Extremal structure of Lipschitz-free spaces

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Spaces of Lipschitz functions and Lipschitz-free spaces

- Given a **complete** metric space (M, d) and a distinguished point $0 \in M$, the space

$$\mathrm{Lip}_0(M) := \{f \colon M \to \mathbb{R} : f \text{ is Lipschitz}, f(0) = 0\}$$

is a dual Banach space when equipped with the norm

$$\|f\|_L := \sup\left\{\frac{|f(x) - f(y)|}{d(x,y)} : x \neq y\right\}.$$

- The canonical predual of $\mathrm{Lip}_0(M)$ is the **Lipschitz-free space** (also called Arens-Eells space)

$$\mathcal{F}(M) = \overline{\mathrm{span}}\{\delta_x : x \in M\} \subset \mathrm{Lip}_0(M)^*,$$

where $\langle f, \delta_x \rangle = f(x)$.

Spaces of Lipschitz functions and Lipschitz-free spaces

Theorem (Arens-Eells, Kadets, Godefroy-Kalton, Weaver)

Let $f : M \to N$ be a Lipschitz map such that $f(0) = 0$. Then there exists an operator $T : \mathcal{F}(M) \to \mathcal{F}(N)$ such that $\|T_f\| = \|f\|_L$ and the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \delta \downarrow & & \delta \downarrow \\ \mathcal{F}(M) & \xrightarrow{T_f} & \mathcal{F}(N) \end{array}$$

Example

- $\mathcal{F}(\mathbb{N}) = \ell_1$ $(\delta_n \mapsto e_1 + \ldots + e_n)$.
- $\mathcal{F}(\mathbb{R}) = L_1$ $(\delta_x \mapsto \chi_{(0,x)})$.

Isometric properties of Lipschitz-free spaces

When is $\mathcal{F}(M)$...

- isometric to a dual space? Sufficient conditions found by Weaver (1999), Kalton (2004), Dalet (2015), GL-P-P-RZ (2018).
- isometric to a subspace of L_1? Godard (2010).
- isometric to a ℓ_1? Dalet-Kaufmann-Procházka (2016).
- octahedral? Becerra-López-Rueda Zoca (2018), Procházka-Rueda Zoca (2018).
- Daugavet property? Ivakhno-Kadets-Werner (2007), GL-P-RZ (2018), Avilés-Martínez Cervantes (2018).
- extreme points? Weaver (1999), Aliaga-Guirao (2017), Aliaga-Pernecka (2018), GL-P-P-RZ, GL-P-RZ (2018).

Some distinguished types of points in B_X

Let X be a Banach space and $x \in B_X$.

- x is an **extreme point** if $x = \frac{y+z}{2}$, $y, z \in B_X$, implies $x = y = z$.
- x is an **exposed point** if there is $f \in X^*$ such that

$$f(x) > f(y) \text{ for all } y \in B_X \setminus \{x\}.$$

- x is a **preserved extreme point** if it is an extreme point of $B_{X^{**}}$. Equivalently, the slices of B_X containing x are a neighbourhood basis for x in the weak topology.
- x is a **denting point** if the slices of B_X containing x are a neighbourhood basis for x in the norm topology.
- x is a **weak-strongly exposed point** if there is $f \in X^*$ providing slices that form a neighbourhood basis for x in the weak topology.
- x is a **strongly exposed point** if there is $f \in X^*$ providing slices that form a neighbourhood basis for x in the norm topology.

How is $B_{\mathcal{F}(M)}$?

- $\|\delta_x - \delta_y\| = d(x, y)$.
- The elements of the form

$$\frac{\delta_x - \delta_y}{d(x, y)}, \ x, y \in M, x \neq y$$

  are called (elementary) **molecules**.
- $B_{\mathcal{F}(M)} = \overline{\text{conv}} \left\{ \frac{\delta_x - \delta_y}{d(x,y)} : x, y \in M \right\}$.

