On subspaces of distributions which are Δ^m-invariant

J. M. Almira
Dedicated to the memory of Prof. Bernardo Cascales

Universidad de Murcia

XVI Encuentros Murcia-Valencia
December, 2018
jmalmira@um.es

Theorem (Jacobi, 1834)

The following claims hold true:

(a) *A meromorphic function* $f:\mathbb{C}\to\widehat{\mathbb{C}}$ *is constant if and only if it solves a system of functional equations of the form*

$$\Delta_{h_1}f(z)=\Delta_{h_2}f(z)=\Delta_{h_3}f(z)=0\ \ (z\in\mathbb{C})$$

for three independent "periods" $\{h_1,h_2,h_3\}\subseteq\mathbb{C}$.

(b) *There exist non-constant meromorphic functions* $f:\mathbb{C}\to\widehat{\mathbb{C}}$ *with two independent periods*
(These are the so called "elliptic functions", which are extremely important in Complex Function Theory).

Note: $\{h_1,h_2,h_3\}$ are independent if $h_i\notin h_j\mathbb{Z}+h_k\mathbb{Z}$ for all (i,j,k) such that $\{i,j,k\}=\{1,2,3\}$.

Montel's theorem

Theorem (P. Montel, 1937)

If $f:\mathbb{C}\to\mathbb{C}$ *is an analytic function satisfying*

$$\Delta_{h_1}^m f(t) = \Delta_{h_2}^m f(t) = \Delta_{h_3}^m f(z) = 0 \textit{ for all } z \in \mathbb{C}$$

for three independent "periods" $\{h_1, h_2, h_3\} \subseteq \mathbb{C}$ *Then*

$$f(z) = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1}$$

for all $z \in \mathbb{C}$ *and certain complex numbers* $a_0, a_1, \cdots, a_{m-1}$.

Montel's theorem

Theorem (P. Montel, 1937)

If $f : \mathbb{R} \to \mathbb{C}$ *is a continuous function satisfying*

$$\Delta_{h_1}^m f(t) = \Delta_{h_2}^m f(t) = 0 \text{ for all } t \in \mathbb{R}$$

and certain $h_1, h_2 \in \mathbb{R} \setminus \{0\}$ *such that* $h_1/h_2 \notin \mathbb{Q}$, *then*

$$f(t) = a_0 + a_1 t + \cdots + a_{m-1} t^{m-1}$$

for all $t \in \mathbb{R}$ *and certain complex numbers* $a_0, a_1, \cdots, a_{m-1}$.

It is important, for our talk, to note that Montel's Theorem claims that certain properties which hold true for Δ_h can be proved for the operators Δ_h^m.

X denotes either the space of continuous functions $f : \mathbb{R} \to \mathbb{C}$ or the space of complex valued Schwartz distributions.

Definition (Translation operator $\tau_h : X \to X$)

$\tau_h(f)(t) = f(t+h)$ if f is an ordinary function and
$\tau_h(f)\{\phi\} = f\{\tau_{-h}(\phi)\}$ if f is a distribution and ϕ is a test function.

Definition

A subspace V of X is translation invariant if for all $h \in \mathbb{R}$ we have that $\tau_h(V) \subseteq V$.

Theorem (P. M. Anselone, J. Korevaar)

Let V be a finite dimensional subspace of X which is translation invariant. Then V is the space of solutions of some homogeneous linear differential equation with constant coefficients

$$x^{(n)} + a_1 x^{(n-1)} + \cdots + a_{n-1} x' + a_n x = 0$$

(here $x : \mathbb{R} \to \mathbb{C}$ and $a_1, \cdots, a_n \in \mathbb{C}$ for some $n \in \mathbb{N}$).

These spaces are generated by a set of monomials of the form

$$t^{k-1} e^{\lambda t},\ k = 1, \cdots, m(\lambda) \text{ and } \lambda \in \{\lambda_0, \lambda_1, \cdots, \lambda_s\} \subset \mathbb{C}, \qquad (0.1)$$

so that their elements are exponential polynomials. We assume, by convention, that $\lambda_0 = 0$ and that $m(\lambda_0) = 0$ means that this set does not contain elements of the form t^k with $k \in \mathbb{N}$.

