

A Robust Model of Bubbles

with Multidimensional Uncertainty

Antonio Doblas-Madrid*

Department of Economics
Michigan State University
110 Marshall-Adams Hall
East Lansing, MI 48824
Phone (517) 355 77 55
Fax (517) 432 10 68
Email: doblasma@msu.edu

Abstract

A series of recent, seminal papers have greatly advanced our understanding of bubbles by modeling *greater fool's bubbles*, where rational but asymmetrically informed investors buy overvalued assets hoping to sell at a profit before the crash. The authors of these papers note that some assumptions made to keep prices from revealing private information may be controversial. Prices are either assumed to be unresponsive to sales, which is difficult to justify, or prices must satisfy certain parameter restrictions exactly, which makes bubbles fragile. To avoid these critiques, I add multidimensional uncertainty, so that price movements due to investors' sales can be mistaken for random day-to-day fluctuations. Temporary confusion allows some investors to sell before the crash, allowing bubbles of arbitrary duration to arise. Thus, this paper supports and advances previous contributions by introducing noisy prices in a tractable way and generating bubbles that are robust to the aforementioned critiques.

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1 Introduction

The perception that bubbles can inflate asset prices beyond their fundamental values appears to be widespread among financial market participants and policymakers alike. For instance, central bankers Greenspan (1996), Bernanke (2002) and Trichet (2005) have given speeches about monetary policy in the presence of bubbles, and famed investor Warren Buffett (2001, 2005) has, in different contexts, expressed the view that prices were “out of line” with fundamentals.¹ On the other hand, the idea of bubbles has proven very difficult to reconcile with rigorous economic theory, and this has led many economists to be skeptical about bubbles, and to favor fundamentals-based asset pricing.²

Standard asset pricing theory is based on the *efficient markets hypothesis* (Fama (1965)), by which prices reflect all public information about fundamentals, thus ruling out bubbles. Proponents of fundamentals-based asset pricing point out that even in episodes widely cited as bubbles, like the Dutch Tulipmania (1634-1637), the Mississippi Bubble (1719-1720) and the South Sea Bubble (1720), fundamentals-based interpretations cannot be ruled out (see Garber (2001)). Furthermore, Santos and Woodford (2000) show that, in models with rational agents and symmetric information, bubble-equilibria are fragile and depend on special assumptions. In environments with asymmetric information, no-trade theorems (Milgrom and Stokey (1982)) rule out bubbles in a wide class of environments.

However, a series of recent papers have provided important support to the view that bubbles are relevant. For example, Brunnermeier and Nagel (2004) document that, in the late 1990s, hedge funds invested heavily in tech stocks, knowing that they were overvalued. Still, many funds succeeded in timing the market, earning large returns for a while, and selling before the crash. There is also a strand of literature (see Lei, Plott, and Noisire (2001)) documenting that, in experimental settings, bubbles are very pervasive.

Moreover, recent models of bubbles have become increasingly compatible with standard economic theory. A particularly influential line of research includes Abreu and

¹ In 1996, Greenspan gave his famous *irrational exuberance* speech. In their 2005 and 2002 speeches, Bernanke and Trichet expressed the view that prices may deviate from fundamentals. It is also well known that Mr. Buffett shunned tech stocks in the 1990s, was criticized during the boom, but vindicated during the bust. Mr. Buffett also expressed the view, in 2005, that several housing markets in the US were overvalued.

² While the focus of this paper is on theory, on an empirical level, there is also an unresolved debate regarding the validity of econometric methods that have been used to support or refute evidence of bubbles in different datasets (see, among others, Flood and Garber (1980), LeRoy and Porter (1981), Kleidon (1986), Diba and Grossman (1988), West (1987), and Evans (1991)).

