

# A class of poverty measures induced by the dual decomposition of aggregation functions

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## Abstract

In this paper we introduce a new family of poverty measures for comparing and ordering social situations. The aggregation scheme of these poverty measures is based on the one-parameter family of exponential means. The poverty measures introduced satisfy interesting properties and the dual decomposition of the underlying exponential means induces a natural decomposition of the proposed poverty indices themselves into three underlying factors: incidence, intensity, and inequality among the poor.

*Keywords:* Economic indices, welfare; inequality, decomposition of poverty indices, aggregation functions, dual decomposition, exponential means.

*JEL Classification:* D63, I32.

## 1 Introduction

In the economic literature there exists a large variety of economic indices (inequality, poverty, social development, gross domestic product, health, welfare, etc.) for comparing and ranking social situations across populations (see, for instance, Sen [16], Dagum [7], and Chakravarty and Muliere [5, 6]). A thorough survey of the literature on inequality, poverty, and welfare can be found in Chakravarty [3, 4]. The main difference between inequality or welfare and poverty lies in their focus. Whereas inequality and welfare measurement involves the whole population, the concern of poverty is focused on those who are worse-off.

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Specifically, any poverty measure includes a method to identify the poor and an aggregative measure. The identification step requires the specification of a poverty line that splits society into two groups: the poor and the non-poor people. With respect to the aggregation scheme, according to Sen [15], any poverty index should be sensitive to the number of people below the poverty line, to the extent of the income short-fall of the poor from the poverty line, and to the exact pattern of the income distribution of the poor. In other words, every poverty measure should be expressed as a function of these three poverty indicators, showing the incidence, the intensity and the inequality of the poverty, respectively. Poverty changes can be more meaningful and easily understandable if poverty indices can be decomposed into these underlying contributing factors.

In this paper we introduce a new class of poverty measures based on the one-parameter family of exponential means, which are used in order to aggregate the normalized income gaps of the poor individuals. The exponential means have already been used to define inequality and welfare measures. However, to our knowledge, no relative poverty indices have been proposed involving the exponential means.

The class of poverty measures introduced in this paper has three interesting features. Firstly, the poverty measures proposed are invariant under changes in the unit in which income is measured. Secondly, given a poverty line and a set of poor individuals, when all poor individuals receive the same amount of extra income the reduction in the overall poverty level depends only on that extra amount. Finally, all the members of the new family of poverty measures are more sensitive to changes at the lower end of the income profile, that is, to what happens to the poorest of the poor. In addition, the family parameter can be interpreted as a measure of this sensitivity. None of the indices introduced in the literature are able to capture these three issues together.

In order to obtain the decomposition of these new poverty indices into the three underlying contribution factors, we use the dual decomposition of an aggregation function into a self-dual core and an anti-self-dual remainder as applied to the family of exponential means (see García-Lapresta and Marques Pereira [9]). We show that these two terms can be interpreted as measures of the intensity and of the inequality among the poor, respectively. This decomposition allows us to know if increasing poverty is due to more people becoming poor, or increasing deprivation of the poor, or because income short-fall below the poverty line has become more unequal, or some combination of the above. Therefore, the decomposition proposed can be useful to have a better understanding of the measurement of poverty, particularly in a complex scenario.

The paper is organized as follows. Section 2 introduces basic notation and properties of aggregation functions. Section 3 is devoted to summarize the dual decomposition of an aggregation

function into a self-dual core and an associated anti-self-dual remainder. In Section 4 we review the class of exponential means and discuss their dual decomposition. In Section 5 we introduce our proposal for measuring poverty using exponential means and propose a decomposition of the indices into incidence, intensity, and inequality. An illustrative example is described. Finally, Section 6 contains some concluding remarks.

## 2 Aggregation functions

In this section we present notation and basic definitions regarding aggregation functions on  $[0, 1]^n$ , with  $n \in \mathbb{N}$  and  $n \geq 2$  throughout the text.

**Notation.** Points in  $[0, 1]^n$  will be denoted by means of boldface characters:  $\mathbf{x} = (x_1, \dots, x_n)$ ,  $\mathbf{1} = (1, \dots, 1)$ ,  $\mathbf{0} = (0, \dots, 0)$ . For  $x \in [0, 1]$ , we have  $x \cdot \mathbf{1} = (x, \dots, x)$ . Given  $\mathbf{x}, \mathbf{y} \in [0, 1]^n$ , by  $\mathbf{x} \geq \mathbf{y}$  we mean  $x_i \geq y_i$  for every  $i \in \{1, \dots, n\}$ ; by  $\mathbf{x} > \mathbf{y}$  we mean  $\mathbf{x} \geq \mathbf{y}$  and  $\mathbf{x} \neq \mathbf{y}$ . Given  $\mathbf{x} \in [0, 1]^n$ , with  $(x_{(1)}, \dots, x_{(n)})$  we denote the increasing ordered version of  $\mathbf{x}$ , i.e.,  $x_{(i)}$  is the  $i$ -th lowest number of  $\{x_1, \dots, x_n\}$ . Moreover,  $x_* = x_{(1)} = \min\{x_1, \dots, x_n\}$  and  $x^* = x_{(n)} = \max\{x_1, \dots, x_n\}$ . Given a permutation on  $\{1, \dots, n\}$ , i.e., a bijection  $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ , with  $\mathbf{x}_\sigma$  we denote  $(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ .

We begin by defining standard properties of real functions on  $[0, 1]^n$ . On this, see Fodor and Roubens [8], Calvo *et al.* [2], Beliakov *et al.* [1], García-Lapresta and Marques Pereira [9] and Grabisch *et al.* [10].

**Definition 1** Let  $A : [0, 1]^n \rightarrow \mathbb{R}$  be a function.

1.  $A$  is idempotent if for every  $x \in [0, 1]$ :

$$A(x \cdot \mathbf{1}) = x.$$

2.  $A$  is symmetric if for every permutation  $\sigma$  on  $\{1, \dots, n\}$  and every  $\mathbf{x} \in [0, 1]^n$ :

$$A(\mathbf{x}_\sigma) = A(\mathbf{x}).$$

3.  $A$  is monotonic if for all  $\mathbf{x}, \mathbf{y} \in [0, 1]^n$ :

$$\mathbf{x} \geq \mathbf{y} \Rightarrow A(\mathbf{x}) \geq A(\mathbf{y}).$$

4.  $A$  is strictly monotonic if for all  $\mathbf{x}, \mathbf{y} \in [0, 1]^n$ :

$$\mathbf{x} > \mathbf{y} \Rightarrow A(\mathbf{x}) > A(\mathbf{y}).$$

5.  $A$  is compensative if for every  $\mathbf{x} \in [0, 1]^n$ :

$$x_* \leq A(\mathbf{x}) \leq x^*.$$

6.  $A$  is self-dual if for every  $\mathbf{x} \in [0, 1]^n$ :

$$A(\mathbf{1} - \mathbf{x}) = 1 - A(\mathbf{x}).$$

7.  $A$  is anti-self-dual if for every  $\mathbf{x} \in [0, 1]^n$ :

$$A(\mathbf{1} - \mathbf{x}) = A(\mathbf{x}).$$

8.  $A$  is invariant for translations if for all  $t \in [-1, 1]$  and  $\mathbf{x} \in [0, 1]^n$ :

$$A(\mathbf{x} + t \cdot \mathbf{1}) = A(\mathbf{x})$$

whenever  $\mathbf{x} + t \cdot \mathbf{1} \in [0, 1]^n$ .