Anselone-Korevaar's Theorem

Theorem (P. M. Anselone, J. Korevaar)

Assume that one of the following conditions hold true:

(a) *V is a finite dimensional subspace of X and $\tau_{h_1}(V) \subseteq V$, $\tau_{h_2}(V) \subseteq V$ for certain non-zero real numbers h_1, h_2 such that $h_1/h_2 \notin \mathbb{Q}$*

(b) *V is a finite dimensional subspace of $\mathbf{C}(0,\infty)$ and $\tau_{h_k}(V) \subseteq V$ for an infinite sequence of positive real numbers $\{h_k\}_{k=1}^{\infty}$ which converges to zero,*

Then V admits a basis of the form

$$t^{k-1}e^{\lambda t}, \; k = 1, \cdots, m(\lambda) \text{ and } \lambda \in \{\lambda_0, \lambda_1, \cdots, \lambda_s\} \subset \mathbb{C},$$

Note. $\Delta_h = \tau_h - I$. Hence

$$V \text{ is } \tau_h\text{-invariant if and only if it is } \Delta_h\text{-invariant.}$$

Δ^m-invariant subspaces: characterization

Question:
What about the finite dimensional Δ_h^m-invariant subspaces of X?

Note

In general, if $L : X \to X$ is a linear operator, the inclusion $L^m(V) \subseteq V$ may be not related to $L(V) \subseteq V$.
For example, if

$$L \neq \lambda I \text{ for any } \lambda \text{ but } L^m \in \{0, I\}$$

then:

- *All subspaces of X are L^m-invariant*
- *There exists $v_0 \in X$ such that $L(v_0) \notin \mathbf{span}\{v_0\}$. Hence $V = \mathbf{span}\{v_0\}$ is not L-invariant.*

Invariant subspaces of linear operators defined on $\mathbb{C}^n$

I. Gohberg, P. Lancaster, L. Rodman

Invariant subspaces of matrices with applications, Classic in Applied Mathematics **51** SIAM, 2006.

Definition

Given $T : \mathbb{C}^n \to \mathbb{C}^n$ a linear transformation and $\lambda \in \sigma(T)$, we define the root subspace associated to λ and T by the formula

$$R_\lambda(T) = \ker(T - \lambda I)^n.$$

Theorem (Characterization of invariant subspaces)

Let $\sigma(T) = \{\lambda_0, \cdots, \lambda_t\}$ be all (pairwise distinct) eigenvalues of $T : \mathbb{C}^n \to \mathbb{C}^n$. The subspace $V \subseteq \mathbb{C}^n$ is T-invariant if and only if

$$V = (V \cap R_{\lambda_0}(T)) \oplus (V \cap R_{\lambda_1}(T)) \oplus \cdots \oplus (V \cap R_{\lambda_t}(T))$$

and each subspace $V_i = (V \cap R_{\lambda_i}(T))$ is T-invariant.

Lemma (Important example)

Let E be a vector space with basis $\beta = \{v_i\}_{i=1}^n$ and let $m \geq 1$ be a natural number. Assume that $T : E \to E$ satisfies

$$A = M_\beta(T) = \lambda I + B,$$

where $\lambda \neq 0$ and B is strictly upper triangular with nonzero entries in the first superdiagonal. Then:

(a) *The full list of T-invariant subspaces of E is given by $V_0 = \{0\}$ and $V_k = \textbf{span}\{v_1, \cdots, v_k\}$, $k = 1, \cdots, n$.*

(b) *T^m has the same invariant subspaces as T.*

Let us prove (a): Assume $T(V) \subset V$ and $v = a_1v_1 + \cdots + a_sv_s \in V$ with $a_s \neq 0$. Then $w = Tv - \lambda v \in V$. Moreover,

$w = \alpha_1 v_1 + \cdots + \alpha_{s-1}v_{s-1}$ with $\alpha_{s-1} = b_{s-1,s}a_s \neq 0$, where $B = (b_{i,j})$

Hence:

If V is T-invariant and $v = a_1v_1 + \cdots + a_sv_s \in V$ with $a_s \neq 0$, then

$$\mathbf{span}\{v_1, \cdots, v_s\} \subseteq V.$$

Take

$$k_0 = \max\{k : \exists v \in V, v = a_1v_1 + \cdots + a_kv_k,\ a_k \neq 0\}$$

Then $V = \mathbf{span}\{v_1, \cdots, v_{k_0}\}$.
Moreover, it is evident that all spaces $V_k = \mathbf{span}\{v_1, \cdots, v_k\}$ are T-invariant.

