Weakly compactly generated Banach spaces and some of their relatives classified by using projectional skeletons

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NonLinear Functional Analysis
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Outline

WCG spaces

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- ℓ_∞ not WCG (every w-compact is $\|\cdot\|$-separable).

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We say that the couple (V, Y) gives an r-bounded projection $P_{V\times Y}$ (on V).

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(v) For $x \in X$, $\alpha \to P_\alpha(x)$ continuous.

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$P_\gamma \circ P_\beta = P_\beta \circ P_\gamma$ commutative.
$\|P_\gamma\| \leq r$ r-skeleton.
(see, e.g., Kakol–Kubiś–López Pellicer book).

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- X <LUR> (modulo S. Troyanski and V. Zizler).

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$\boxed{(N, \Phi)}$, where $N \subset X^*$ 1-norming, $\mathbb{Q}$-linear, $\Phi : N \to 2^X$, at most countably-valued, st $\forall B \subset N,\ B \neq \emptyset,\ \mathbb{Q}$-linear,

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Theorem (Orihuela, Valdivia)

(N, Φ) *PG* $\Rightarrow$ *PRI.*

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Particular case: X with M-basis st $\forall x^* \in X^*$, supp $_M x^*$ is countable.
This class is exactly WLD, i.e., X with (B_{X^*}, w^*) Corson.

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$\mathcal{R} \subset \mathcal{S}_\square(X, X^*)$ rich subfamily (directed upwards, cofinal, σ-complete).

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Corollary

X with r-PG. Then X has an r-PS.

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- *If moreover D is r-norming, Γ rich in $\mathcal{S}_{\square}(X \times D)$ given by (D, Φ). Then PS generated by $\mathcal{R} \cap \Gamma$ is commutative.*

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Characterizing WCG by skeletons

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“Internal” characterization of SWCG

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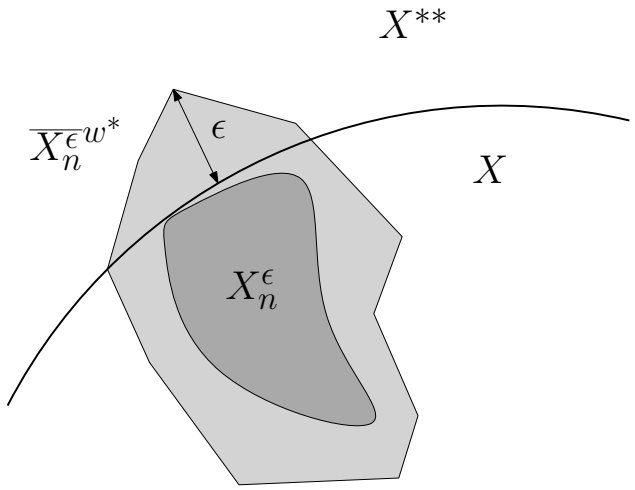

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Theorem

X WCG Asplund $\Leftrightarrow \exists$ shrinking M-basis.

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an M-basis is σ-shrinking if $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$ st $\forall U \in \mathcal{N}(0)$ in $(X^{**}, \|\cdot\|)$, $\forall \gamma \in \Gamma$, $\exists n \in \mathbb{N}$ st $\gamma \in \Gamma_n$, $\Gamma_n' \subset U$ (derivative in (X^{**}, w^*)).

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Moreover, every M-basis is σ-shrinking.

Three more open problems

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Problem

Characterize Banach spaces simultaneously Asplund and 1-Plichko by using skeletons.

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