9.  $A$  is stable for translations if for all  $t \in [-1, 1]$  and  $\mathbf{x} \in [0, 1]^n$ :

$$A(\mathbf{x} + t \cdot \mathbf{1}) = A(\mathbf{x}) + t$$

whenever  $\mathbf{x} + t \cdot \mathbf{1} \in [0, 1]^n$ .

**Definition 2** Consider the binary relation  $\succsim$  on  $[0, \infty)^n$  defined as

$$\mathbf{x} \succsim \mathbf{y} \Leftrightarrow \sum_{i=1}^n x_i = \sum_{i=1}^n y_i \text{ and } \sum_{i=1}^k x_{(i)} \leq \sum_{i=1}^k y_{(i)} \text{ for every } k \in \{1, \dots, n-1\}.$$

1.  $A$  is  $S$ -convex if for all  $\mathbf{x}, \mathbf{y} \in [0, 1]^n$ :

$$\mathbf{x} \succsim \mathbf{y} \Rightarrow A(\mathbf{x}) \geq A(\mathbf{y}).$$

2.  $A$  is  $S$ -concave if for all  $\mathbf{x}, \mathbf{y} \in [0, 1]^n$ :

$$\mathbf{x} \succsim \mathbf{y} \Rightarrow A(\mathbf{x}) \leq A(\mathbf{y}).$$

**Definition 3** Given  $\mathbf{x}, \mathbf{y} \in [0, \infty)^n$ , we say that  $\mathbf{y}$  is obtained from  $\mathbf{x}$  by a progressive transfer if there exist two individuals  $i, j \in \{1, \dots, n\}$  and  $h > 0$  such that  $x_i < x_j$ ,  $y_i = x_i + h \leq x_j - h = y_j$  and  $y_k = x_k$  for every  $k \in \{1, \dots, n\} \setminus \{i, j\}$ .

A classical result (see Marshall and Olkin [13, Ch. 4, Prop. A.1]) establishes that  $\mathbf{x} \succsim \mathbf{y}$  if and only if  $\mathbf{y}$  can be derived from  $\mathbf{x}$  by means of a finite sequence of permutations and/or progressive transfers.

**Definition 4** Let  $\{A^{(k)}\}_{k \in \mathbb{N}}$  be a sequence of functions, where  $A^{(k)} : [0, 1]^k \rightarrow \mathbb{R}$  and  $A^{(1)}(x) = x$  for every  $x \in [0, 1]$ .

1.  $\{A^{(k)}\}_{k \in \mathbb{N}}$  is invariant for replications if for all  $\mathbf{x} \in [0, 1]^n$  and any number of replications  $m \in \mathbb{N}$  of  $\mathbf{x}$ :

$$A^{(mn)}(\overbrace{\mathbf{x}, \dots, \mathbf{x}}^m) = A^{(n)}(\mathbf{x}).$$

2.  $\{A^{(k)}\}_{k \in \mathbb{N}}$  is decomposable (or separable) if, for any given subset of variables, every variable in the subset can have its value replaced by the partial aggregated value of the subset without altering the overall aggregated value of the full set of variables, for instance if

$$\begin{aligned} A^{(n)}(x_1, \dots, x_m, x_{m+1}, \dots, x_n) &= \\ &= A^{(n)}\left(A^{(m)}(x_1, \dots, x_m), \dots, A^{(m)}(x_1, \dots, x_m), x_{m+1}, \dots, x_n\right), \end{aligned}$$

for all  $\mathbf{x} \in [0, 1]^n$  and  $m \leq n$ .

**Definition 5** A function  $A : [0, 1]^n \rightarrow [0, 1]$  is called an  $n$ -ary aggregation function if it is monotonic and satisfies  $A(\mathbf{0}) = 0$  and  $A(\mathbf{1}) = 1$ . For the sake of simplicity, the  $n$ -arity is omitted whenever it is clear from the context. An aggregation function is said to be strict if it is strictly monotonic.

Self-duality and stability for translations are important properties of aggregation functions. In turn, anti-self-duality and invariance for translations are incompatible with the defining properties of aggregation functions, namely with the boundary conditions  $A(\mathbf{0}) = 0$  and  $A(\mathbf{1}) = 1$ . Nevertheless, anti-self-duality and invariance for translations play an important role in this paper as far as they are properties of important functions associated with aggregation functions, such as we shall discuss later. The following are standard facts concerning aggregation functions (see García-Lapresta and Marques Pereira [9]).

**Proposition 1** Let  $A : [0, 1]^n \rightarrow [0, 1]$  be an aggregation function.

1.  $A$  is idempotent if and only if  $A$  is compensative.
2. If  $A$  is strict, then  $A(\mathbf{x}) = 0$  if and only if  $\mathbf{x} = \mathbf{0}$  and, analogously,  $A(\mathbf{x}) = 1$  if and only if  $\mathbf{x} = \mathbf{1}$ .
3. If  $A$  is stable for translations, then  $A$  is idempotent.

### 3 Dual decomposition

In this section we briefly recall the so-called *dual decomposition* of an aggregation function into its self-dual core and associated anti-self-dual remainder, due to García-Lapresta and Marques Pereira [9]. First we introduce the concepts of self-dual core and anti-self-dual remainder of an aggregation function, establishing which properties are inherited in each case from the original aggregation function. Particular emphasis is given to the properties of stability for translations (self-dual core) and invariance for translations (anti-self-dual remainder).

**Definition 6** *Let  $A : [0, 1]^n \rightarrow [0, 1]$  be an aggregation function. The aggregation function  $A^* : [0, 1]^n \rightarrow [0, 1]$  defined as*

$$A^*(\mathbf{x}) = 1 - A(\mathbf{1} - \mathbf{x})$$

*is known as the dual of the aggregation function  $A$ .*

Clearly,  $(A^*)^* = A$ , which means that dualization is an *involution*. An aggregation function  $A$  is self-dual if and only if  $A^* = A$ .

**Proposition 2** *Let  $A : [0, 1]^n \rightarrow [0, 1]$  be an aggregation function. The dual  $A^*$  inherits from the aggregation function  $A$  the properties of continuity, idempotency (hence, compensativeness), symmetry, strict monotonicity, self-duality, stability for translations, invariance for replications, and decomposability, whenever  $A$  has these properties. In addition,  $A^*$  is  $S$ -convex (resp.  $S$ -concave) whenever  $A$  is  $S$ -concave (resp.  $S$ -convex).*

**Proof.** We only prove the last property: if  $A$  is  $S$ -concave, then  $A^*$  is  $S$ -convex, and viceversa (the proofs of the previous properties are straightforward). It is easy to check that  $\mathbf{x} \succcurlyeq \mathbf{y} \Rightarrow \mathbf{1} - \mathbf{x} \succcurlyeq \mathbf{1} - \mathbf{y}$ . If  $A$  is  $S$ -concave, then  $\mathbf{x} \succcurlyeq \mathbf{y}$  implies  $A(\mathbf{x}) \leq A(\mathbf{y})$ , and thus also  $A(\mathbf{1} - \mathbf{x}) \leq A(\mathbf{1} - \mathbf{y})$ . Consequently,  $A^*(\mathbf{x}) = 1 - A(\mathbf{1} - \mathbf{x}) \geq 1 - A(\mathbf{1} - \mathbf{y}) = A^*(\mathbf{y})$ , i.e.,  $A^*$  is  $S$ -convex. ■

#### 3.1 The self-dual core

Aggregation functions are not in general self-dual. However, a self-dual aggregation function can be associated to any aggregation function in a simple manner. The construction of the so-called *self-dual core* of an aggregation function  $A$  is as follows.