To demonstrate (b) we take into account that $M_\beta(T^m) = A^m$, so that

$$
\begin{aligned}
M_\beta(T^m) &= A^m \\
&= (\lambda I + B)^m \\
&= \sum_{k=0}^{m} \binom{m}{k} \lambda^{m-k} B^k \\
&= \lambda^m I + m\lambda^{m-1} B + \sum_{k=2}^{m} \binom{m}{k} \lambda^{m-k} B^k \\
&= \lambda^m I + C
\end{aligned}
$$

with $C = m\lambda^{m-1}B + \sum_{k=2}^{m} \binom{m}{k} \lambda^{m-k} B^k$ strictly upper triangular with nonzero entries in the first upperdiagonal (that came from $m\lambda^{m-1}B$, with $\lambda \neq 0$) Hence we can apply (a) to T^m and both operators share the same invariant subspaces. This ends the proof.

Finite dimensional Δ^m-invariant subspaces are formed by exponential polynomials

Theorem

Assume that

- *V is a finite dimensional subspace of X*
- *$\Delta_{h_1}^m(V) \subseteq V$, $\Delta_{h_2}^m(V) \subseteq V$ for certain non-zero real numbers h_1, h_2 such that $h_1/h_2 \notin \mathbb{Q}$.*

Then there exists a finite dimensional subspace W of X which is invariant by translations and contains V.
Consequently, all elements of V are exponential polynomials.

Finite dimensional Δ^m-invariant subspaces are formed by exponential polynomials

Lemma

Let E be a vector space and $L : E \to E$ be a linear operator defined on E. If $V \subset E$ is an L^m-invariant subspace of E, then the space

$$\square_L^m(V) = V + L(V) + L^2(V) + \cdots + L^m(V)$$

is L-invariant. Furthermore, $\square_L^m(V)$ is the smallest L-invariant subspace of E containing V.

Finite dimensional Δ^m-invariant subspaces are formed by exponential polynomials

Proof :
The linearity of L implies that

$$L(\square_L^m(V)) = L(V) + L^2(V) + L^3(V) + \cdots + L^m(V) + L^{m+1}(V).$$

Now, $L^{m+1}(V) = L(L^m(V)) \subseteq L(V)$ and $L(V) + L(V) = L(V)$, so that $L(\square_L^m(V)) \subseteq \square_L^m(V)$.
On the other hand, let us assume that $V \subseteq F \subseteq E$ and F is an L-invariant subspace of E. If $\{v_k\}_{k=0}^m \subseteq V$, then $L^k(v_k) \in F$ for all $k \in \{0, 1, \cdots, m\}$, so that $v_0 + L(v_1) + \cdots + L^m(v_m) \in F$. This proves that $\square_L^m(V) \subseteq F$.

Finite dimensional Δ^m-invariant subspaces are formed by exponential polynomials

Lemma

Let E be a vector space and $L, S : E \to E$ be two linear operators defined on E. Assume that $LS = SL$. If $V \subset E$ is a vector subspace of E which satisfies $L^m(V) \cup S^m(V) \subseteq V$, then

$$S^m(\Box_L^m(V)) \subseteq \Box_L^m(V).$$

Consequently, the space

$$\diamond_{L,S}^m(V) = \Box_S^m(\Box_L^m(V))$$

is L-invariant, S-invariant, and contains V.