Brunnermeier (2003) (AB henceforth), Allen, et al. (1993), and Conlon (2004). In these models, asymmetric information deactivates the backward induction mechanism that typically precludes bubbles in other environments. The key idea is that of a “greater fool’s bubble”, by which it is rational to be a fool and invest in an overvalued asset, as long as there is a good chance of finding a greater fool who will pay even more later.³ Investors chase profits understanding that they may end up being the greater fool, unable to sell before the crash. In AB, rational agents hold a rapidly appreciating asset, and at some point, observe a private signal revealing that it is overvalued. Importantly, they do not know when others observe the signal. In equilibrium, some sell before the crash and make profits, and others suffer losses. Still, the probability of being in the former group and the growth rate of the bubble are high enough to entice agents to take their chances and knowingly hold an overpriced asset. While details differ in Allen et al. (1993) and Conlon (2004), the core ideas are similar. Asymmetrically informed, rational agents know there is a bubble, but want to ride it, because expected profit is positive.⁴

But the authors of these seminal papers note that the assumptions they make to prevent prices from fully revealing private information may be controversial. Allen et al. (1993) and Conlon (2004) assume that parameters, such as the probabilities of different states of the world and dividends at those states, satisfy exact proportions. A small change in one parameter, holding others constant, makes bubbles collapse. In AB, bubbles are robust to small changes in parameters, but prices are, to some extent, independent of selling pressure. Even though those who sell before the crash do so gradually over an interval of time, the price is insensitive to these sales, and the bubble continues to grow as if nobody was selling. Only when sales reach a certain threshold, the price reacts, by abruptly collapsing. As AB note, this “invisibility-of-sales” assumption is necessary to generate bubbles. Without it, prices would reveal private information as soon as sales

³ Other approaches to modeling bubbles include, among others, bubbles that may last indefinitely in expected value (Blanchard and Watson (1982)), overconfidence (Scheinkman and Xiong (2003)), noise agent risk (DeLong, et al (1990)), and asymmetric information with call-option-type compensation for fund managers (Allen and Gorton (1993)). For a survey, see Brunnermeier (2001).

⁴ In reality, fund managers who stay out of bubbles post lower returns than their peers, and thus may lose customers and/or face pressure to resign. Actively betting against bubbles can be even worse, as illustrated by the experience of fund manager Michael Berger, who bought put options on *dotcom* stocks during the second half of the 1990s. As prices kept rising, quarter after quarter, his options expired worthless. In 1996, faced with horrific losses, he began falsifying performance reports, hoping that the crash would come soon. In January of 2000, only a few months before the crash, he could no longer hide his fraud and went out of business. To avoid jail, he fled to Austria in 2002, where he was finally arrested in 2007.

began, effectively reducing the measure of agents who can sell before the crash to zero. Given this, agents would sell as soon as they observed the signal.

AB, Allen et al. (1993) and Conlon (2004) also coincide in suggesting that if their models were extended to allow for multidimensional uncertainty, these assumptions could be avoided and the main results would still hold. In this paper, I follow this hint by constructing a discrete-time model that is heavily based on AB, but introduces two new elements.⁵ First, prices always reflect selling pressure, and second, prices have an unobservable noisy component. As in AB, the growth of the bubble is fueled by demand from behavioral agents. But I assume that this demand, and thus the growth rate of the price, has some randomness. Also, I assume that as rational agents start selling, the absorption capacity of behavioral agents is progressively satisfied, and thus, the expected price growth rate falls. But prices are not fully revealing since slow price growth may mean that rational agents are selling, or that the realization of the random shock is low.

I begin the analysis by considering the case without noise, in which as soon as one type sells (a type includes those who observed the overvaluation signal in the same period), all uncertainty is revealed, triggering a crash in the next period. I derive a parameter restriction under which, without noise, agents sell as soon as they observe the signal. Maintaining this restriction, I increase the amount of noise so that it may conceal sales of one type, but not more. Then, price growth can be high, revealing that nobody has sold, medium, revealing that maybe one type has sold and maybe not, or low, revealing that at least one type has sold. I show that, even with this minimal amount of noise, there is a region of the parameter space where bubbles of arbitrary duration arise. In this class of equilibria, agents wait the longest before selling if price growth is high, but sell immediately if it is low. If price growth is medium, they also wait for a while, but not as long as with high growth. This strategy is optimal because, at all times, agents who are not supposed to sell think that their type may have been first to observe the signal, in which case, selling preemptively would mean foregoing large gains.

