On a theorem of Helson for general Dirichlet series

A talk dedicated
to my friend Bernardo Cascales

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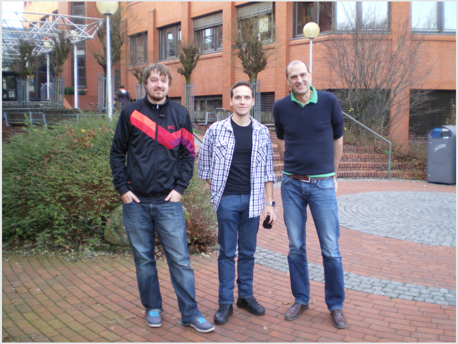
Bernardo with Klaus and Andreas, 1987

Antonio with Sunke and Andreas, 2014



Carleson-Hunt theorem, 1966

The Fourier series of every $f \in L_p(\mathbb{T}), 1 < p < \infty$ converges almost everywhere on $\mathbb{T}$.

Bohr's fundamental theorem, 1914

Let $D = \sum a_n n^{-s}$ be a Dirichlet series. Then the abscissa of uniform convergence and the abscissa of boundedness coincide.

Helson's theorem.....

Appetizer – Riemann's conjecture is true 'almost everywhere':

For almost all completely multiplicative arithmetic functions $\chi : \mathbb{N} \to \mathbb{T}$ the radomized ζ-function

$$\sum \chi(n) \frac{1}{n^s}$$

converges on $[\mathrm{Re} > \frac{1}{2}]$ and has no zeros there.

Almost everywhere?

$$\Xi \longrightarrow \mathbb{T}^\infty, \quad \chi \mapsto (\chi(p_k)) \text{ group isomorphism}$$

Helson's theorem, 1969

Let $D = \sum a_n n^{-s}$ be a Dirichlet series such that $(a_n) \in \ell_2$. Then

$$\sum \chi(n) a_n n^{-s}$$

converges for almost all $\chi \in \Xi$ on $[\mathrm{Re} > 0]$.

Carleson-Hunt theorem, 1966

The Fourier series of every $f \in L_p(\mathbb{T}), 1 < p < \infty$ converges almost everywhere on $\mathbb{T}$.

Bohr's theorem, 1914

Let $D = \sum a_n n^{-s}$ be a Dirichlet series. Then the abscissa of uniform convergence and the abscissa of boundedness coincide.

Helson's theorem, 1969

Let $D = \sum a_n n^{-s}$ be a Dirichlet series such that $(a_n) \in \ell_2$. Then $\sum \chi(n) a_n(D) n^{-s}$ converges for almost all χ on $[\mathrm{Re} > 0]$.

These three apparently very different results are in fact linked!

General Dirichlet series

$$\begin{tikzcd}
& \begin{array}{c}\text{general Dirichlet series}\\ \sum a_n e^{-\lambda_n s}\end{array} \arrow[dl, "\lambda_n = \log n"'] \arrow[dr, "{\lambda_n = n,\ z = e^{-s}}"] & \\
\begin{array}{c}\text{ordinary case}\\ \sum a_n \frac{1}{n^s}\end{array} & & \begin{array}{c}\text{power series}\\ \sum a_n z^n\end{array}
\end{tikzcd}$$

A natural class of λ-Dirichlet series

$$\mathcal{D}_\infty(\lambda) = \Big\{ D = \sum a_n e^{-\lambda_n s} \colon \; D \text{ is bdd and holo on } [\mathrm{Re} > 0] \Big\}$$

Theorem

$\mathcal{D}_\infty(\lambda)$ is a Banach space whenever λ satisfies Bohr's condition (BC):

$$\exists l = l(\lambda) > 0 \forall \delta > 0 \; \exists \; C > 0 \; \forall \; n \in \mathbb{N} : \lambda_{n+1} - \lambda_n \geq C e^{-(l+\delta)\lambda_n}$$

Obvious

There is a bijective isometry

$$\mathcal{D}_\infty((n)) = H_\infty(\mathbb{T})\,,\ \ D = \sum a_n e^{-ns} \mapsto f = \sum a_n z^n\,.$$