**Definition 7** *Let  $A : [0, 1]^n \rightarrow [0, 1]$  be an aggregation function. The function  $\widehat{A} : [0, 1]^n \rightarrow [0, 1]$  defined as*

$$\widehat{A}(\mathbf{x}) = \frac{A(\mathbf{x}) + A^*(\mathbf{x})}{2} = \frac{A(\mathbf{x}) - A(\mathbf{1} - \mathbf{x}) + 1}{2}$$

is called the core of the aggregation function  $A$ .

Since  $\widehat{A}$  is self-dual, we say that  $\widehat{A}$  is the *self-dual core* of the aggregation function  $A$ . Notice that  $\widehat{A}$  is clearly an aggregation function, verifying the boundary conditions  $\widehat{A}(\mathbf{0}) = 0$ ,  $\widehat{A}(\mathbf{1}) = 1$  and monotonicity.

It is interesting to note that the self-dual core reduces to the arithmetic mean in the simple case of  $n = 2$ , but not in higher dimensions.

The following results<sup>1</sup> can be found in García-Lapresta and Marques Pereira [9].

**Proposition 3** *An aggregation function  $A : [0, 1]^n \rightarrow [0, 1]$  is self-dual if and only if  $\widehat{A}(\mathbf{x}) = A(\mathbf{x})$  for every  $\mathbf{x} \in [0, 1]^n$ .*

**Proposition 4** *The self-dual core  $\widehat{A}$  inherits from the aggregation function  $A$  the properties of continuity, idempotency (hence, compensativeness), symmetry, strict monotonicity, stability for translations, and invariance for replications, whenever  $A$  has these properties.*

### 3.2 The anti-self-dual remainder

We now introduce the *anti-self-dual remainder*  $\widetilde{A}$ , which is simply the difference between the original aggregation function  $A$  and its self-dual core  $\widehat{A}$ .

**Definition 8** *Let  $A : [0, 1]^n \rightarrow [0, 1]$  be an aggregation function. The function  $\widetilde{A} : [0, 1]^n \rightarrow \mathbb{R}$  defined as  $\widetilde{A}(\mathbf{x}) = A(\mathbf{x}) - \widehat{A}(\mathbf{x})$ , that is*

$$\widetilde{A}(\mathbf{x}) = \frac{A(\mathbf{x}) - A^*(\mathbf{x})}{2} = \frac{A(\mathbf{x}) + A(\mathbf{1} - \mathbf{x}) - 1}{2},$$

is called the remainder of the aggregation function  $A$ .

Since  $\widetilde{A}$  is anti-self-dual, we say that  $\widetilde{A}$  is the *anti-self-dual remainder* of the aggregation function  $A$ . Clearly,  $\widetilde{A}$  is not an aggregation function. In particular,  $\widetilde{A}(\mathbf{0}) = \widetilde{A}(\mathbf{1}) = 0$  violates the boundary conditions and implies that  $\widetilde{A}$  is either non monotonic or everywhere null. Moreover,  $-0.5 \leq \widetilde{A}(\mathbf{x}) \leq 0.5$  for every  $\mathbf{x} \in [0, 1]^n$ .

The following results<sup>2</sup> can be found in in García-Lapresta and Marques Pereira [9].

<sup>1</sup>Excepting that invariance for replications is inherited by the core (the proof is immediate).

<sup>2</sup>Excepting that invariance for replications is inherited by the remainder (the proof is immediate) and that  $S$ -convexity and  $S$ -concavity are also inherited by the remainder (the proof is similar to that of Proposition 2).

**Proposition 5** *An aggregation function  $A : [0, 1]^n \rightarrow [0, 1]$  is self-dual if and only if  $\tilde{A}(\mathbf{x}) = 0$  for every  $\mathbf{x} \in [0, 1]^n$ .*

**Proposition 6** *The anti-self-dual remainder  $\tilde{A}$  inherits from the aggregation function  $A$  the properties of continuity, symmetry, invariance for replications, plus also  $S$ -convexity and  $S$ -concavity, whenever  $A$  has these properties.*

Summarizing, every aggregation function  $A$  decomposes additively  $A = \hat{A} + \tilde{A}$  in two components: the self-dual core  $\hat{A}$  and the anti-self-dual remainder  $\tilde{A}$ , where only  $\hat{A}$  is an aggregation function. The so-called *dual decomposition*  $A = \hat{A} + \tilde{A}$  clearly shows some analogy with other algebraic decompositions, such as that of square matrices and bilinear tensors into their symmetric and skew-symmetric components.

The following result concerns two more properties of the anti-self-dual remainder based directly on the definition  $\tilde{A} = A - \hat{A}$  and the corresponding properties of the self-dual core (see García-Lapresta and Marques Pereira [9]).

**Proposition 7** *Let  $A : [0, 1]^n \rightarrow [0, 1]$  be an aggregation function.*

1. *If  $A$  is idempotent, then  $\tilde{A}(x \cdot \mathbf{1}) = 0$  for every  $x \in [0, 1]$ .*
2. *If  $A$  is stable for translations, then  $\tilde{A}$  is invariant for translations.*

These properties of the anti-self-dual remainder are suggestive. The first statement applies to the class of idempotent aggregation functions. In such case, self-dual cores are idempotent and therefore anti-self-dual remainders are null on the main diagonal. The second statement applies to the subclass of stable aggregation functions. In such case, self-dual cores are stable and therefore anti-self-dual remainders are invariant for translations. In other words, if the aggregation function  $A$  is stable for translations, the value  $\tilde{A}(\mathbf{x})$  does not depend on the average value of the  $\mathbf{x}$  coordinates, but only on their numerical deviations from that average value. These properties of the anti-self-dual remainder  $\tilde{A}$  suggest that it may give some indication on the dispersion of the  $\mathbf{x}$  coordinates.

In Maes *et al.* [12], the authors propose a generalization of the dual decomposition framework introduced in García-Lapresta and Marques Pereira [9], based on a family of binary aggregation operators satisfying a form of twisted self-duality condition. Each binary aggregation operator in that family corresponds to a particular way of combining an aggregation operator  $A$  with its dual  $A^*$  for the construction of the self-dual core  $\hat{A}$ . As particular cases of the general framework proposed in Maes *et al.* [12], one obtains García-Lapresta and Marques Pereira's

construction, based on the arithmetic mean, and Silvert's construction, based on the symmetric sums formula (see Silvert [17]). However, the dual decomposition framework introduced in García-Lapresta and Marques Pereira [9] remains the only one which preserves stability under translations, a crucial requirement in the present construction of poverty measures based on exponential means.

## 4 Exponential means

Quasiarithmetic means are aggregation functions that satisfy interesting properties. They were simultaneously characterized in 1930 by Kolmogoroff [11] and Nagumo [14] (see also Fodor and Roubens [8, pp. 112-114]) through five properties: continuity, idempotency, symmetry, strict monotonicity, and decomposability. Exponential means are quasiarithmetic means that are stable for translations. Thus, their self-dual cores are also stable for translations, and their anti-self-dual remainders are invariant for translations, together with other interesting properties. This is the reason why we consider exponential means and their dual decomposition in our approach to the construction of appropriate poverty measures.