In this first environment, agents cannot reenter the market after selling. Naturally, this opens the question of whether the model's ability to generate bubbles depends on this

⁵ The discrete-time assumption, in addition to increasing tractability, might be of independent interest, since it implies that agents have a finite number of trading opportunities before the time of the crash. In some models (e.g. Allen and Gorton (1993)) continuous time is necessary, precisely because it generates infinitely many trading opportunities.

restriction. Intuitively, the possibility of reentry may play a role, since it weakens the opportunity-cost argument that deters preemptive sales. With forbidden reentry, agents who sell preemptively miss out on large gains if their type turns out to be first. With allowed reentry, if agents see that their type was first, they can repurchase their shares, foregoing only part of the gains. When I extend the model to allow for reentry, however, the main results do not change qualitatively. Some equilibria are discarded because they are not “reentry-proof”, but for appropriate parameter values, the opportunity cost associated with preemptive selling is still large enough to deter such sales.

Finally, I generalize the analysis by letting noise hide sales by $\bar{z} \geq 1$ types. In this case, strategies that condition actions only on the most recent price ratio are no longer optimal. This is partly because, with minimal noise, if an agent knows that her type is selling, this greatly increases chances that price growth will be low and the crash will happen next period. But as noise increases, sales by one type have a diminishing marginal impact on the probability that the bubble will burst, and thus, it becomes increasingly important for agents to infer how many other types could be selling. Thus, I construct a class of equilibria where agents sell only if the price history implies that up to $\bar{z} + 1$ types could be out of the market by the end of the period. The analysis remains tractable, because it is possible to construct a single variable summarizing the price history. Given this variable, strategies still satisfy a Markov property. Numerical results show that these strategies are indeed optimal for some parameter values. Furthermore, within this class of strategies, the degree of equilibrium multiplicity is less than in the minimal-noise case, and it is possible to rule out the possibility that bubbles last less than a certain positive threshold. Thus, bubbles are rationalized in a strong sense.

In sum, this paper takes models of bubbles one step further by providing microfoundations for assumptions that were previously exogenous, and by validating the conjecture that a model with noisy prices can generate robust bubbles even if prices respond to selling pressure at all times. Results largely confirm previous findings, since asymmetric information allows bubbles of arbitrary length to arise, although bubbles are less deterministic, in the sense that there is always a range of possible bursting times, and the realizations of noise determine where in that range the bubble bursts, and how many types manage to sell before the crash.

Since bubbles have traditionally been so difficult to reconcile with theory, models of bubbles are still, in spite of enormous recent progress, less established than other asset pricing theories. Showing that recent theories of bubbles do not depend on special assumptions shall further increase their appeal, and promote their use in analyzing applied topics, such as optimal monetary policy, or the emergence of bubbles in markets besides the stock market, such as real estate and foreign exchange.⁶

The paper is organized as follows. In sections 2 and 3, respectively, I describe the environment and define equilibrium. In section 4, I construct bubbles with minimal noise and in section 5, I allow noise to hide sales by more than one type. Section 6 concludes.

2 The Model

Time is discrete and infinite with periods labeled $t = \dots, -1, 0, 1, \dots$. There are two assets, a risky asset that trades at the price p_t and a risk-free asset with gross return $R > 1$. For $t \leq 0$, p_t is given by R^t , i.e. up to time zero the risky asset appreciates at the risk-free rate, and p_0 is normalized to 1. After time 0 news begin arriving about events that increase the fundamental value of the risky asset. Price growth accelerates in response to the news, and thus, for $t \geq 1$, p_t / p_{t-1} is a random variable with expected value $G > R$.⁷

There are two kinds of agents, rational and behavioral. The only role of behavioral agents is to fuel the risky asset's appreciation. Every period, they become more enthusiastic about prospective returns, and no matter how high the price gets, they continue to demand shares until they have bought up to their absorption capacity, which equals $\kappa \in (0, 1)$ shares. Then, there is a unit mass of rational agents with discount factor for future utility given by $1/R$. Rational agents are risk neutral, and the positions they can take in the risky asset are limited by short sales constraints. Concretely, their positions range between a maximum of 1 and a minimum of 0. At $t = 0$, rational agents are fully invested in the asset, i.e. holdings equal one. The model's ability to generate bubbles will depend on whether these agents keep holding the asset even after they learn that it is overvalued.

⁶ There are already some applications of models of bubbles. For example, Minguez-Afonso (2007) applies the AB model to currency crises.