Extremal structure of $B_{\mathcal{F}(M)}$ and molecules

Question

$\frac{\delta_x - \delta_y}{d(x,y)}$ is extreme if and only if $d(x,z) + d(z,y) > d(x,y)\ \forall z \in M \setminus \{x,y\}$.

True if M:

- compact (Aliaga-Guirao).
- bounded and uniformly discrete (GL-P-P-RZ).

The general case has just been solved by Aliaga and Pernecka. Moreover, Petitjean and Procházka have shown that $\frac{\delta_x - \delta_y}{d(x,y)}$ is exposed whenever it is an extreme point.

Question

If $\mu \in \mathcal{F}(M)$ is a extreme point, then $\mu = \frac{\delta_x - \delta_y}{d(x,y)}$ for some $x, y \in M$.

True if M is compact and the metric is of the form d^α, $0 < \alpha < 1$ (GL-P-P-RZ).

Extremal structure of $B_{\mathcal{F}(M)}$ and molecules

Theorem (Weaver, 1995)

Every preserved extreme point of $B_{\mathcal{F}(M)}$ is a molecule.

Definition

Let $f \in S_{\mathrm{Lip}_0(M)}$. We say that f is **peaking at** (x, y) if

$$\frac{f(x) - f(y)}{d(x,y)} = 1 \text{ and } \lim_{n\to\infty} \frac{f(u_n) - f(v_n)}{d(u_n, v_n)} = 1 \Rightarrow u_n \to x, v_n \to y$$

Theorem (Weaver, 1999)

Assume that there is a Lipschitz function f peaking at (x, y). Then $\frac{\delta_x - \delta_y}{d(x,y)}$ is a preserved extreme point.

Strongly exposed points in $B_{\mathcal{F}(M)}$

Theorem (G.-L. – Procházka – Rueda Zoca)

Let $x, y \in M$, $x \neq y$. The following are equivalent.

(i) *The molecule $\frac{\delta_x - \delta_y}{d(x,y)}$ is a strongly exposed point of $B_{\mathcal{F}(M)}$.*

(ii) *There is $f \in \mathrm{Lip}_0(M)$ peaking at (x, y).*

(iii) *There is $\varepsilon > 0$ such that for every $z \in M \setminus \{x, y\}$,*

$$d(x,z) + d(y,z) > d(x,y) + \varepsilon \min\{d(x,z), d(y,z)\}$$

Condition (iii) comes from a paper by Ivakhno, Kadets and Werner (2007). This result extends a characterisation of peaking functions in subsets of $\mathbb{R}$-trees due to by Dalet, Kaufmann and Procházka (2016).

Let M be a **compact** metric space. Then $\mathrm{Lip}_0(M)$ has the Daugavet property if and only if $B_{\mathcal{F}(M)}$ does not have any strongly exposed point.

Recently, Avilés and Martínez-Cervantes have shown that this result holds for complete metric spaces.

Preserved extreme points in $B_{\mathcal{F}(M)}$

Theorem (Aliaga-Guirao, 2018)

$\frac{\delta_x - \delta_y}{d(x,y)}$ is a preserved extreme point if and only if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $z \in M \setminus \{x, y\}$,

$$(1-\delta)(d(x,z)+d(z,y)) < d(x,y) \Rightarrow \min\{d(x,z), d(y,z)\} < \varepsilon.$$

This solves a problem posed by Weaver and implies that if M is compact then every molecule which is an extreme point of $B_{\mathcal{F}(M)}$ is also a preserved extreme point.

Preserved extreme vs denting

Theorem (G.-L. - Petitjean - Procházka - Rueda Zoca)

Every preserved extreme point of $B_{\mathcal{F}(M)}$ is a denting point.

Lemma

Assume $\frac{\delta_{x_\alpha}-\delta_{y_\alpha}}{d(x_\alpha,y_\alpha)}$ converges weakly to $\frac{\delta_x-\delta_y}{d(x,y)}$. Then $x_\alpha \to x$ and $y_\alpha \to y$. Therefore $\frac{\delta_{x_\alpha}-\delta_{y_\alpha}}{d(x_\alpha,y_\alpha)}$ converges in norm to $\frac{\delta_x-\delta_y}{d(x,y)}$.