Theorem, Bohr-Hedenmalm-Lindqvist-Seip, 1998

There is a bijective isometry

$$\mathcal{D}_\infty((\log n)) = H_\infty(\mathbb{T}^\infty)\,,\ \ D \mapsto f$$

which preserves the coefficients, i.e. $a_n(D) = \hat{f}(\alpha)$ whenever $n \in \mathbb{N}$ and $\alpha = (\alpha_1, \ldots, \alpha_N, 0, \ldots) \in \mathbb{N}_0^{(\mathbb{N})}$ are such that

$$n = p_1^{\alpha_1} \times \ldots \times p_N^{\alpha_N}\,.$$

One of the crucial tools

The continuous group homomorphism

$$\beta : \mathbb{R} \to \mathbb{T}^\infty \,, \quad t \mapsto (p_k^{-it})_{k=1}^\infty$$

has dense range, and for each n and α with $n = \mathfrak{p}^\alpha$ the following diagram commutes

$$\begin{tikzcd}
\mathbb{T}^\infty \arrow[r, "\boldsymbol{z}^\alpha"] & \mathbb{T} \\
\mathbb{R} \arrow[u, "\beta"] \arrow[ur, "e^{-i\boldsymbol{t}\log n}"'] &
\end{tikzcd}$$

Bayart's Hardy spaces of Dirichlet series, 2002

For $1 \leq p \leq \infty$

$$\mathcal{H}_p := \Big\{ D = \sum a_n n^{-s} \colon \exists f \in H_p(\mathbb{T}^\infty) : a_n(D) = \hat{f}(\alpha) \text{ if } n = \mathfrak{p}^\alpha \Big\}$$

together with $\|D\|_p = \|f\|_p$ defines a Banach space.

The Bohr-Hedenmalm-Lindqvist-Seip theorem revisited

$$\mathcal{D}_\infty((\log n)) = \mathcal{H}_\infty$$

From ordinary to general Dirichlet series

- . . . a big step!
- A few heros in alphabetical order: Bohr, Besikovitch, Bohnenblust, Hardy, Helson, Hille, Kahane, Landau, Perron, M. Riesz, Neder,. . .

Under construction

. . . an $\mathcal{H}_p$-theory of λ-Dirichlet series
modelled along Bayart's 'ordinary theory'

λ-Dirichlet groups

Given a frequency λ, we call a pair (G, β) a λ-Dirichlet group if

- G is a compact abelian group and $\beta : \mathbb{R} \to G$ a continuous group homomorphism with dense range.
- For each character $e^{-i\lambda_n \boldsymbol{t}}$ there is some character $h_{\lambda_n}(\boldsymbol{\omega})$ (then unique) such that

$$\begin{tikzcd}
G \arrow[r, "h_{\lambda_n}(\boldsymbol{\omega})"] & \mathbb{T} \\
\mathbb{R} \arrow[u, "\beta"] \arrow[ru, "e^{-i\lambda_n \boldsymbol{t}}"'] &
\end{tikzcd}$$

Definition – Hardy spaces of general Dirichlet series

Let (G, β) be a λ-Dirichlet group, and $1 \leq p \leq \infty$. Then

$$\mathcal{H}_p(\lambda)$$

consists of all $D = \sum a_n e^{-\lambda_n s}$ for which there is some $f \in L_p(G)$ (then unique) such that

- $\hat{f} : \widehat{G} \to \mathbb{C}$ is supported by all $h_{\lambda_n},\ n \in \mathbb{N}$
- $a_n(D) = \hat{f}(h_{\lambda_n})$ for all n

Essential

- The $\mathcal{H}_p(\lambda)$'s are Banach spaces which are independent of the chosen λ-Dirichlet group.
- For $\lambda_n = \log n$ we may choose $G = \mathbb{T}^\infty$ and the Kronecker flow $\beta : \mathbb{R} \to \mathbb{T}^\infty\,, \quad t \mapsto (p_k^{-it})_{k=1}^\infty$. Hence Bayart's $\mathcal{H}_p$-theory is incorporated.
- There are plenty of ways to 'realize' the groups in this result! For arbitrary λ's the Bohr compactification $\overline{\mathbb{R}}$ or $\widehat{\mathbb{Q}}^\infty$ always do the job, and for certain classes of 'nice' λ's the groups $\mathbb{T}^\infty = \widehat{\mathbb{Z}}^\infty$ and Ξ...
- The general motto is: Choose the group which fits with your frequency and your problem!

The Bohr-Hedenmalm-Lindqvist-Seip theorem for general Dirichlet series

Let λ satisfy (BC). Then

$$\mathcal{D}_\infty(\lambda) = \mathcal{H}_\infty(\lambda)$$

The proof needs an extension of Helson's theorem ....

Recall that the idea of this talk was to explain how the following results are linked?

Carleson-Hunt theorem

The Fourier series of every $f \in L_p(\mathbb{T}), 1 < p < \infty$ converges almost everywhere on $\mathbb{T}$.

Bohr's theorem

Let $D = \sum a_n n^{-s}$ be a Dirichlet series. Then the abscissa of uniform convergence and the abscissa of bounedness coincide.