**Definition 9** *Let  $A : [0, 1]^n \rightarrow [0, 1]$  be an aggregation function. We say that  $A$  is a quasiarithmetic mean if there exists an order automorphism (bijective and increasing function)  $\varphi : [0, 1] \rightarrow [0, 1]$  such that*

$$A(\mathbf{x}) = \varphi^{-1} \left( \frac{\varphi(x_1) + \dots + \varphi(x_n)}{n} \right),$$

where  $\varphi$  is said to generate the quasiarithmetic mean  $A$ .

The exponential means are the quasiarithmetic means generated by the order automorphisms  $\varphi_\alpha$ , with  $\alpha \neq 0$ ,

$$\varphi_\alpha(x) = \frac{e^{\alpha x} - 1}{e^\alpha - 1}.$$

The limit case,  $\varphi_0(x) = x$ , generates the arithmetic mean. The exponential means, together with the arithmetic mean, are the only quasiarithmetic means which are stable for translations.

**Definition 10** *Given  $\alpha \neq 0$ , the exponential mean  $A_\alpha$  is the aggregation function defined by*

$$A_\alpha(\mathbf{x}) = \frac{1}{\alpha} \ln \frac{e^{\alpha x_1} + \dots + e^{\alpha x_n}}{n}.$$

**Proposition 8** *For every  $\alpha \neq 0$ , the exponential mean  $A_\alpha$  is continuous, idempotent (hence compensative), symmetric, strictly monotonic, stable for translations, invariant for replications,*

and decomposable. In addition,  $A_\alpha$  is  $S$ -convex for every  $\alpha > 0$ , and  $S$ -concave for every  $\alpha < 0$ .

**Proof.** Continuity, idempotency, symmetry, strict monotonicity, decomposability, and stability for translations are well-known properties of exponential means, as mentioned above. Invariance for replications is straightforward. Let  $\partial_i A_\alpha$  denote the partial derivative of  $A_\alpha$  with respect to  $x_i$ , with  $i \in \{1, \dots, n\}$ . It is routinary to check that the exponential mean  $A_\alpha$  satisfies *Schur's condition* for continuously differentiable functions:  $(x_i - x_j)(\partial_i A_\alpha(\mathbf{x}) - \partial_j A_\alpha(\mathbf{x})) \geq 0$  for every  $i, j \in \{1, \dots, n\}$  when  $\alpha > 0$ , and the reverse inequality when  $\alpha < 0$ . Accordingly,  $A_\alpha$  is  $S$ -convex for  $\alpha > 0$ , and  $S$ -concave for  $\alpha < 0$  (see Marshall and Olkin [13, Ch. 3, Prop. A.4]).

■

**Proposition 9** For every  $\alpha \neq 0$ , the dual of the exponential mean  $A_\alpha$  is the aggregation function  $A_\alpha^*$  given by

$$A_\alpha^*(\mathbf{x}) = -\frac{1}{\alpha} \ln \frac{e^{-\alpha x_1} + \dots + e^{-\alpha x_n}}{n}.$$

Notice that  $A_\alpha^* = A_{-\alpha}$ , which means that taking the dual corresponds to inverting the parameter sign.

**Proposition 10** For every  $\alpha \neq 0$ , the dual  $A_\alpha^*$  is continuous, idempotent (hence compensative), symmetric, strictly monotonic, stable for translations, invariant for replications, and decomposable. In addition,  $A_\alpha^*$  is  $S$ -concave for every  $\alpha > 0$ , and  $S$ -convex for every  $\alpha < 0$ .

**Proof.** Since  $A_\alpha^* = A_{-\alpha}$  as in Proposition 9, it follows from Proposition 8. ■

**Proposition 11** For every  $\alpha \neq 0$ , the self-dual core of the exponential mean  $A_\alpha$  is the aggregation function  $\widehat{A}_\alpha = (A_\alpha + A_\alpha^*)/2$  given by

$$\widehat{A}_\alpha(\mathbf{x}) = \frac{1}{2\alpha} \ln \frac{e^{\alpha x_1} + \dots + e^{\alpha x_n}}{e^{-\alpha x_1} + \dots + e^{-\alpha x_n}}.$$

**Proposition 12** For every  $\alpha \neq 0$ , the self-dual core  $\widehat{A}_\alpha$  is continuous, idempotent (hence compensative), symmetric, strictly monotonic, stable for translations, and invariant for replications.

**Proof.** It is a direct consequence of Propositions 4 and 8. ■

It is important to note that  $\widehat{A}_\alpha$  is not decomposable (see García-Lapresta and Marques Pereira [9, Remark 37]). This implies that the self-dual core of an exponential mean is not itself an exponential mean.

**Proposition 13** *For every  $\alpha \neq 0$ , the anti-self-dual remainder of the exponential mean  $A_\alpha$  is the function  $\widetilde{A}_\alpha = (A_\alpha - A_\alpha^*)/2$  given by*

$$\widetilde{A}_\alpha(\mathbf{x}) = \frac{1}{2\alpha} \ln \frac{(e^{\alpha x_1} + \dots + e^{\alpha x_n})(e^{-\alpha x_1} + \dots + e^{-\alpha x_n})}{n^2}.$$

**Proposition 14** *For every  $\alpha \neq 0$ , the anti-self-dual remainder  $\widetilde{A}_\alpha$  is continuous, symmetric, anti-self-dual, invariant for translations, and invariant for replications. In addition,  $\widetilde{A}_\alpha$  is  $S$ -convex for every  $\alpha > 0$ , and  $S$ -concave for every  $\alpha < 0$ .*

**Proof.** It is a direct consequence of Propositions 6, 7 and 8. ■

**Proposition 15** *For every  $\alpha \neq 0$ , we have  $\widetilde{A}_\alpha(\mathbf{x}) = 0$  if and only if  $x_1 = \dots = x_n$ . Moreover, we have  $0 \leq \widetilde{A}_\alpha(\mathbf{x}) \leq b_\alpha$  for  $\alpha > 0$ , and  $b_\alpha \leq \widetilde{A}_\alpha(\mathbf{x}) \leq 0$  for  $\alpha < 0$ , for every  $\mathbf{x} \in [0, 1]^n$ . For every  $\alpha \neq 0$ , the upper (resp. lower) bound  $b_\alpha$  is given by*

$$b_\alpha = \frac{1}{\alpha} \ln \frac{1 + e^\alpha}{2} - \frac{1}{2} \quad \text{for } \alpha > 0, \text{ and } \quad b_\alpha = \frac{1}{\alpha} \ln \frac{1 + e^{-\alpha}}{2} + \frac{1}{2} \quad \text{for } \alpha < 0$$

where  $0 < b_\alpha < 0.5$  for  $\alpha > 0$ , and  $-0.5 < b_\alpha < 0$  for  $\alpha < 0$ , respectively. Finally, we have the following limits:  $\lim_{\alpha \rightarrow \pm\infty} b_\alpha = \pm 0.5$  and  $\lim_{\alpha \rightarrow 0} b_\alpha = 0$ .