⁷ In order to facilitate comparison with the literature, I have kept the model as close as possible to AB. Besides having discrete time, the only new ingredients are noise and price responsiveness to sales.

Fast appreciation of the asset is justified by fundamentals for some time. However, behavioral agents keep fueling rapid growth of p_t even after it surpasses the level justified by fundamentals. This generates a mispricing that is corrected only when the bubble bursts. The first period in which the asset becomes overvalued is denoted by t_0 , which is a random variable with probability function φ given by

$$\varphi(t_0) = e^{-\lambda t_0} (e^\lambda - 1) \quad \text{for all } t_0 = 1, 2, \dots, \quad (1)$$

with $\lambda > 0$.⁸ A crucial ingredient of the model is that when t_0 is realized, it is not perfectly observable. Instead, every period from t_0 to $t_0 + N - 1$, a mass $1/N$ of rational agents observe a signal revealing that the overvaluation has begun. This divides the unit mass of rational agents into N different types, indexed by $n \in \{t_0, \dots, t_0 + N - 1\}$. Rational agents know when they observed their signal, but not when others did. That is, once they get their signal they know n , but not t_0 . Given the signal n , the probability that $t_0 = j$ is given by $\varphi(j | n) = e^{-\lambda j} / (e^{-\lambda(\max\{1, n - (N-1)\})} + \dots + e^{-\lambda n})$, if $\max\{1, n - (N-1)\} \leq j \leq n$ and zero otherwise. Agents also update beliefs as they observe prices, which reflect a mixture of noise and sales by others. Thus, $I_{n,t}$ —which denotes information available to type- n agents at time t —is given by the price history $p^{t-1} = (p_0, p_1, \dots, p_{t-1})$ before observing the signal, and by the signal and price history (n, p^{t-1}) after observing the signal. Also, let $h_{n,t}$ denote asset holdings of type- n agents at the end of period t (and the beginning of $t+1$) and $H_t = (h_{t_0,t} + \dots + h_{t_0+N-1,t})/N$ denote aggregate holdings across rational agents. As mentioned above, for all n , $h_{n,0} = 1$ and because of short sales constraints, for all n and t , $h_{n,t} \in [0, 1]$.

From $t = 1$ until the bubble bursts, p_t / p_{t-1} is given by $G + \alpha[\varepsilon_t - (1 - H_t)]$, where $\alpha > 0$ is a parameter capturing the extent to which prices respond to sales. The noisy component of the price ε_t is uniformly distributed over $[-\bar{\varepsilon}, \bar{\varepsilon}]$, and $1 - H_t$ denotes rational agents' cumulative sales (net of repurchases). As sales start, expected price growth falls, but because of noise, prices reflect these sales imperfectly. For example, if

⁸ This is a discrete version of the distribution with cdf $\Phi(t_0) = 1 - e^{-\lambda t_0}$, $\forall t_0 \in [0, \infty)$ used by AB. φ assigns the same probability to t_0 that Φ assigns to $[t_0 - 1, t_0)$, i.e. $\forall t_0 \geq 1$, $\varphi(t_0) \equiv \Phi(t_0) - \Phi(t_0 - 1)$.

$1/N < 2\bar{\varepsilon} < 2/N$, noise can hide sales by, at most, one type. Price ratios p_t/p_{t-1} above $G + \alpha(\bar{\varepsilon} - 1/N)$ reveal that, as of time t , nobody has sold. p_t/p_{t-1} below $G - \alpha\bar{\varepsilon}$ reveals that at least one type has sold, and if p_t/p_{t-1} is between these two thresholds, maybe one type sold, and maybe none.

The bubble bursts once $1 - H_t \geq \kappa$. If the bubble bursts at T , the post-crash price p_T is given by $p_{T-1}R(G/R)^{-(T-t_0)}$. Note that if α is small—so that from t_0 until T , $p_t/p_{t-1} \approx G$ —this post-crash price approximately equals the price that would be observed if there never was a bubble, in which case prices would be $p_t = p_{t-1}(G + \alpha\varepsilon_t)$ while $1 \leq t \leq t_0 - 1$ and $p_t = p_{t-1}R$ for $t \geq t_0$. After the crash, the price grows at the risk-free rate. In sum, for $t > 0$, the price process is given by

$$p_t = \begin{cases} p_{t-1}(G + \alpha[\varepsilon_t - (1 - H_t)]) & \text{if } 1 - H_s < \kappa \quad \forall s \in \{0, 1, \dots, t\} \\ p_{t-1}R \left(\frac{G}{R}\right)^{-(t-t_0)} & \text{if } 1 - H_t \geq \kappa \text{ and } 1 - H_s < \kappa \quad \forall s \in \{0, 1, \dots, t-1\} \\ p_{t-1}R & \text{if } 1 - H_s \geq \kappa \text{ for at least one } s \in \{0, 1, \dots, t-1\}. \end{cases} \quad (2)$$

