Biholomorphic functions on dual of Banach Space

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Conference on Non Linear Functional Analysis.
Workshop on Infinite Dimensional Analysis Valencia -SPAIN
2017

Notation

- *Let E be a complex Banach space.*
- $\mathcal{L}(E)$ *space of all bounded linear operator*
- *Let V an open set of E, $\mathcal{H}(V, E)$ the space of all holomorphic functions from V into E.*
- *If f belongs to $\mathcal{H}(V, E)$, df_p denotes the derivative of f at the point $p \in V$*

Theorem (Carathéodory- Cartan–Kaup-Wu)

If a holomorphic mapping $f : \Omega \longrightarrow \Omega$ of a bounded domain $\Omega \subset \mathbb{C}^n$, satisfies $f(p) = p$ for some $p \in \Omega$ and $|det\{df_p\}| = 1$, then f is a biholomorphic mapping.

Definition

Let $T \in \mathcal{L}(E)$ be a bounded linear operator. T is triangularizable if there is a total, i. e., with dense span, linearly independent sequence $\{e_1, e_2, ...e_n, ...\}$ of E such that for all $x \in span\{e_1, e_2, ..., e_n\}$

$$T(x) \in span\{e_1, e_2, ..., e_n\}$$

for every $n \in \mathbb{N}$. In this case, $T(e_k) = \sum_{j=1}^{k} \beta_j^k e_j$, for all $k = 1, 2, \cdots, n$.

So, the matrix of T when restricted the subspace generated by $\{e_1, e_2, ..., e_k\}$ is an upper-triangular matrix, whose main diagonal is given by $\beta_1^1, \beta_2^2, \cdots \beta_k^k$. We shall refer to the sequence (β_k^k) as the diagonal entries.

J.A. Cima, I. Graham, K. T. Kim and S. Krantz, The Carathéodory-Cartan-Kaup-Wu Theorem on an infinite dimensional Hilbert Space. Nagoya Math. J. Vol.185 (2007) *17-30.*

Theorem

Let H be a separable Hilbert space and $\Omega \subset H$, be a bounded convex domain. Fix a point $p \in \Omega$. Let $f : \Omega \longrightarrow \Omega$ be a holomorphic mapping such that

- $f(p) = p$;
- *the differential df_p is triangularizable;*
- $\sigma(df_p) \subset S(0,1)$.

Then f is biholomorphic

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H.Carrion, P. Galindo and M.L. Lourenço, Biholomorphic functions in dual Banach spaces, Complex Anal. Oper. Theory, **(7)***(2013),107-114.*

Theorem

Let E be the dual of a Banach space and $G \subset E$ a bounded domain with the separation property E such that its weak closure coincides with its norm closure. Let $f : G \to G$ be a holomorphic mapping such that*

1. $f(p) = p$.
2. df_p *is triangularizable with diagonal entries of modulus* 1.

Then f is a biholomorphic mapping.

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Definition

An operator is said to be **power bounded** *if* $\sup_n \|T^n\| < \infty$.

where $T^n = T \circ T \circ \cdots \circ T$, *is the n-times self composition of* T.

If E *is a Banach space, this is equivalent to* $\sup_n \|T^n(x)\| < \infty$ *for each* $x \in E$ *according to the Uniform Boundedness Principle.*

Theorem

Let E be a Banach space. Let $T \in \mathcal{L}(E)$ be a power bounded closed range operator. Assume that:

- *There is a subsequence $\left(T^{n(k)}\right) \subset \left(T^{n}\right)$ which pointwise converges to an operator $S \in \mathcal{L}(E)$.*
- *If S is onto and T one-to-one, or just S is invertible.*

Then T is an invertible operator and $\sigma(T) \subset S(0,1)$.
Further, if $m(k) = n(k+1) - n(k)$, then $\left(T^{m(k)}\right)$ pointwise converges to the identity mapping.

If E is a Hilbert space, then T is equivalent to a unitary operator.