**Proof.** These results derive from the standard properties of the continuously differentiable function  $f(\mathbf{y}) = (y_1 + \dots + y_n)(y_1^{-1} + \dots + y_n^{-1})/n^2$  on the domain  $y_i > 0$ , for  $i \in \{1, \dots, n\}$ . The critical points are those for which  $y_1 = \dots = y_n$  and there the function takes its minimum value 1. A small displacement  $t\mathbf{v} = t(v_1, \dots, v_n)$  with respect to a critical point  $\mathbf{y} = (y, \dots, y)$ , for any given  $\mathbf{v} \in \mathbb{R}^n$ , produces a non negative increment in the function value whose dominant term (second order in  $t$ ) is proportional to the variance of the displacement vector  $\mathbf{v}$ , and inversely proportional to  $y^2$ . Finally, on the restricted domain  $0 < m \leq y_i \leq M$ , for every  $i \in \{1, \dots, n\}$ , the following inequality holds:  $1 \leq f(\mathbf{y}) \leq (m + M)^2/4mM$  (see Marshall and Olkin [13, Ch. 3, Example D.3.a]). The corresponding results in the exponential means case are obtained considering the transformation  $y = e^{\alpha x}$  with  $\alpha \neq 0$  and  $x \in [0, 1]$ , that is  $m = 1$  and  $M = e^\alpha$ . ■

The following result presents the parameter limits of the exponential mean and those of the associated dual, self-dual core, and anti-self-dual remainder. The proof is by straightforward application of l'Hospital's rule (see García-Lapresta and Marques Pereira [9]).

**Proposition 16** *The following statements hold:*

1.  $\lim_{\alpha \rightarrow \infty} A_\alpha(\mathbf{x}) = x^*$ .
2.  $\lim_{\alpha \rightarrow -\infty} A_\alpha(\mathbf{x}) = x_*$ .
3.  $\lim_{\alpha \rightarrow 0} A_\alpha(\mathbf{x}) = \frac{x_1 + \cdots + x_n}{n}$ .
4.  $\lim_{\alpha \rightarrow \infty} A_\alpha^*(\mathbf{x}) = x_*$ .
5.  $\lim_{\alpha \rightarrow -\infty} A_\alpha^*(\mathbf{x}) = x^*$ .
6.  $\lim_{\alpha \rightarrow 0} A_\alpha^*(\mathbf{x}) = \frac{x_1 + \cdots + x_n}{n}$ .
7.  $\lim_{\alpha \rightarrow \infty} \widehat{A}_\alpha(\mathbf{x}) = \lim_{\alpha \rightarrow -\infty} \widehat{A}_\alpha(\mathbf{x}) = \frac{x_* + x^*}{2}$ .
8.  $\lim_{\alpha \rightarrow 0} \widehat{A}_\alpha(\mathbf{x}) = \frac{x_1 + \cdots + x_n}{n}$ .
9.  $\lim_{\alpha \rightarrow \infty} \widetilde{A}_\alpha(\mathbf{x}) = \frac{x^* - x_*}{2}$ .
10.  $\lim_{\alpha \rightarrow -\infty} \widetilde{A}_\alpha(\mathbf{x}) = -\frac{x^* - x_*}{2}$ .
11.  $\lim_{\alpha \rightarrow 0} \widetilde{A}_\alpha(\mathbf{x}) = 0$ .

## 5 Poverty measures

We consider a population consisting of  $n$  individuals, with  $n \geq 2$ . An *income distribution* is represented by a vector  $\mathbf{x} = (x_1, \dots, x_n) \in [0, \infty)^n$ , where  $x_i$  is the income of individual  $i$ .

Since Sen [15], any poverty measure consists of a method to identify the poor together with an aggregative measure. Thus, the first step to define a poverty measure is the identification of the poor people in society. This step requires the specification of a *poverty line*  $z \in (0, \infty)$  which represents the necessary income to maintain a minimum level of living. For an income

distribution  $\mathbf{x}$ , person  $i$  is considered to be poor if  $x_i < z$ . Otherwise the person is non-poor or rich.

We denote the set of poor people by

$$Q(\mathbf{x}, z) = \{i \in \{1, \dots, n\} \mid x_i < z\},$$

and by  $q(\mathbf{x}, z)$  the number of the poor, i.e.,  $q(\mathbf{x}, z) = \#Q(\mathbf{x}, z)$ .

Once the poor people have been identified, the second step to determine the extent of poverty involves the aggregation scheme. In what follows, a poverty measure is a non-constant function  $P(\mathbf{x}, z)$  of the income distribution  $\mathbf{x}$  and the poverty line  $z$ .

## 5.1 Axioms

A number of axioms are usually assumed for a poverty measure.

- *Poverty Focus (PF)*: For all  $\mathbf{x}, \mathbf{y} \in [0, \infty)^n$  and  $z \in (0, \infty)$ , if  $Q(\mathbf{x}, z) = Q(\mathbf{y}, z) = Q$  and  $x_i = y_i$  for every  $i \in Q$ , then  $P(\mathbf{x}, z) = P(\mathbf{y}, z)$ .
- *Poverty Monotonicity (PM)*: For all  $\mathbf{x}, \mathbf{y} \in [0, \infty)^n$  and  $z \in (0, \infty)$ , if  $Q(\mathbf{x}, z) = Q(\mathbf{y}, z) = Q$  and  $\mathbf{x} = \mathbf{y}$  except for  $x_i > y_i$  with  $i \in Q$ , then  $P(\mathbf{x}, z) < P(\mathbf{y}, z)$ .

Since poverty measurement is concerned with the deprivations of poor people, these two properties, postulated by Sen [15], are considered as the basic axioms for a poverty measure. Thus, axiom **PF** requires that a poverty index should not depend on the income of the non-poor people, i.e., the poverty level should not vary if the rich incomes change, as long as the set of poor people remains unchanged. On the other hand, axiom **PM** demands that poverty should increase if the income of a poor person decreases.

The following axiom is concerned with inequality among the poor. In the inequality field, the Pigou-Dalton transfer principle establishes that a progressive transfer, that is a transfer from a richer person to a poorer one, should decrease inequality. Accordingly, a progressive transfer among the poor should decrease inequality among the poor. Sen [15] introduces the counterpart of this principle in the poverty field, requiring poverty also to decrease. This is captured by the Transfer Sensitive axiom below.

- *Transfer Sensitivity (TS)*: For all  $\mathbf{x}, \mathbf{y} \in [0, \infty)^n$  and  $z \in (0, \infty)$ , if  $\mathbf{y}$  is obtained from  $\mathbf{x}$  by a progressive transfer among the poor, then  $P(\mathbf{y}, z) < P(\mathbf{x}, z)$ .

A progressive transfer among the poor entails an increment of income for one poor individual, and a decrement for another poor person, the richer of the two. This **TS** axiom goes beyond

**PM** and demands that greater weight should be placed on the poorer person and that poverty should decrease if inequality among the poor decreases.

A normalization condition is also usually assumed. This property requires that if all the individuals are non-poor, then the society deprivation level is equal to 0.

- *Normalization (N)*: For all  $\mathbf{x}, \mathbf{y} \in [0, \infty)^n$  and  $z \in (0, \infty)$ ,  $P(\mathbf{x}, z) = 0$  if and only if  $Q(\mathbf{x}, z) = \emptyset$ , that is  $x_i \geq z$  for every  $i \in \{1, \dots, n\}$ .

