}^n)$ with $f(0) > 0$. Then

$$\tilde{K}(f) := \left\{ x \in \mathbb{R}^n : n \int_0^\infty f(rx) r^{n-1} dr \geq f(0) \right\}$$

fulfills $\tilde{K}(f) \in \mathcal{K}^n$ and $|\tilde{K}(f)| = \|f\|_1 / f(0)$.

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Lemma 3

Let $g \in \mathcal{F}(\mathbb{R}^n)$ with $g(0) > 0$. If $0 \leq t \leq n/e$ then

$$\frac{t}{(n!)^{\frac{1}{n}}} \tilde{K}(g) \subset K_t(g).$$

"=" iff $g(x) = e^{-\|x\|_K}$ for some $K \in \mathcal{K}^n$ with $0 \in K$.

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$$\bigcap_{0<\lambda<\lambda_0} \frac{-\log(1-\lambda)}{(n!)^{\frac{1}{n}}\lambda}\tilde{K}(g) \subset 2\|f\|_1\Pi^*(f).$$

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$$\begin{aligned} 2^n \|f\|_1^n |\Pi^*(f)| &\geq \frac{1}{n!} |\tilde{K}(g)| \\ &= \frac{1}{n!} \frac{1}{g(0)} \int_{\mathbb{R}^n} g(x) dx \end{aligned}$$

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2^n\|f\|_1^n|\Pi^*(f)| &\geq \frac{1}{n!}|\tilde{K}(g)| \\
&= \frac{1}{n!}\frac{1}{g(0)}\int_{\mathbb{R}^n} g(x)dx \\
&= \frac{1}{n!}\frac{1}{\int \frac{f}{\|f\|_\infty}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \min\left\{\frac{f(y)}{\|f\|_\infty}, \frac{f(x)}{\|f\|_\infty}\right\} dydx.
\end{aligned}$$

□

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Thank you for your attention!!