Finite dimensional Δ^m-invariant subspaces are formed by exponential polynomials

Proof:

$$\begin{aligned} S^m(\Box_L^m(V)) &= S^m(V + L(V) + L^2(V) + \cdots + L^m(V)) \\ &= S^m(V) + L(S^m(V)) + L^2(S^m(V)) + \cdots + L^m(S^m(V))) \\ &\subseteq V + L(V) + L^2(V) + \cdots + L^m(V) = \Box_L^m(V), \end{aligned}$$

since S, L commute. This proves that $\Box_L^m(V)$ is S^m-invariant, and Lemma above implies that $\diamond_{L,S}^m(V) = \Box_S^m(\Box_L^m(V))$ is S-invariant.

$$\begin{aligned} L(\diamond_{L,S}^m(V)) &= L(\Box_L^m(V) + S(\Box_L^m(V)) + \cdots + S^m(\Box_L^m(V))) \\ &= L(\Box_L^m(V)) + S(L(\Box_L^m(V))) + \cdots + S^m(L(\Box_L^m(V))) \\ &\subseteq \Box_L^m(V) + S(\Box_L^m(V)) + \cdots + S^m(\Box_L^m(V)) = \diamond_{L,S}^m(V), \end{aligned}$$

so that $\diamond_{L,S}^m(V)$ is L-invariant. Finally, $V \subseteq \Box_L^m(V) \subseteq \diamond_{L,S}^m(V)$.

Finite dimensional Δ^m-invariant subspaces are formed by exponential polynomials: The proof

- We apply the second Lemma with $E = X$, $L = \Delta_{h_1}$ and $S = \Delta_{h_2}$ to conclude that $V \subseteq W = \diamond^m_{\Delta_{h_1},\Delta_{h_2}}(V)$ and W is a finite dimensional subspace of X satisfying $\Delta_{h_i}(W) \subseteq W$, $i = 1, 2$.
- Hence we can apply Anselone-Korevaar's Theorem to W and conclude that this space admits an algebraic basis of the form

$$t^{k-1}e^{\lambda t},\ k = 1, \cdots , m(\lambda) \text{ and } \lambda \in \{\lambda_0, \lambda_1, \cdots , \lambda_s\} \subset \mathbb{C},$$

- In particular, all elements of V are exponential polynomials.

Application: Montel's Theorem for distributions

Corollary (Montel's Theorem for distributions)

Assume that f *is a complex valued distribution such that* $\Delta_{h_1}^m f = \Delta_{h_2}^m f = 0$ *for certain non-zero real numbers* h_1, h_2 *such that* $h_1/h_2 \notin \mathbb{Q}$*. Then* f *is an ordinary polynomial of degree* $\leq m-1$.

Assume that $\Delta_{h_1}^m f = \Delta_{h_2}^m f = 0$. Then $V = \textbf{span}\{f\}$ is a 1-dimensional space of complex valued distributions and:

$$\Delta_{h_1}^m(V) = \Delta_{h_2}^m(V) = \{0\} \subseteq V$$

Hence all elements of V are exponential polynomials. In particular, f is an exponential polynomial,

$$f(t) = \sum_{k=0}^{m(0)-1} a_{0,k}t^k + \sum_{i=1}^{s} \sum_{k=0}^{m(\lambda_i)-1} a_{i,k}t^k e^{\lambda_i t}$$

and we can assume that $m(0) \geq m$ with no loss of generality.

Let

$$\beta = \{t^{k-1}e^{\lambda_i t}, \;\; k = 1, \cdots, m(\lambda_i) \text{ and } i = 0, 1, 2, \cdots, s\}$$

and $E = \textbf{span}\{\beta\}$ be such that $V \subseteq E$.
Let us consider the linear map $\Delta_h : E \to E$ induced by the operator Δ_h when restricted to E.

Application: Montel's Theorem for distributions

The matrix associated to this operator with respect to the basis β is block diagonal, $A = \textbf{diag}[A_0, A_1, \cdots, A_s]$, with

$$A_0 = \begin{bmatrix} 0 & h & h^2 & \cdots & h^{m(0)-1} \\ 0 & 0 & 2h & \cdots & \binom{m(0)-1}{2} h^{m(0)-2} \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{m(0)-1}{m(0)-2} h \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

and

$$A_i = \begin{bmatrix} e^{\lambda_i h} - 1 & he^{\lambda_i h} & h^2 e^{\lambda_i h} & \cdots & h^{m(i)-1} e^{\lambda_i h} \\ 0 & e^{\lambda_i h} - 1 & 2he^{\lambda_i h} & \cdots & \binom{m(i)-1}{2} h^{m(i)-2} e^{\lambda_i h} \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{m(i)-1}{m(i)-2} he^{\lambda_i h} \\ 0 & 0 & 0 & \cdots & e^{\lambda_i h} - 1 \end{bmatrix},$$

for $i = 1, 2, \ldots, s$.