Helson's theorem

Let $D = \sum a_n n^{-s}$ be a Dirichlet series such that $(a_n) \in \ell_2$. Then $\sum \chi(n) a_n(D) n^{-s}$ converges for almost all χ on $[\mathrm{Re} > 0]$.

Assume that λ satisfies (BC) and (G, β) be a λ-Dirichlet group.

Helson's theorem in $\mathcal{H}_p(\lambda)$'s

Let $1 \leq p < \infty$. Then for every $D \in \mathcal{H}_p(\lambda)$ the Dirichlet series

$$\sum h_{\lambda_n}(\omega) a_n(D) e^{-\lambda_n s}$$

for almost all $\omega \in G$ converges on $[\text{Re} > 0]$.

Assume that λ satisfies (BC) and (G, β) be a λ-Dirichlet group.

Helson's theorem in $\mathcal{H}_p(\lambda)$'s

Let $1 \leq p < \infty$. Then for every $u > 0$ and every $D \in \mathcal{H}_p(\lambda)$ the series

$$\sum a_n(D) e^{-\lambda_n u} h_{\lambda_n}$$

converges almost everywhere on G .

Maximal inequality

For every $u > 0$ there is some $C = C(u) > 0$ such that for every $D \in \mathcal{H}_p(\lambda)$, $1 \leq p < \infty$

$$\left\| \sup_N \left| \sum_{n=1}^{N} a_n(D) e^{-\lambda_n u} h_{\lambda_n} \right| \right\|_{L_p(G)} \leq C \|D\|_{\mathcal{H}_p(\lambda)} .$$

Credits on the 'almost everywhere part' in the ordinary case:

- $p = 2$: due to Helson, and Bayart gives a proof with the Menchoff-Rademacher theorem….
- $1 \leq p < \infty$: due to Bayart, and his proof uses so-called hypercontractivity….

In the ordinary case no maximal inequalities were known so far. First application: If in our maximal inequality we let $p \to \infty$, then Bohr's fundamental theorem appears in a natural way.

What about convergence on the imaginary axis – the case $u = 0$?

Theorem

Let λ be a frequency of integer type and (G, β) a λ-Dirichlet group. Then for every $1 < p < \infty$ there is a constant $C = C(p) > 0$ such that for every $D \in \mathcal{H}_p(\lambda)$

$$\Big\| \sup_N \Big| \sum_{n=1}^{N} a_n h_{\lambda_n} \Big| \Big\|_{L_p(G)} \leq C \|D\|_{\mathcal{H}_p(\lambda)} \,.$$

In particular, the series

$$\sum a_n(D) h_{\lambda_n}$$

converges almost everywhere on G.

Credits

- For $1 < p < \infty$ and $\lambda = (n)$ this is a reformulation of the Carleson-Hunt theorem.
- For $p = 2$ and $\lambda = (\log n)$ the result is due to Hedenmalm and Saksman – and its proof is based on Carleson's maximal inequality and a technique of Fefferman.
- For $1 < p < \infty$ our proof follows similar ideas – in particular using the Carleson-Hunt maximal inequality and Feffermans technique.

What does this mean for functions on $\mathbb{T}^\infty$?

Carleson-Hunt type theorem for the infinite dimensional torus

Let $1 < p < \infty$. Then for every $f \in H_p(\mathbb{T}^\infty)$ and for almost all $z \in \mathbb{T}^\infty$

$$f(z) = \lim_N \sum_{\mathfrak{p}^\alpha \leq N} \hat{f}(\alpha) z^\alpha$$

and

$$\left\| \sup_N \left| \sum_{\mathfrak{p}^\alpha \leq N} \hat{f}(\alpha) z^\alpha \right| \right\|_{L_p(\mathbb{T}^\infty)} \leq C(p) \|f\|_{H_p(\mathbb{T}^\infty)} .$$

Helson's theorem for the infinite dimensional torus

Let $1 \leq p < \infty$. Then for every $f \in H_p(\mathbb{T}^\infty)$, every $u > 0$ and almost all $z \in \mathbb{T}^\infty$

$$\lim_N \sum_{\mathfrak{p}^\alpha \leq N} \hat{f}(\alpha) \left(\frac{z}{\mathfrak{p}^u}\right)^\alpha \quad \text{exists}$$

and

$$\left\| \sup_N \left| \sum_{\mathfrak{p}^\alpha \leq N} c_\alpha \left(\frac{w}{\mathfrak{p}^u}\right)^\alpha \right| \right\|_{L_p(\mathbb{T}^\infty)} \leq C(u) \|f\|_p \,.$$