The two following invariance axioms are also standard requirements for a poverty measure:

- *Poverty Symmetry (PS)*: For all  $\mathbf{x} \in [0, \infty)^n$ ,  $z \in (0, \infty)$ , and permutation  $\sigma$  on  $\{1, \dots, n\}$ , it holds that  $P(\mathbf{x}_\sigma, z) = P(\mathbf{x}, z)$ .
- *Replication Invariance (RI)*: For all  $\mathbf{x} \in [0, \infty)^n$  and  $z \in (0, \infty)$ , if  $\mathbf{y}$  is obtained from  $\mathbf{x}$  by a replication, that is  $\mathbf{y} = (\overbrace{\mathbf{x}, \dots, \mathbf{x}}^m)$  for some  $m \in \mathbb{N}$ , then  $P(\mathbf{y}, z) = P(\mathbf{x}, z)$ .

**PS** establishes that no other characteristic apart from the income deprivation matters in defining a poverty index. In turn, **RI** allows us to compare populations of different sizes.

The first poverty measure introduced in the literature has been the *headcount ratio*  $H : [0, \infty)^n \times (0, \infty) \rightarrow [0, 1]$  defined as

$$H(\mathbf{x}, z) = \frac{q(\mathbf{x}, z)}{n},$$

which measures the percentage of poor people in the society.

This is a crude index, which is able to capture the incidence of poverty. However, it is able to capture neither the intensity nor the inequality among the poor. In fact it violates both **PM** and **TS**, since it does not change if the income of a poor decreases, and under progressive transfers among the poor.

In most cases, measuring poverty involves gauging the extent of the deprivation felt by each individual, once the income poverty line has been determined. One of the most used procedures to measure individual  $i$ 's shortfall is to consider the normalized gap of individual  $i$ .

**Definition 11** For all  $\mathbf{x} \in [0, \infty)^n$  and  $z \in (0, \infty)$ , the normalized gap of individual  $i$  is defined as

$$g_i = \max \left\{ \frac{z - x_i}{z}, 0 \right\}.$$

Notice that  $g_i \in [0, 1]$ ,  $g_i = 0 \Leftrightarrow x_i \geq z$ , and  $g_i = 1 \Leftrightarrow x_i = 0$ .

Moreover, the normalized gaps are invariant under proportional income changes. In other words, the function  $G : [0, \infty)^n \times (0, \infty) \rightarrow [0, 1]^n$  defined by  $G(\mathbf{x}, z) = (g_1, \dots, g_n)$  is homogeneous of degree 0:  $G(\lambda x_i, \dots, \lambda x_n, \lambda z) = G(x_i, \dots, x_n, z)$  for every  $\lambda > 0$ .

On the other hand, a progressive transfer among the poor people lead to an increment in the richer individual gap whereas the poorer person gap decreases. Since the richer gap is smaller than the poorer one, then the progressive transfers among the poor incomes are equivalent to the progressive transfers among the poor gaps. Then, according to Marshall and Olkin [13, Ch. 4, Prop. A.1], the **TS** axiom is to be fulfilled whenever the function is  $S$ -convex either in incomes or in gaps.

## 5.2 A new poverty measure

We now introduce the following single parameter class of poverty measures, using the headcount ratio and the exponential means of the normalized gaps of the poor.

**Definition 12** Given  $\alpha > 0$ , the poverty measure associated with  $A_\alpha$  is the function  $P_\alpha : [0, \infty)^n \times (0, \infty) \rightarrow [0, 1]^n$  defined by

$$P_\alpha(\mathbf{x}, z) = \begin{cases} H(\mathbf{x}, z) \cdot A_\alpha(g'_1, \dots, g'_q) = \frac{q}{n} \cdot \frac{1}{\alpha} \ln \frac{e^{\alpha g'_1} + \dots + e^{\alpha g'_q}}{q}, & \text{if } q \neq 0, \\ 0, & \text{if } q = 0, \end{cases}$$

where  $q = q(\mathbf{x}, z)$  and  $g'_1, \dots, g'_q$  are the non-null elements of  $g_1, \dots, g_n$ .

**Proposition 17** For every  $\alpha > 0$ ,  $P_\alpha$  satisfies **PF**, **PM**, **TS**, **N**, **PS**, and **RI**.

**Proof.** By construction,  $P_\alpha$  satisfies **PF** and **N**.

**PM:** By hypothesis,  $H(\mathbf{x}, z) = H(\mathbf{y}, z)$ . From  $x_i > y_i$  we have  $\frac{z-x_i}{z} < \frac{z-y_i}{z}$ . Then, by the strict monotonicity of  $A_\alpha$ , we obtain  $P(\mathbf{x}, z) < P(\mathbf{y}, z)$ .

**TS:** Given  $i, j \in Q(\mathbf{x}, z)$  such that  $x_i < x_j$ ,  $y_i = x_i + h \leq x_j - h = y_j$  for some  $h > 0$ , and  $y_k = x_k$  for every  $k \in \{1, \dots, n\} \setminus \{i, j\}$ , we have  $y_j < x_j < z$  and  $Q(\mathbf{x}, z) = Q(\mathbf{y}, z)$ , hence  $H(\mathbf{x}, z) = H(\mathbf{y}, z)$ . By Proposition 8,  $A_\alpha$  is  $S$ -convex and, consequently, satisfies **TS**. Then, we have  $P(\mathbf{y}, z) < P(\mathbf{x}, z)$ .

**PS:** Given a permutation  $\sigma$  on  $\{1, \dots, n\}$ , we have  $Q(\mathbf{x}_\sigma, z) = Q(\mathbf{x}, z)$ , hence  $H(\mathbf{x}_\sigma, z) = H(\mathbf{x}, z)$ . Taking into account that  $A_\alpha$  is symmetric, we obtain **PS**.

**RI:** It is easy to see that if  $\mathbf{y}$  is obtained from  $\mathbf{x}$  by a replication, then  $H(\mathbf{y}, z) = H(\mathbf{x}, z)$ . Taking into account that  $A_\alpha$  satisfies invariance for replications, we obtain  $P(\mathbf{x}, z) = P(\mathbf{y}, z)$ . ■

Proposition 17 above establishes that for every  $\alpha > 0$ ,  $P_\alpha$  satisfies the basic poverty properties. Moreover, since  $P_\alpha$  is defined directly on normalized income gaps, it is homogeneous of degree 0. This is an interesting property that ensures that the poverty level does not change when the units in which income is measured change. All the poverty indices referred to as “relative” satisfy this property. In addition, it is easy to show that if the same amount of extra income, say  $k$ , is given to each poor person as long as the poor set does not change, the poverty level is reduced by  $\frac{q}{n} \frac{k}{z}$ . This property is also quite interesting, since it provides a starting point to analyze poverty policies.

Among the poverty measures proposed in the literature, only the Gini-type indices such as the Sen index, the Sen modified index, the Thon index and so on (see for instance Chakravarty [4]) fulfill these two properties. However, none of the Gini-type poverty measures are more sensitive to progressive transfers at the lower end than at the upper end of the income profile. This property, often invoked in the inequality and poverty framework, ensures that the measure is focused on the poorest of the poor and may be formalized as follows.

- *Diminishing Transfer Sensitivity (DTS):* For all  $\mathbf{x}, \mathbf{y} \in [0, \infty)^n$  and  $z \in (0, \infty)$ , if  $Q(\mathbf{x}, z) = Q(\mathbf{y}, z)$  and  $\mathbf{y}$  is obtained from  $\mathbf{x}$  by a progressive transfer from the poor person with income  $x_i + c$  to the poor person with income  $x_i$ , for some  $c > 0$ , then the magnitude of decrease in poverty  $P(\mathbf{x}, z) - P(\mathbf{y}, z)$  is higher the lower  $x_i$ .

The DTS principle requires that the poverty reduction effect of a “poor to poorer” progressive transfer should decrease as the income of the poorer person increases.