It follows that the matrix associated to $(\Delta_h^m)_{|E}$ with respect to the basis β is given by $A^m = \mathbf{diag}[A_0^m, A_1^m, \cdots, A_s^m]$. Obviously, the matrices A_i^m $(i = 1, 2, \cdots, s)$ are invertible since the corresponding A_i are so. On the other hand, $\mathbf{rank}(A_0^m) = m(0) - m$ and

$$\mathbf{ker}(A_0^m) = \mathbf{span}\{(0, 0, \cdots, 0, 1^{\text{(i-th position)}}, 0, \cdots, 0) : i = 1, 2, \cdots, m\}.$$

It follows that $\mathbf{rank}(A^m) = \dim_{\mathbb{C}} E - m$, so that $\dim_{\mathbb{C}} \mathbf{ker}(A^m) = m$. On the other hand, a simple computation shows that the space of ordinary polynomials of degree $\leq m - 1$, which we denote by Π_{m-1}, is contained into $\mathbf{ker}(\Delta_h^m)$. Hence $\mathbf{ker}(\Delta_h^m) = \Pi_{m-1}$, since both spaces have the same dimension. This, in conjunction with $f \in \mathbf{ker}(\Delta_h^m)$, ends the proof.

Theorem

Assume that V is a finite dimensional subspace of X which satisfies $\Delta_h^m(V) \subseteq V$ for all $h \in \mathbb{R}$. Then there exist vector spaces $\mathcal{P} \subset \mathbb{C}[t]$ and $\mathcal{E} \subset \mathbf{C}(\mathbb{R})$ such that

$$V = \mathcal{P} \oplus \mathcal{E}$$

and $\mathcal{E}$ is invariant by translations. Consequently,

V is invariant by translations if and only if $\mathcal{P}$ is so.

Proof: We use the Characterization of invariant subspaces of any linear operator based on root subspaces.
We have already shown that $V \subseteq E$ with $E = \mathbf{span}\{\beta\}$ as above, and that

$$M_\beta((\Delta_h)_{|E}) = \mathbf{diag}[A_0, A_1, \cdots, A_s]$$

with matrices A_i as shown in the previous slides. In particular,

$$M_\beta((\Delta_h^m)_{|E}) = \mathbf{diag}[A_0^m, A_1^m, \cdots, A_s^m]$$

Δ^m-invariant subspaces: structure

An easy computation shows that:

$$R_0((\Delta_h^m)_{|E}) = \Pi_{m(0)-1} = \textbf{span}\{t^k\}_{k=0}^{m(0)-1}$$

and

$$R_{(e^{\lambda_i h}-1)^m}((\Delta_h^m)_{|E}) = \Pi_{m(0)-1} = \textbf{span}\{t^k e^{\lambda_i t}\}_{k=0}^{m(\lambda_i)-1} =: E_i$$

Hence V is Δ_h^m-invariant if and only if

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_s$$

with $V_0 \subseteq \Pi_{m(0)-1}$ and $V_i \subset E_i$ Δ_h^m-invariant subspaces (for all i).

Δ^m-invariant subspaces: structure

Now:

- $\beta_i = \{t^k e^{\lambda_i t}\}_{k=0}^{m(\lambda_i)-1}$ is a basis of E_i and $M_{\beta_i}((\Delta_h)_{|E_i}) = A_i$, which is a matrix of the special form $\lambda I + B$ that we studied in Lemma (Important Example).
- Hence $V_i \subseteq E_i$ is Δ_h^m-invariant if and only if it is Δ_h-invariant.
- This proves the result with $\mathcal{P} = V_0$ and $\mathcal{E} = V_1 \oplus V_2 \oplus \cdots \oplus V_s$.

