Ergodic properties of convolution operators

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Dedicated to the memory of Bernardo Cascales.

Definitions

X is a Banach space.

$L(X)$ is the space of all continuous linear operators on X.

Power bounded operators

An operator $T \in L(X)$ is said to be power bounded if $\{T^m\}_{m=1}^{\infty}$ is an equicontinuous subset of $L(X)$ if and only if $\sup_m ||T^m|| < \infty$ if and only if the orbits $\{T^m(x)\}_{m=1}^{\infty}$ of all the elements $x \in X$ are bounded (uniform boundedness principle).

Definitions

Mean ergodic operators

An operator $T \in L(X)$ is said to be mean ergodic if

$$\exists Px := \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} T^m x, \quad \forall x \in X.$$

If the limit converges in $(L(X), \|\cdot\|)$, then T is called uniformly mean ergodic.

Remark

If T is mean ergodic then $\lim_n T^n(x)/n = 0$ for all $x \in X$ and $\|T^n\|/n$ is bounded by Banach Steinhauss. If T is uniformly mean ergodic then $\lim_n \|T^n\|/n = 0$.

Results for mean ergodicity in Banach spaces

Let T be a power bounded operator.

$$T \text{ is mean ergodic} \Longleftrightarrow X = \mathrm{Ker}(I - T) \oplus \overline{\mathrm{Im}(I - T)}$$

Here, $\mathrm{Ker}(P) = \overline{\mathrm{Im}(I - T)}$ and $\mathrm{Im}P = \mathrm{Ker}(I - T)$.

Relation with the spectral radius

Let T be a bounded linear operator on a Banach space X:

1. If $r(T) < 1$ then T is uniformly mean ergodic (further, T^n converges to 0 in the norm operator topology).
2. If $r(T) > 1$ then $\|T^n\|/n$ is not bounded. Hence T is not mean ergodic.
3. If T is power bounded and $\|T^n - K\| < 1$ for some $n \in \mathbb{N}$ and $K \in K(X)$ then T is uniformly mean ergodic (Yosida-Kakutani).

Classic results for mean ergodicity in Banach spaces

Let X be a Banach space and $T \in L(X)$.

Theorem (Lotz, 1985)

Let X be a Banach space with the GDP property. Let T be an operator such that $\|T^n\|/n \to 0$:

$$T \text{ mean ergodic} \iff T \text{ is uniformly mean ergodic.}$$

$H^\infty(\mathbb{D})$, $\mathcal{B}$, ℓ_∞ and L^∞ have the GDP property.

Yosida, 1960 (it contains classical Von Neumann theorem and Lorch's theorem

Let X be a Banach space. The operator $T \in L(X)$ is mean ergodic if and only if it $\lim_{n\to\infty} \frac{1}{n} T^n = 0$, in $L_s(X)$ and

$$\{T_{[n]}x\}_{n=1}^\infty \textit{ is relatively } \sigma(X, X^*)\textit{–compact, } \forall x \in X. \tag{1}$$

As a corollary, if $(T^n)_n$ converges to T_0 in the WOT then $T_{[n]}$ is convergent to T_0 in the SOT.

Hille, 1945

There exist mean ergodic operators T on $L_1([0,1])$ which are not power bounded.

Results for mean ergodicity in Banach spaces

Dunford-Lin Theorem

Let $T \in L(X)$. The following are equivalent:

1. T is uniformly mean ergodic.
2. $(I - T)^2(X)$ is closed and $\lim_n \frac{\|T^n\|}{n} = 0$.
3. Either $1 \in \varrho(T)$ or 1 is a pole of order 1 of the resolvent mapping $\varrho(T) \to (L(X), \|\cdot\|)$, $z \mapsto R(z, T) := (zI - T)^{-1}$ and $\lim_n \frac{\|T^n\|}{n} = 0$.(this implies that 1 cannot be an accumulation point of $\sigma(T)$).
4. $(I - T)(X)$ is closed and $\lim_n \frac{\|T^n\|}{n} = 0$.
5. $(I - T)(X)$ is closed, $X = (I - T)(X) \oplus \text{Ker}(I - T)$ and $\lim_n \frac{\|T^n\|}{n} = 0$.

A consequence of Dunford Lin theorem

Proposition

Let H be a Hilbert space and let $T \in B(H)$ be a normal operator.

(a) The operator T is mean ergodic if and only if T is power bounded if and only if $\|T\| \leq 1$

(b) T is uniformly mean ergodic if and only if $\|T\| \leq 1$ and 1 is not an accumulation point of $\sigma(T)$.

Proof

(a) Is a consequence $\|T\| = r(T)$ and mean ergodicity of power bounded operators in reflexive spaces.

