Approximation of Lipschitz functions preserving boundary values.

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Problem. Let Ω be an open subset of a Banach space and $u_0 : \overline{\Omega} \to I\!R$ be a Lipschitz function.

Is it possible to approximate u_0 uniformly on Ω by a function $u : \overline{\Omega} \to I\!R$ having the same Lipschitz constant as u_0, such that u is C^k-smooth and $u = u_0$ on $\partial\Omega$?

Theorem 1. Let X be a finite dimensional Banach space. Let Ω be an open subset of X and let $u_0 : \overline{\Omega} \to I\!R$ be a Lipschitz function such that $Lip(u_0, \partial\Omega) < Lip(u_0, \overline{\Omega})$.

Given $\varepsilon > 0$, there exists a function $v : \overline{\Omega} \to I\!R$ such that
1) v is of class $C^\infty(\Omega)$,
2) v is Lipschitz on Ω with $Lip(v, \overline{\Omega}) = Lip(u_0, \overline{\Omega})$,
3) $v = u_0$ on $\partial\Omega$,
4) $\|v - u_0\| \leq \varepsilon$ on $\overline{\Omega}$.

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The theorem remains true if X is a separable Hilbert space, or if $X = c_0(\Gamma)$.

Whenever X is a non separable Hilbert space, it is only possible to guarantee that the function v is C^1.

Proof of Theorem 1.

Lipschitz extension. *Let* (X, d) *be a metric space.*
Let E *and* F *be two nonempty closed sets such that* $F \subset E$*,*
let $u_0 : E \to \mathbb{R}$ *be a* 1*-Lipschitz such that* $\lambda_0 := Lip(u_0, F) < 1$*.*
Given $\varepsilon > 0$*, there exists a function* $u : E \to \mathbb{R}$ *such that*
1) $|u - u_0| \le \varepsilon$ *on* E*,*
2) $u = u_0$ *on* F*,*
3) $Lip(u, B) < 1$ *for every bounded subset* B *of* E*.*

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An approximation property. *Let* X *be a separable Hilbert space or a finite dimensional normed space. Given a Lipschitz function* $f : X \to \mathbb{R}$ *and* $\varepsilon > 0$*, there exists a function g of class* $C^\infty(X)$ *such that* $|g - f| \leq \varepsilon$ *on* X *and* $Lip(g, B(x_0, r)) \leq Lip(f, B(x_0, r + \varepsilon)) + \varepsilon$ *for every ball* $B(x_0, r) \subset X$*.*

If $dim(X) < +\infty$*,* $g = f * \theta$ *with* θ *approximate* C^∞ *unit.*

If X *is a non-separable Hilbert space, the statement holds replacing* C^∞ *smoothness with* C^1*.*

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Let E and F be two nonempty closed sets such that $F \subset E$,
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Given $\varepsilon > 0$, there exists a function $u : E \to I\!R$ such that
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A better approximation property. *Let X be a separable Hilbert space or a finite dimensional normed space, and $\Omega \subset X$ be open. Let $u : X \to I\!R$ be 1-Lipschitz and such that $Lip(u, B) < 1$ for any bounded subset B of Ω and let $\varepsilon : \Omega \to (0, +\infty)$.*

Then there exists $v \in C^\infty(\Omega)$ such that, for all $x \in \Omega$,

$$|u(x) - v(x)| \leq \varepsilon(x) \qquad \text{and} \qquad \|Dv(x)\| < 1.$$

Technical lemma. *Let E and F be two nonempty closed subsets such that $F \subset E$ and $E \backslash F$ is bounded and non empty. Let $u_0 : E \to \mathbb{R}$ be a 1-Lipschitz mapping, let $u_\mu : F \to \mathbb{R}$ be μ-Lipschitz, with $\mu < 1$, let $\delta \geq 0$ and assume that $|u_\mu - u_0| \leq \delta$ on F.*
For every $\mu < \lambda < 1$, there exists a function $u_\lambda : E \to \mathbb{R}$ such that u_λ is λ-Lipschitz on E with $u_\lambda = u_\mu$ on F and $u_0 - u_\lambda \leq \delta + \varepsilon(\lambda, \mu, E, F)$ on E, where $\varepsilon = \varepsilon(\lambda, \mu, E, F) = \frac{1-\lambda}{\lambda-\mu}(\lambda + \mu)\Big(diam(\overline{E\backslash F}) + dist(\overline{E\backslash F}, F)\Big) > 0$.