Within-period timing is as follows. At the start of period t , having observed previous prices $p^{t-1} = \{\dots, p_{t-2}, p_{t-1}\}$ (and, if $t \geq n$, the signal n), a type- n agent submits orders to buy/sell, or does nothing. Once agents have submitted their orders and ε_t has been realized, p_t is determined and orders are executed.⁹

The assumption that behavioral agents fuel bubble growth keeps the model close to AB, and helps introduce changes in a parsimonious way. However, I conjecture that the

⁹ Agents are thus submitting *market orders*, since they know that their orders will be executed, but they do not know exactly at what price. See Chapter 3 in Brunnermeier (2001) for a description of microstructure models and types of orders (market orders, limit orders, stop orders, etc). During the period of the crash there are so many orders to sell that the execution price ends up falling far below the last price that agents saw. In these situations, which practitioners call *fast markets*, from the time an order is submitted until it is executed, the price may have skyrocketed or plummeted. While I assume that, in the crash period, all orders are executed at the post-crash price, nothing would qualitatively change if, instead, I followed AB and let some shares be sold at the pre-crash price and others at the post-crash price.

presence of behavioral agents may not be necessary in order to generate bubbles. Imagine, for instance, that there were no behavioral agents, and that rational agents received an endowment every period and invested it in the risky asset while they believed that it would keep appreciating. This would fuel price growth. Supply could be modeled as follows. Every period, a randomly chosen fraction of agents would be hit by a shock (representing unforeseen events that increase liquidity needs, such as medical emergencies) that would force them to sell. Individuals would know whether they themselves had been hit by the shock, but not the total fraction of agents hit by the shock, and this fraction would vary randomly from period to period. In such a model, sales as rational agents left the market could be mistaken for the realization of the random shock, giving rise to same kind of uncertainty as the model with behavioral agents.

3 Equilibrium

The equilibrium concept is Perfect Bayesian Nash Equilibrium, and consists of strategies $h_{n,t}(h_{n,t-1} | I_{n,t})$, which determine next-period asset holdings as a function of current holdings and information, and beliefs $\mu(t_0 | I_{n,t})$, which are probability distributions over values of t_0 , conditional on prices and, if $t \geq n$, the signal.

I first define equilibrium beliefs taking strategies $h_{n,t}(\cdot|\cdot)$ as given. $\mu(t_0 | I_{n,t})$ assigns positive probability only to values of t_0 that are consistent with $I_{n,t}$. The set of such values is the support of t_0 given $I_{n,t}$ and denoted by $\text{supp}(t_0 | I_{n,t})$. As agents observe signals and prices, they progressively eliminate values of t_0 . The signal n reduces the support of t_0 from the set of positive integers to the set $\{\max\{1, n - (N - 1)\}, \dots, n\}$. And the price history p^{t-1} rules out values of t_0 as follows: Given p^{t-1} and (2), agents know $\varepsilon_s - (1 - H_s)$ for $s = 1, \dots, t-1$. Given this, initial holdings $h_{n,0} = 1$ for all n , and $h_{n,t}(\cdot|\cdot)$, every value of t_0 implies realizations of ε_s for $s = 1, \dots, t-1$, and holdings $h_{m,s}$ for all types $m \in \{t_0, \dots, t_0 + N - 1\}$.¹⁰ A value of t_0 is ruled out by $I_{n,t}$ if, for at least one s , the