(b) Assume $\|T\| \leq 1$.
If T is uniformly mean ergodic it follows from Dunford-Lin that 1 cannot be an accumulation point of $\sigma(T)$.
Let see the converse. If $1 \in \varrho(T)$ then $(I - T)(H) = H$ and T is uniformly mean ergodic by Dunford Lin. We assume now that 1 is isolated in $\sigma(T)$. Thus $I - T$ is a normal operator which contains 0 as an isolated point in the spectrum. Hence $(I - T)(H)$ is closed, and then T is uniformly mean ergodic again by Dunford Lin.

Let G be a locally compact **metrizable and separable** group with Haar measure $\mathbf{m}_G$.

Definition

Let $\mu_1, \mu_2 \in M(G)$. We define:

- The *convolution of measures*:

$$\langle \mu_1 * \mu_2, f \rangle = \int\int f(xy)\mathrm{d}\mu_1(x)\mathrm{d}\mu_2(y), \text{ for every } f \in C_{00}(G).$$

- The *convolution operator*: $\lambda^p(\mu)\colon L_p(G) \to L_p(G)$ given by

$$\lambda^p(\mu)(f)(s) = (\mu * f)(s) := \int f(x^{-1}s)\mathrm{d}\mu(x), \quad \text{for all } f \in L_p(G) \text{ and all } s \in G.$$

Some examples, G Abelian, $p = 2$, frequency domain

- If G is Abelian, convolution operators for $p = 2$ are multiplication operators, via Fourier-Stieltjes transforms

$$\widehat{\lambda^2(\mu)f} = \widehat{(\mu * f)} = \widehat{f} \cdot \widehat{\mu} = M_{\widehat{\mu}} \widehat{f},$$

where $\widehat{\mu} \in CB(\widehat{G})$ with $\widehat{G} = \{\chi \colon G \to \mathbb{T} \colon \chi \text{ a continuous homomorphism}\}$. Every $s \in G$ determines $\chi_s \colon \widehat{G} \to \mathbb{T}$ with $\chi_s(\psi) = \psi(s)$.

- Put $s \in G$, then $\lambda^2(\delta_s)$ corresponds with M_{χ_s}.
- Now, $\widehat{\mathbb{R}} = \mathbb{R}$, $\widehat{\mathbb{T}} = \mathbb{Z}$ and $\widehat{\mathbb{Z}} = \mathbb{T}$ and:
For $G = \mathbb{R}$, $\chi_s \colon \mathbb{R} \to \mathbb{R}$, $\chi_s(t) = e^{2\pi ist}$. For $G = \mathbb{Z}$, $\chi_s \colon \mathbb{T} \to \mathbb{T}$, $\chi_s(e^{2\pi it}) = e^{2\pi ist}$.And for $G = \mathbb{T}$, $\chi_s \colon \mathbb{Z} \to \mathbb{T}$, $\chi_s(n) = e^{2\pi isn}$.
- If K is a compact subgroup of G, then $m_K \in M(G)$ and $\widehat{m_K} = \mathbf{1}_{K^\perp}$.

Definition

A measure $\mu \in M(G)$ is said to be normal if $\lambda^2(\mu)$ is a normal operator.If G is abelian $\lambda^2(\mu)$ is normal for every $\mu \in M(G)$.

(a) If G abelian always $\sigma(\lambda^2(\mu)) = \overline{\{\widehat{\mu}(\chi) : \ \chi \in \widehat{G}\} \subseteq \sigma(\lambda^p(\mu))}$ for any $1 \le p \le \infty$, $p \neq 2$.

(b) If μ is normal then $\lambda^2(\mu)$ is mean ergodic if and only if $\|\mu\| \le 1$ and $\lambda^2(\mu)$ is uniformly mean ergodic if and only if 1 is not an accumulation point of $\sigma(\lambda^2(\mu))$. If G is an abelian compact group and $\widehat{\mu} \in C_0(\widehat{G})$ $(\mu \in M_0(G))$ then $\lambda^2(\mu)$ is mean ergodic if and only if it is uniformly mean ergodic.

Independent of p

Proposition

Let G be a locally compact group and let $\mu \in M(G)$. Assume that $\lambda^2(\mu)$ is uniformly mean ergodic. Let $1 \le p < \infty$. Then $\lambda^p(\mu)$ is uniformly mean ergodic if and only if $\lim \|\lambda^p(\mu^{*,n})\|/n = 0$ and 1 is an isolated point of $\sigma(\lambda^p(\mu))$.