Other results

Theorem (Δ^m-invariant subspaces of $\mathbf{C}(0,\infty)$)

Let V be a finite dimensional subspace of the space of continuous complex valued functions defined on the semi-infinite interval $(0,\infty)$ and assume that $\Delta_{h_k}^m(V) \subseteq V$ for an infinite sequence of positive real numbers $\{h_k\}_{k=1}^{\infty}$ which converges to zero. Then all elements of V are exponential polynomials.

Other results

Theorem

Assume that V is a finite dimensional subspace of X. Then the following statements are equivalent:

(i) $\Delta_h^m(V) \subseteq V$ *for all* $h \in \mathbb{R}$.

(ii) $\Delta_{h_1 h_2 \cdots h_m}(V) \subseteq V$ *for all* $(h_1, h_2, \cdots, h_m) \in \mathbb{R}^m$.

Theorem

Given $\sigma : \mathbb{R} \to \mathbb{R}$ continuous, consider the space:

$$W(\sigma) = \textbf{span}\{\sigma(w \cdot x^t + b) : w \in \mathbb{R}^d, \|w\|_2 = 1 \text{ and } b \in \mathbb{R}\},$$

where $a \cdot b^t = \sum_{k=1}^d a_k b_k$ is the dot product of vectors. There are exactly two possibilities: Either

- *σ is an ordinary polynomial (In which case $\dim W(\sigma) < \infty$).*

or

- *$W(\sigma)$ is a dense subset of $C(\mathbb{R}^d)$ with the topology of uniform convergence on compact subsets of $\mathbb{R}^d$.*

Other results

There is a similar result substituting σ by a function of several variables and $W(\sigma)$ by

$$\Gamma(\sigma) = \textbf{span}\{\sigma(w \circ x + b) : b, w \in \mathbb{R}^d\},$$

where $a \circ b = (a_1 b_1, a_2 b_2, \cdots, a_d b_d)$ is the componentwise product of the vectors.
In that case, either

- $\dim \Gamma(\sigma) < \infty$ (and σ is a polynomial in d variables)

or

- $\Gamma(\sigma)$ is dense in $C(\mathbb{R}^d)$.

Some open problems

- Many of the results above hold true in several variables setting. In particular, this is so for Montel's Theorem and for the Characterizations of Δ_h^m-invariant subspaces of continuous functions and distributions.
- On the other hand, our proof that Δ_h^m-invariant and $\Delta_{h_1 h_2 \cdots h_m}$-invariant finite dimensional subspaces are the same, does not apply for the multivariate setting (and it is still open question to know the corresponding result)

The paper:

M. Engert

Finite dimensional translation invariant subspaces, Pacific J. Math. 32 (2) (1970) 333-343

contains an analogous result to Anselone-Korevaar's theorem for the case of measurable functions on σ-compact locally compact abelian groups.

- Do the results of this talk hold true for measurable functions?
  (We conjeture: No.
  In any case, the proofs should be different!!)

Some open problems

- What about the questions of this talk if formulated for Lie Groups? (Here the functions can be of the form $f : (G, *) \to (H, \circ)$ and the associated operators are $\tau_h g(x) = g(x * h)$, $\Delta_h g(x) = g(x * h) \circ g(x)^{-1}$).
- It would be of interest to study spaces of functions which are invariant under certain geometrically motivated operators. For example, given $N > 0$, we can consider the operators

$$L_h(f)(z) = \frac{1}{N} \sum_{k=0}^{N-1} f(z + w^k h),$$

  with w any primitive N-th root of 1.
- Is there any natural way to characterize the spaces of solutions of linear EDO's $x'(t) = A(t)x(t)$ with $A(t)$ periodic matrix function?

Some open problems

- Is it possible to prove a Montel's type theorem for Popoviciu's functional equation?

$$det\begin{bmatrix} f(x) & f(x+h) & \cdots & f(x+nh) \\ f(x+h) & f(x+2h) & \cdots & f(x+(n+1)h) \\ \vdots & \vdots & \ddots & \vdots \\ f(x+nh) & f(x+(n+1)h) & \cdots & f(x+2nh) \end{bmatrix} = 0$$

Thank you for your attention!!