Proof.

Test the weak convergence against the function

$$f(t) = \max\{\varepsilon - d(x,t), 0\}$$

□

Preserved extreme vs denting

Theorem (G.-L. - Petitjean - Procházka - Rueda Zoca)

Every preserved extreme point of $B_{\mathcal{F}(M)}$ is a denting point.

Proof.

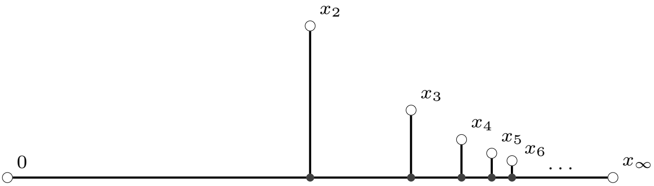
Denote Mol(M) the set of molecules and let $\mu \in \mathrm{Mol}(M)$ be a preserved extreme point. Assume there is $\varepsilon > 0$ such that every slice of $B_{\mathcal{F}(M)}$ containing μ has diameter at least ε.
There must be a slice S of $B_{\mathcal{F}(M)}$ such that $\mathrm{diam}(\mathrm{Mol}(M) \cap S) < \varepsilon/2$. Otherwise, there would be a net $\{\mu_\alpha\}$ of molecules that converges weakly to μ but not in norm, a contradiction. Note that

$$B_{\mathcal{F}(M)} = \overline{\mathrm{conv}}(\mathrm{Mol}(M)) = \overline{\mathrm{conv}}(\overline{\mathrm{conv}}(\mathrm{Mol}(M) \cap S) \cup \overline{\mathrm{conv}}(\mathrm{Mol}(M) \setminus S))$$

Now, a variation of Asplund–Bourgain–Namioka superlemma provides a slice of $B_{\mathcal{F}(M)}$ containing μ of diameter less than ε, a contradiction. □

Example

There is a compact countable metric space M with a denting point of $B_{\mathcal{F}(M)}$ which is not strongly exposed.



Example

Consider the sequence in c_0 given by $x_0 = 0, x_1 = 2e_1$, and $x_n = e_1 + (1 + 1/n)e_n$ for $n \geq 2$. Let $M = \{0\} \cup \{x_n : n \in \mathbb{N}\}$. Aliaga and Guirao showed that the molecule $\frac{\delta(x_1)}{2}$ is not a preserved extreme point of $B_{\mathcal{F}(M)}$. However, it is an extreme point.

Proposition (G.L.-Procházka-Rueda Zoca)

Every weak-strongly exposed point of $B_{\mathcal{F}(M)}$ *is a strongly exposed point.*

Corollary

The norm of $\mathrm{Lip}_0(M)$ *is Gâteaux differentiable at* f *if and only if it is Fréchet differentiable at* f.

Corollary (Ivakhno-Kadets-Werner, GL-P-P-RZ and Avilés-Martínez Cervantes)

Let M be a complete metric space. TFAE:

1. *M is length, i.e., $d(x, y)$ is the infimum of the length of the curves joining x, y for all $x, y \in M$.*
2. $\mathrm{Lip}_0(M)$ *has the Daugavet property.*
3. *$\mathcal{F}(M)$ has the Daugavet property.*
4. *The unit ball of $\mathcal{F}(M)$ does not have any preserved extreme point.*
5. *The unit ball of $\mathcal{F}(M)$ does not have any strongly exposed point.*
6. *The norm of* $\mathrm{Lip}_0(M)$ *does not have any point of Gâteaux / Fréchet differentiability.*

If moreover M is compact then these conditions are also equivalent to:

7. *M is geodesic.*
7. $\forall x, y \in M\ \exists z \in M \setminus \{x, y\}$ *such that* $d(x, y) = d(x, z) + d(z, y)$.

M geodesic $\Leftrightarrow$ $\mathrm{ext}\, B_{\mathcal{F}(M)} = \emptyset$?

Thank you for your attention