Proof

- We only have to show that if 1 is isolated in $\sigma(\lambda^p(\mu))$ then $B(1,r) \setminus \{0\} \to L(L_p(G))$, $z \mapsto R(z) := (z-1)(zI - \lambda^p(\mu))^{-1}$ can be extended holomorphically to 1.
- R is holomorphic on $B(1,r) \setminus \{0\}$. From uniform mean ergodicity in $L^2(G)$ and Theorem of Dunford-Lin we get

$$\langle (z-1)R(z, \lambda^q(\mu)(f), g\rangle$$

admits a holomorphic extension for each $f, g \in C_{00}(G)$.
- We conclude from $R(z, \lambda^q(\mu)(f) = R(z, \lambda^p(\mu)(f))$ for each $f \in C_{00}(G)$ that the function $\langle (z-1)R(z, \lambda(\mu)(f), g\rangle$ admits a holomorphic extension to 1 for each $f, g \in C_{00}(G)$. Thus R can be holomorphically extended by a criterion due to Grosse-Erdmann.

Positive measures in amenable groups

If $\mu \in M(G)_+$ for G amenable then $r(\lambda^p(\mu)) = \|\lambda^p(\mu)\| = \|\mu\|$ for every $1 \leq p \leq \infty$.

Theorem

If $\mu \in M(G)_+$ is normal and G is amenable then

1. $\lambda^p(\mu)$ is mean ergodic if and only if it is power bounded if and only if $\|\mu\| \leq 1$ for $1 < p < \infty$.
2. $\lambda^p(\mu)$ is uniformly mean ergodic if and only if $\|\mu\| \leq 1$ and 1 is not an accumulation point of $\sigma(\lambda^p(\mu))$.. If G is abelian and $\mu \in M_0(G)_+$ then $\lambda^p(\mu)$ is uniformly mean ergodic if and only if $\lambda^2(\mu)$ is mean ergodic if and only if 1 is not in the accumulation of $\{\widehat{\mu}(\chi) : \ \chi \in \widehat{G}\}$.
3. $\lambda^1(\mu)$ is mean ergodic if and only if $\langle$ supp $\mu\rangle$ is compact and $\|\mu\| \leq 1$.
4. $\lambda^\infty(\mu)$ is (uniformly) mean ergodic if and only if $\lambda^1(\mu)$ is uniformly mean ergodic if and only $\langle$ supp $\mu\rangle$ is compact and 1 is not an accumulation point of $\sigma(\mu)(\sigma(\mu)$ denotes the spectrum of μ in $M(G))$.

L_1-type positive measures in relevant groups

$G = \mathbb{R}^n$

Let $\mu \in M_0(\mathbb{R}^n)_+$ and $1 < p < \infty$. Then $\lambda^p(\mu)$ is mean ergodic if and only if $\mu \leq 1$ and $\lambda^p(\mu)$ is uniformly ergodic if and only if $\|\mu\| < 1$ if and only if $1 \notin \{\widehat{\mu}(\chi) : \ \chi \in \mathbb{R}^n\}$.

$G = \mathbb{T}$

Let $\mu \in M_0(\mathbb{T})_+$ and $1 < p < \infty$. Then $\lambda^p(\mu)$ is mean ergodic if and only if $\lambda^p(\mu)$ is uniformly mean ergodic if and only if $\|\mu\| < 1$ if and only if $1 \notin \{\widehat{\mu}(\chi) : \ \chi \in \mathbb{Z}\}$.

Not true for non positive measures

If $\mu = (1/2)(\delta_1 - \delta_2) \in M(\mathbb{Z})$, then $\sigma(\lambda^2(\mu)) \cap \mathbb{T} = \{-1\}$ and $\lambda^p(\mu)$ is uniformly mean ergodic for every $1 < p < \infty$.

measures with big support

p-harmonic measures

Let $\mu \in M(G)$ with $\|\mu\| \leq 1$ and $H := \langle \text{supp}\mu \rangle$ is not compact. Then $1 \notin \sigma_p(\lambda^p(\mu))$ for each $1 < p < \infty$.

measures with amenable support

If $\mu \in M(G)_+$ and $H := \langle \text{supp}\mu \rangle$ is amenable then $1 \in \sigma(\lambda^p(\mu))$

Measures with big support

Proposition

Let $\mu \in M(G)_+$ such that $H := \langle \text{supp}\mu \rangle$ is not compact. Then $\lambda^p(\mu)$ is uniformly mean ergodic if and only if $r(\lambda^p(\mu)) < 1$.

Proposition

Let $\mu \in M(G)_+$ such that $\|\mu\| \leq 1$ and suppose that $H := \langle \text{supp}\mu \rangle$ is not compact. Then $\lambda^p(\mu)$ is uniformly mean ergodic if and only if $\|\mu\| < 1$.

This is a consequence of $Ker(I - \lambda^p(\mu)) = \{0\}$, and Dunford-Lin theorem implies $(I - \lambda^p(\mu))(L_p(G)) = L_p(G)$

Other concepts involving ergodicity

Definition

A measure $\mu \in M(G)$ is said to be ergodic if and only if $(\sum_{n=1}^{N} \mu^{*,n})/N$ is convergent in the $\sigma(M(G), C_0(G))$ topology to a measure μ_c.