**Proposition 18** *In the parameter range  $\alpha > 0$ , the poverty measure  $P_\alpha$  satisfies the DTS principle and the poverty reduction effect of a “poor to poorer” progressive transfer increases as the parameter value  $\alpha$  increases.*

**Proof.** It is straightforward taking into account the exponential function is increasing and convex. ■

This proposition enables us to interpret the  $\alpha$  parameter as the degree of sensitivity to the poorest. It may be worth noting that

$$\lim_{\alpha \rightarrow \infty} P_\alpha(\mathbf{x}, z) = \frac{q}{n} g^* = \frac{q}{n} \frac{z - x_*}{z}.$$

**Remark 1** When  $\alpha > 0$ , the core  $\widehat{A}_\alpha$  is idempotent, symmetric, strictly monotonic, and stable for translations (see Proposition 10). The strictly monotonicity axiom implies that  $\widehat{A}_\alpha$  is increasing in the gap of a poor person. The stability for translations means that equal absolute changes in all poor gaps lead to the same absolute change in  $\widehat{A}_\alpha$ . These properties can be regarded as basic properties of a poverty intensity index. Nevertheless,  $\widehat{A}_\alpha$  is not  $S$ -convex, and then it goes against the Pigou-Dalton transfer principle. For this, it suffices to consider two gaps distributions  $(g - c, g, \dots, g, g + c)$  and  $(g, g, \dots, g, g)$ . Obviously the second distribution is obtained from the former by a progressive transfer among the poor. Hence the Pigou-Dalton transfer principle would require  $\widehat{A}_\alpha$  to decrease, however it does not change:  $\widehat{A}_\alpha(g - c, g, \dots, g, g + c) = \widehat{A}_\alpha(g, g, \dots, g, g) = g$ .

In turn, the remainder  $\widetilde{A}_\alpha$  is symmetric, fulfils that  $\widetilde{A}_\alpha(g_1, \dots, g_q) = 0$  if and only if  $g_1 = \dots = g_q$ , and Propositions 6 and 8 ensure that it is  $S$ -convex, and consequently the Pigou-Dalton transfer principle is satisfied. Hence, we can obtain a direct interpretation of  $\widetilde{A}_\alpha$  as a measure of inequality among the poor individuals. In addition  $\widetilde{A}_\alpha$  is invariant for translations (Proposition 14), thus it measures inequality from an absolute point of view and remains invariant if the gaps of all the poor are increased by the same amount. However,  $\widetilde{A}_\alpha$  does not fulfil **PM**.

The next proposition, which involves the self-dual core and the anti self-dual remainder of  $A_\alpha$ , provides a decomposition of the poverty measure  $P_\alpha$  in three components:  $H$ ,  $\widehat{A}_\alpha$  and  $\widetilde{A}_\alpha$ .

**Proposition 19** *For every  $\alpha > 0$ , the poverty measure  $P_\alpha$  associated with  $A_\alpha$  satisfies the following decomposition:*

$$P_\alpha(\mathbf{x}, z) = \begin{cases} H(\mathbf{x}, z) \cdot \left( \widehat{A}_\alpha(g'_1, \dots, g'_q) + \widetilde{A}_\alpha(g'_1, \dots, g'_q) \right), & \text{if } q \neq 0, \\ 0, & \text{if } q = 0, \end{cases}$$

where  $q = q(\mathbf{x}, z)$  and  $g'_1, \dots, g'_q$  are the non-null elements of  $g_1, \dots, g_n$ .

When all the poor individuals have the same gap and, thus, there is not inequality among the poor, the extent of the poverty is given by the product of the headcount ratio and  $\widehat{A}_\alpha$ . If there are progressive transfers of income among the poor, both  $P_\alpha$  and  $\widetilde{A}_\alpha$  are bound to decrease, whereas  $\widehat{A}_\alpha$  can vary in any way. Therefore, societies with the same proportion of poor individuals,  $H$ , and the same location measure,  $\widehat{A}_\alpha$ , may be ordered by considering the dispersion of the gap distribution,  $\widetilde{A}_\alpha$ .

On the other hand, the effect of an increment of the gap in a poor individual implies an increase in  $P_\alpha$  and in  $\widehat{A}_\alpha$ , since both of them satisfy **PM**. In contrast, the change in  $\widetilde{A}_\alpha$

depends on the effect of this change for the distribution dispersion. Consequently, societies with the same proportion of poor individuals,  $H$ , and the same dispersion of the gap distributions  $\tilde{A}_\alpha$  may be ordered by considering the location measure,  $\hat{A}_\alpha$ .

Hence, the three components in the proposition above can be interpreted as follows:

1.  $H$  is a measure of the incidence of poverty.
2. The intensity of poverty as the poverty depths of poor individuals in the society is summarized by the location measure  $\hat{A}_\alpha$ .
3. The inequality of the poor's gap distribution is captured by the dispersion measure,  $\tilde{A}_\alpha$ , which provides the sensitivity to the inequality among the poor.

According to Sen [15], these three indicators can give an adequate image of poverty.

Similarly to Definition 12, it is possible to introduce a family of poverty measures for  $\alpha < 0$  with the same results (recall that  $A_\alpha^* = A_{-\alpha}$ ).

**Definition 13** *Given  $\alpha < 0$ , the poverty measure associated with  $A_\alpha$  is the function  $P_\alpha : [0, \infty)^n \times (0, \infty) \rightarrow [0, 1]^n$  defined by*

$$P_\alpha(\mathbf{x}, z) = \begin{cases} H(\mathbf{x}, z) \cdot A_\alpha^*(g'_1, \dots, g'_q) = \frac{q}{n} \cdot \frac{-1}{\alpha} \ln \frac{e^{-\alpha g'_1} + \dots + e^{-\alpha g'_q}}{q}, & \text{if } q \neq 0, \\ 0, & \text{if } q = 0, \end{cases}$$

where  $q = q(\mathbf{x}, z)$  and  $g'_1, \dots, g'_q$  are the non-null elements of  $g_1, \dots, g_n$ .

**Proposition 20** *For every  $\alpha < 0$ ,  $P_\alpha$  satisfies **PF**, **PM**, **TS**, **N**, **PS**, and **RI**.*

**Proposition 21** *For every  $\alpha < 0$ , the poverty measure  $P_\alpha$  associated with  $A_\alpha$  satisfies the following decomposition:*

$$P_\alpha(\mathbf{x}, z) = \begin{cases} H(\mathbf{x}, z) \cdot \left( \hat{A}_\alpha(g'_1, \dots, g'_q) - \tilde{A}_\alpha(g'_1, \dots, g'_q) \right), & \text{if } q \neq 0, \\ 0, & \text{if } q = 0, \end{cases}$$

where  $q = q(\mathbf{x}, z)$  and  $g'_1, \dots, g'_q$  are the non-null elements of  $g_1, \dots, g_n$ .

**Remark 2** An alternative class of poverty measures can be proposed using the exponentials means of the normalized gap distribution:  $A_\alpha(g_1, \dots, g_n)$ , whenever  $\alpha > 0$ , or  $A_\alpha^*(g_1, \dots, g_n)$ , whenever  $\alpha < 0$ . The indices in this families satisfy **PF**, **PM**, **TS**, **N**, **PS**, and **RI**. They are sensitive to the headcount ratio, the intensity, and the inequality among the poor. However, contrary to our proposal, they do not admit a straightforward decomposition in the three basic components  $H$ ,  $\hat{A}_\alpha$  and  $\tilde{A}_\alpha$ , but only in the two last ones.