The idea of the proof:

T is an isomorphism into. Let $y \in E$ arbitrary and consider $x \in E$ such that $Sx = y$. Then $y = \lim T^{n(k)}x = \lim T\left(T^{n(k)-1}x\right)$. Now, for $y_k = T^{n(k)-1}x$, the sequence (y_k) is a Cauchy sequence since T is an isomorphism. Let $y_0 \in E$ be the limit of (y_k). Then $T(y_0) = \lim T(y_k) = \lim T\left(\left(T^{n(k)-1}\right)(x)\right) = \lim T^{n(k)}(x) = y$. Therefore T is an onto mapping, thus invertible by the Open Mapping Principle.

In the particular case that S is invertible, also T is injective since if $T(x) = 0$, also $T^{n(k)}(x) = 0$, so $S(x) = 0$.

It is possible to prove

$$\sup_m \left\| T^{-m} \right\| < \infty$$

Indeed, $C := \sup_{n\in\mathbb{Z}} \|T^n\| < \infty$.

Let $m(k) = n(k+1) - n(k)$. *It is possible to check that* $\lim_k T^{m(k)}x = x$.

Since $||T^n|| \leq C$ *for all* $n \in \mathbb{Z}$, *we have* $r(T) \leq 1$ *and* $r(T^{-1}) \leq 1$. *So* $\sigma(T) \subset \Delta(0,1)$ *and* $\sigma(T^{-1}) \subset \Delta(0,1)$. *Thus* $\sigma(T) \subset S(0,1)$.

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If E is a Hilbert space, there is an invertible self-adjoint operator Q such that $QTQ^{-1} = U$ is unitary.

B. Sz. Nagy, On uniformly bounded linear transformations in Hilbert space. Acta Univ. Szeged. Sect. Sci. Math. 11, (1947). 152-157

Theorem

For each power bounded linear operator T in a Hilbert space there is a self-adjoint operator Q such that $Q^{-1}TQ$ is a unitary operator.

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Example

Let $(\lambda_n)_n \subset S(0,1)$. *Consider the Hilbert space* $E = \ell_2(\mathbb{R})$. *Put* $\mathbb{R} \equiv I \times \mathbb{N}$ *for an uncountable set* I. *For every element in the canonical basis* $\{e_{i,n} : I \times \mathbb{N}\}$ *define*

$$T(e_{i,n}) = \lambda_n e_{i,n}.$$

$T \in \mathcal{L}(E)$, *and* $\|T\| = 1$, *and hence* $\|T^m\| \leq 1$.
Since $(\lambda_n)_n \subset S(0,1)$, *by the Cantor's diagonalization process there exists a subsequence of positive integer numbers* $(m(k))_k$ *such that*

$$\lim_k \lambda_n^{m(k)} = 1$$

for all n. *It turns out that* T^{m_k} *pointwise converges to* $Id_{|_E}$ *since for all pairs* (i,n), $\lim_k T^{m_k}(e_{i,n}) = \lim_k (\lambda_n)^{m_k} e_{i,n} = e_{i,n}$.
Notice that T *is not triangularizable.*

Definition

We say that an open set $A \subset E$ has the separation property if for every $u \in \overline{A} \setminus A$, there is an analytic function h in a neighborhood of $\overline{A}$ such that $h(u) = 1$ and $|h(x)| < 1$ for all $x \in A$.

Examples

- *Any convex domain Ω has the separation property.*
- *For each A a relatively compact strictly pseudoconvex open set in $\mathbb{C}^n$ with a $\mathcal{C}^2$ boundary has the separation property*

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Theorem

Let E be a dual Banach space and let $U \subset E$ a bounded domain with the separation property such that its weak closure coincides with its norm closure. Let $f \in \mathcal{H}(U, U)$ and $p \in U$ such that*

- $f(p) = p$.
- df_p *is a one-to-one closed range operator*
- *there is a subnet* $\left(df_p^{n(k)}\right)$ *which converges pointwise to an onto operator.*

Then f is a biholomorphic mapping.

Remark

Observe that Theorem does not provide a necessary condition for f to be biholomorphic: Consider $E = \ell_2(\mathbb{Z})$ and f the shift operator in E,

$$f((x_n)) = (x_{n+1}).$$

- *f is an automorphism of the unit ball,*
- *$f = df_0$, and $f^m(e_n) = e_{n+m}$,*

so the sequence $(f^k(e_n))_k$ does not have a Cauchy subsequence and, therefore, $(f^k)_k$ can not have a pointwise convergent subsequence. In addition, f is known to be not a triangularizable operator.