Grenander's theorem

If G is a separable and metrizable locally compact group and μ is a probability measure then μ is ergodic.

Proposition

Let G be a locally compact group. The mapping $\lambda_p : (M(G), w^*) \to (L(L_p(G)), WOT)$, $\mu \mapsto \lambda^p(\mu)$ is sequentially continuous for every $1 < p < \infty$.

Proof

- If (μ_n) be convergent in the vague topology to $\mu \in M(G)$. It follows that $(\|\mu_n\|)$ is bounded (we assume that by 1) .
- Thus also $(\|\lambda^p(\mu_n)\|)$ is consequently also bounded, i.e. $(\lambda^p(\mu_n))$ is equicontinuous.
- Since $C_{00}(G)$ is strongly dense in $L_p(G)$ we only need to show that $(\lambda^p(\mu_n)(f))$ is weakly convergent to $\lambda^p(\mu)(f)$ in $L_p(G)$ for each $f \in C_{00}(G)$.
- We fix $f \in C_{00}(G)$. Since $(\lambda^p(\mu_n)(f))$ is a weakly bounded sequence it is relatively weakly compact, and hence weak convergence of the sequence is equivalent to convergence in $\sigma(L_p(G), C_{00}(G))$, since $C_{00}(G)$ is dense in $L_q(G)$.
- Let $g \in C_{00}(G)$ be fixed and let $M := \max\{f(x) : \ x \in G\}$. We get

$$\left| \int_G f(x^{-1}s) d\mu_n(x) g(s) \right| \leq \int_G \left| f(x^{-1}s) \right| d|\mu_n|(x) |g(s)| \leq M |g(s)|$$

for each $s \in G$, $n \in \mathbb{N}$.

By hypothesis $(\int_G f(x^{-1}s) d\mu_n(x))$ is convergent to $\int_G f(x^{-1}s) d\mu(x)$ for each $s \in G$. By Lebesgue theorem we conclude

$$\lim_n \int_G \int_G f(x^{-1}s) d\mu_n(x) g(s) dm(s) = \int_G \int_G f(x^{-1}s) d\mu(x) g(s) dm(s).$$

- If $\mu \in M(G)$ satisfies that $(\|\mu^n\|)$ is bounded then μ is ergodic.From the continuity of λ and the mean ergodicity of $\lambda^2(\mu)$ we get that (μ^n) has a unique w^*-cluster point (bounded sets of $(M(G), w^*)$ are metrizable in our context).
- Let G be amenable and let $\mu \in M(G)_+$. A measure μ is ergodic if and only if $\lambda^p(\mu)$ is mean ergodic for each $1 < p < \infty$ if and only if $r(\lambda(\mu)) = \|\mu\| \leq 1$.

NonAmenable groups

Theorem

Let $F(X)$ be a free discrete group and let $S \subseteq F(X)$. Consider $\mu_S = \sum_{s\in S} \delta_s \in M(F(X))_+$. Then:

1. *(Akemann and Ostrand, 1976) If S is a free set, then $\|\lambda^2(\mu_S)\|_{\mathcal{L}(L_2(G))} = 2\sqrt{|S|-1}$.*
2. *(strong Haagerup inequality. Kemp and Speicher, 2007) If S consists of words of length n in the semigroup generated by X: $\|\lambda^2(\mu_S)\|_{\mathcal{L}(L_2(G))} \leq e\sqrt{n+1}\|\mu_S\|_2$.*

Example (Mean ergodic convolution operators of large norm.)

Let $G = F(x_1, x_2, x_3)$ a **free group**. Put $\nu = (\delta_{x_1} + \delta_{x_2} + \delta_{x_3})$, and $\mu := \alpha\nu$. Then:

- $\|\lambda^2(\mu^n)\|_{\mathcal{L}(L_2(G))} \leq \left(\sqrt{3}\alpha\right)^n e\sqrt{n+1}$.
- $\|\mu^n\|_{M(G)} = (3\alpha)^n$.
- $\|\mu\|_{M(G)} = 3\alpha > \|\lambda^2(\mu)\|_{\mathcal{L}(L_2(G))} = 2\sqrt{2}\alpha > 1 > \sqrt{3}\alpha \geq r(\lambda^2(\mu))$.

With $\frac{1}{2\sqrt{2}} < \alpha < \frac{1}{\sqrt{3}}$, $\lambda^2(\mu)$ is **UME** while $\|\lambda^2(\mu)\|_{\mathcal{L}(L_2(G))} > 1$ and $\lim_n \frac{\|\mu^n\|_{M(G)}}{n} = \infty$, hence μ **is not ergodic**.