Proof of the technical lemma. The set

$$C_\lambda = \{u : E \to \mathbb{R} : \ \lambda - \text{Lipschitz}, u \leq u_0 + \delta + \varepsilon, \, u_{|_F} = u_\mu\}$$

is nonempty, and if $u_\lambda(x) := \sup\{u(x) : u \in C_\lambda\}$, for $x \in E$, u_λ is the required solution.

Proof of the Lipschitz extension statement.
Set $E_n = \Big(E \cap B(p, n)\Big) \cup F$ and $F_n = E_{n-1}$, $(\lambda_n) \uparrow 1$ such that $\varepsilon(\lambda_n, \lambda_{n-1}, E_n, F_n) \leq \varepsilon/2^n$, and define inductively u_n on E_n by applying the technical lemma. The function $u : E \to \mathbb{R}$ is defined by $u(x) = u_n(x)$ whenever $x \in E_n$.

Problem. Let $(X, \|\cdot\|)$ be a finite dimensional normed space with $dim(X) \geq 2$, Let Ω be an open subset of X, and let $u_0 : \overline{\Omega} \to I\!\!R$ be a 1-Lipschitz function.

Given $\varepsilon > 0$, does there exist a 1-Lipschitz function $w : \overline{\Omega} \to I\!\!R$ such that w is differentiable on Ω with $\|Dw(x)\| = 1$ almost everywhere on Ω, $w = u_0$ on $\partial\Omega$ and $\|u_0 - w\| \leq \varepsilon$ on Ω ?

Theorem . (Deville-Matheron)
X is a finite dimensional Banach space with $dim(X) \geq 2$,
Ω is an open subset of X,

Given $\varepsilon > 0$, there exists a function $v : \overline{\Omega} \to \mathbb{R}$ such that
1) v is differentiable on Ω,
2) v is 1-Lipschitz on $\overline{\Omega}$ and $\|Dv(x)\| = 1$ almost everywhere on Ω,
3) $v = 0$ on $\partial\Omega$,

Theorem 2. (Deville-Mudarra)
X is a finite dimensional Banach space with $dim(X) \geq 2$,
Ω is an open subset of X,
$u_0 : \overline{\Omega} \to I\!R$ be a 1-Lipschitz function such that $Lip(u_0, \partial\Omega) < 1$.

Given $\varepsilon > 0$, there exists a function $v : \overline{\Omega} \to I\!R$ such that
1) v is differentiable on Ω,
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Proof. Let $v : \overline{\Omega} \to I\!R$ be C^∞, such that $Lip(v, \overline{\Omega}) = Lip(u_0, \overline{\Omega})$, $v = u_0$ on $\partial\Omega$ and $\|v - u_0\| \leq \varepsilon/2$ on $\overline{\Omega}$.

If $F(x, \Lambda) = \|\Lambda + Dv(x)\| - 1$ the equation $F(x, Du(x)) = 0$ on Ω with $u = 0$ on $\partial\Omega$ has an almost classical solution u with $\|u\| \leq \varepsilon/2$.

$w = v + u$ satisfies theorem 2.

The limiting case $Lip(u_0, \partial\Omega) = 1$.

Proposition. *If* $\Omega \subset I\!R^2$ *is open and* $u_0 : \partial\Omega \to I\!R$ *is* 1*-Lispchitz for the usual euclidean distance, then there exists* $w : \overline{\Omega} \to I\!R$ *such that :*
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Counterexample Assume $X = (\mathbb{R}^2, \|\cdot\|_1)$,
$\Omega = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 1\}$ and $u_0(x, y) = |x| - |y|$ on $\partial\Omega$.
Then u_0 is 1-Lipschitz, but there is no 1-Lipschitz function $u : \overline{\Omega} \to \mathbb{R}$ such that $u = u_0$ on $\partial\Omega$ and u is differentiable at each point of Ω.

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Open problems. What happens whenever $X = (\mathbb{R}^2, \|\cdot\|)$, where $\|\cdot\|$ is an arbitrary norm ?

What happens whenever $X = (\mathbb{R}^d, \|\cdot\|)$, $d > 2$, where $\|\cdot\|$ is the euclidian norm ?

Proof of the proposition.

By the theory of Δ_∞, there exists a unique absolutely minimizing Lipschitz extension $v : \overline{\Omega} \to I\!R$ of u_0.

In particular v is a 1-Lipschitz extension of u_0 on $\overline{\Omega}$.

By Savin's result (which holds only in dimension 2), v is C^1 on Ω.

According to Deville and Jaramillo, there exists an almost classical solution of $\|\nabla u + \nabla v\| = 1$ on Ω vanishing on $\overline{\Omega}$.

$w = u + v$ is the desired function.