### 5.3 An illustrative example

We now illustrate the possibilities of the decomposition of the family proposed in this paper. First, consider eight income distributions displayed in the first column of Table 1. The second column includes their corresponding normalized gaps when the poverty line is equal to 1000. Notice that distributions  $\mathbf{x}^2$ ,  $\mathbf{x}^3$ ,  $\mathbf{x}^5$ ,  $\mathbf{x}^6$ ,  $\mathbf{x}^7$  and  $\mathbf{x}^8$  share the same average income of the poor and the averages of the others two are close to this amount. However their poverty levels are quite different and the decomposition of the poverty index  $P_\alpha$  in its three contribution components,  $H$ ,  $\hat{A}_\alpha$  and  $\tilde{A}_\alpha$ , displayed in the last four columns of the Table 1 for  $\alpha = 1$ , allows us to determine where the differences stem from. For instance, notice that  $\mathbf{x}^1$  and  $\mathbf{x}^2$  have the same inequality and the same intensity, and the difference in their poverty levels arises from the different percentages of poor people. Distributions  $\mathbf{x}^2$  and  $\mathbf{x}^3$  have the same headcount ratio and the poverty intensity level, nevertheless income among the poor are more equally distributed in  $\mathbf{x}^2$  than in  $\mathbf{x}^3$ . By contrast,  $\mathbf{x}^3$  and  $\mathbf{x}^4$  share the headcount ratio and the inequality levels, being different their poverty intensities.

In general, we may compare any pair of distributions and analyze its poverty components to better understand their differences. For example, if we concentrate on distributions  $\mathbf{x}^6$ ,  $\mathbf{x}^7$  and  $\mathbf{x}^8$ , we may conclude that  $\mathbf{x}^6$  exhibits the lowest headcount ratio, while  $\mathbf{x}^7$  and  $\mathbf{x}^8$  have the lowest values of inequality and intensity respectively.

As already mentioned, all the measures in the poverty family  $P_\alpha$  fulfils **DTS**. Distributions  $\mathbf{x}^2$ ,  $\mathbf{x}^3$  and  $\mathbf{x}^5$  help us to better understand the meaning of this principle. Notice that both  $\mathbf{x}^2$  and  $\mathbf{x}^5$  may be obtained from  $\mathbf{x}^3$  by a progressive transfer. The richest of the poor is the donor in the two transfers. The receivers are the poorest person and the second poorest person respectively. The main point is that whereas in the first case the receiver is the poorest person, in the last case is the second poorest. **DTS** guarantees that the decrease in poverty is greater under the first transfer.

Table 2 highlights the role of  $\alpha$ . In the comparison of  $\mathbf{x}^3$  and  $\mathbf{x}^8$ , it can be noted that whereas for  $\alpha = 1$  distribution  $\mathbf{x}^3$  displays less poverty, when  $\alpha = 4$  the situation is just the opposite, since more weight is assigned to the poorest of the population. In other words, the greater the value of  $\alpha$  the more sensitive to the poorest incomes. In addition, with respect to **DTS**, when  $\alpha = 4$  in the same cases mentioned above, the decrease in poverty is even greater.

## 6 Concluding remarks

In literature on economics there exists a well established tradition of aggregating different economic indicators for the purpose of comparing societies or countries. In this issue, we have

Table 1: Indices values for  $\alpha = 1$ .

Incomes	$z = 1000$	$P_\alpha$	$H$	$\widehat{A}_\alpha$	$\widetilde{A}_\alpha$
$\mathbf{x}^1 = (170, 500, 729, 605)$	$g^1 = (0.830, 0.500, 0.271, 0.395)$	0.521	1.000	0.500	0.021
$\mathbf{x}^2 = (250, 500, 750, 1300)$	$g^2 = (0.750, 0.500, 0.250, 0.000)$	0.391	0.750	0.500	0.021
$\mathbf{x}^3 = (150, 500, 850, 1300)$	$g^3 = (0.850, 0.500, 0.150, 0.000)$	0.405	0.750	0.500	0.040
$\mathbf{x}^4 = (330, 350, 940, 2800)$	$g^4 = (0.670, 0.650, 0.060, 0.000)$	0.373	0.750	0.457	0.040
$\mathbf{x}^5 = (150, 600, 750, 1300)$	$g^5 = (0.850, 0.400, 0.250, 0.000)$	0.400	0.750	0.501	0.032
$\mathbf{x}^6 = (217, 783, 1100, 3000)$	$g^6 = (0.783, 0.217, 0.000, 0.000)$	0.270	0.500	0.500	0.040
$\mathbf{x}^7 = (290, 500, 600, 610)$	$g^7 = (0.710, 0.500, 0.400, 0.390)$	0.509	1.000	0.500	0.008
$\mathbf{x}^8 = (250, 300, 950, 1400)$	$g^8 = (0.750, 0.700, 0.050, 0.000)$	0.410	0.750	0.496	0.050

Table 2: Indices values for  $\alpha = 4$ .

Incomes	$z = 1000$	$P_\alpha$	$H$	$\widehat{A}_\alpha$	$\widetilde{A}_\alpha$
$\mathbf{x}^1 = (170, 500, 729, 605)$	$g^1 = (0.830, 0.500, 0.271, 0.395)$	0.593	1.000	0.512	0.081
$\mathbf{x}^2 = (250, 500, 750, 1300)$	$g^2 = (0.750, 0.500, 0.250, 0.000)$	0.433	0.750	0.500	0.077
$\mathbf{x}^3 = (150, 500, 850, 1300)$	$g^3 = (0.850, 0.500, 0.150, 0.000)$	0.482	0.750	0.500	0.142
$\mathbf{x}^4 = (330, 350, 940, 2800)$	$g^4 = (0.670, 0.650, 0.060, 0.000)$	0.427	0.750	0.431	0.138
$\mathbf{x}^5 = (150, 600, 750, 1300)$	$g^5 = (0.850, 0.400, 0.250, 0.000)$	0.474	0.750	0.517	0.116
$\mathbf{x}^6 = (217, 783, 1100, 3000)$	$g^6 = (0.783, 0.217, 0.000, 0.000)$	0.317	0.500	0.500	0.134
$\mathbf{x}^7 = (290, 500, 600, 610)$	$g^7 = (0.710, 0.500, 0.400, 0.390)$	0.537	1.000	0.504	0.032
$\mathbf{x}^8 = (250, 300, 950, 1400)$	$g^8 = (0.750, 0.700, 0.050, 0.000)$	0.475	0.750	0.463	0.170

considered poverty measures associated with exponential means, the only family of quasiarithmetic means satisfying stability for translations.

The transformation from non negative income values  $x$  to unit interval valued income gaps  $g$  has a central role in the paper. As usual, it is obtained by means of a threshold  $z$  which defines the subset or coalition of the poor individuals. In the future, we intend to consider the possibility of extending the concept of coalition of the poor, by assigning to each coalition of individuals a Choquet capacity measuring the fraction of poor individuals it contains, on the basis of some unit valued individual degree of poverty (in the present model an individual is either poor, with poverty degree 1, or non-poor, with poverty degree 0, but naturally this depends strongly on the

exact poverty threshold). The Choquet capacity value of each coalition would then constitute a generalized (and possibly non additive) measure of the headcount ratio  $H$ .

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