The idea of the proof:

- *Let $G \subset E$ be a bounded domain and $f : G \to G$ be a holomorphic mapping such that there is $p \in G$ with $f(p) = p$. For each $m \in \mathbb{N}$ we have*

$$\|(df^m(p))\| \leq \frac{\sup_{x \in G} \|f(x)\|}{d}$$

where $d = dist(p, \partial G)$ and $f^m = f \circ f \circ \cdots \circ f$, is the m-times self composition of f.

There is a constant $C > 0$ such that $\left\|df_p^n\right\| < C$ for all n. Theorem shows that df_p is invertible, which implies that df_p^{-1} does exist, and also that there is a subsequence $\left(df_p^{n(k)}\right)$ pointwise convergent to the identity.

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Let $A = \{f^{n(k)}, k \in I\} \subset \mathcal{H}(U, U)$.

Lemma

If $E = X^*$ *is a dual Banach space, then for every* $\mathcal{A} \subset \mathcal{H}(\Omega, E)$ *that is bounded for the compact-open topology, there is a compact-open-weak* convergent net to some point* $g \in \mathcal{H}(\Omega, E)$.

There is a compact-open-weak* *convergent subnet* $\left(f^{n(k_i)}\right)_i$ *to some function, say,* $g \in \mathcal{H}(U, E)$. *Then* $g(z) \in \overline{U}^{w^*} = \overline{U}$, *for all* $z \in U$ *and* $g(p) = p$. *Since* U *is an open set with separation property we have that* $g(U) \subset U$. *Moreover, we have that* $\left((df_p)^{n(k_i)}\right)$ *is compact-open-weak** *convergent to* dg_p. *Therefore* $dg_p = I_E$.

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Theorem

(Cartan) Let $G \subset E$ be an open bounded set, $h \in \mathcal{H}(G.G)$ and $p \in G$, such that $h(p) = p$ and $dh_p = I_E$. Then $h = I_G$.

Using Cartan's Theorem $g = I_U$.

The bounded subnet $\left(f^{n(k_i)-1}\right)_i$ has a subnet that converges to a holomorphic function $h \in \mathcal{H}(U, E)$ in the compact-open-weak topology;*

Clearly, $h(p) = p$ and $h(z) \in \overline{U}^{w^*} = \overline{U}$ for all $z \in U$, we have that $h \in \mathcal{H}(U, U)$. Moreover, for $z \in U$,

$$z = \lim f\left(f^{n(k_i)-1}\right)(z) = \lim f^{n(k_i)-1}(f(z)) = h(f(z)),$$

which shows that $h \circ f = I_U$.

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Finally, we show that $f \circ h = I_U$.
Since $h \circ f = I_U$, *we have*

$$dh_p \circ df_p = I_E$$

and since df_p^{-1} *exists, it follows that* $dh_p = df_p^{-1}$.

Therefore $df_p \circ dh_p = I_E$, *using Cartan's Theorem again, we obtain*

$$f \circ h = I_U.$$

Corollary

Let E be a reflexive Banach space and let $U \subset E$ be a convex bounded domain. Let $f \in \mathcal{H}(U, U)$ and $p \in U$ such that

- $f(p) = p$,
- df_p *is a one-to-one operator*
- *There is a subsequence* $\left((df_p)^{m_k}\right)_k$ *that pointwise converges to an onto operator in* $\mathcal{L}(E)$.

Then f is biholomorphic.

Since df_p is weakly compact, the result follows.

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Then f is biholomorphic.

Since df_p is weakly compact, the result follows.

Theorem

Let $\Omega \subset \ell_p$ $(p > 1,\ p \neq 2)$ be a bounded convex subset and $f : \Omega \to \Omega$, holomorphic mapping with a fixed point $a \in \Omega$. Suppose that df_a is similar a isometric isomorphism and there is an eigenvector $v = (v_1, v_2, ..., v_i, ..) \in \ell_p$ such that $v_i \neq 0$ for all $i \in \mathbb{N}$, then f is biholomorphic.

GRACIAS