¹⁰ Intermediate steps to see whether $j \in \text{supp}(t_0 | I_{n,t})$ are as follows: Fix j and given $h_{m,0} = 1$ for all $m \in \{j, \dots, j + N - 1\}$, strategies imply $h_{m,1}(1 | I_{m,1})$ for all m . Summing across types yields H_1 . Given H_1 , p_1 / p_0 and (2), the implied ε_1 can be computed. Given $h_{m,1}$ for all $m \in \{j, \dots, j + N - 1\}$, the same steps can be repeated to find ε_2 , and so on up to period $t - 1$.

absolute value of the implied ε_s exceeds $\bar{\varepsilon}$. The set $\text{supp}(t_0 | I_{n,t})$ collects all the values of t_0 not ruled out by this process. Since all values of ε_s with $|\varepsilon_s| \leq \bar{\varepsilon}$ are equally likely, by Bayes' rule, the probabilities $\mu(t_0 | I_{n,t})$ are given by

$$\mu(t_0 | I_{n,t}) = \frac{\varphi(t_0)}{\sum_{t_0 \in \text{supp}(t_0 | I_{n,t})} \varphi(t_0)} \quad \text{for all } t_0 \in \text{supp}(t_0 | I_{n,t}). \quad (3)$$

Next, I define the equilibrium strategy $h_{n,t}(\bullet)$ taking μ as given. This strategy maximizes $V(h_{n,t-1} | I_{n,t})$, the value of holdings given $I_{n,t}$. Maximization is greatly simplified by risk neutrality. If the expected current price $E_{n,t} p_t$ is at least as large as the expected discounted future value $E_{n,t} V(1 | I_{n,t+1}) / R$ per share—where $E_{n,t}$ denotes expected value given $I_{n,t}$ —then $h_{n,t} = 0$ is optimal, and otherwise, $h_{n,t} = 1$ is optimal. The expected value $E_{n,t} V(1 | I_{n,t+1})$ is well defined because we know that the value of holdings after the burst is the post-crash price. This gives us an expected terminal value, from which to iterate backwards.¹¹ In sum, the equilibrium strategy $h_{n,t}(h_{n,t-1} | I_{n,t})$ solves

$$V(h_{n,t-1} | I_{n,t}) = \max_{h_{n,t}} E_{n,t} p_t (h_{n,t-1} - h_{n,t}) + \frac{1}{R} E_{n,t} V(h_{n,t} | I_{n,t+1}) \quad (4)$$

subject to (2) and $0 \leq h_{n,t} \leq 1$.

4 Bubble Equilibria with Minimal Noise

In the presence of noisy prices, multiple equilibria are unavoidable, since a given price history can either trigger sales, or not, and frequently, both responses are optimal for an individual agent, as long as others respond to that price history in the same way. For this reason, I will not attempt to characterize all possible equilibria, and will instead focus on a series of examples that illustrate how bubbles arise.

I will begin, in subsection 4.1, by studying the case without noise, i.e. the case where prices reveal sales as soon as one type exits the market. Restricting attention to a particular class of strategies, I derive a parameter condition, namely that G/R is below a

¹¹ As will be shown later, I only consider strategies where, for any price history, agents sell a finite number of periods after observing the signal. This implies that there is a well-defined maximum-possible duration of the bubble and that it is always possible to iterate backwards from that finite period in order to calculate V . We could also assume, as AB do, that there is an exogenous maximum duration of the bubble.

threshold Γ , that ensures that agents sell as soon as they observe the signal.¹² In subsection 4.2, maintaining $G/R < \Gamma$, so that there are no bubbles without noise, I raise $\bar{\varepsilon}$ so that noise may conceal sales of one type, but not more. Then, price ratios can be *high*, revealing that nobody has sold, *medium*, revealing that maybe one type has sold and maybe not, or *low*, revealing that at least one type has sold. I show that this minimal amount of noise is enough for arbitrarily long bubbles to arise. While this result, at first, relies on the assumption that agents cannot reenter the market after selling, in subsection 4.3, I allow for reentry and show that, although some equilibria vanish, it is still possible to generate arbitrarily long bubbles that are “reentry-proof”.

Before presenting the examples, I make two more assumptions about parameter values that will greatly simplify algebraic expressions. First, I assume that the sensitivity of prices to selling pressure α is strictly positive, so that prices reveal information, but small, so that from $t = 1$ until the crash, $p_t / p_{t-1} \approx G$. Second, I let $\lambda \approx 0$, which implies that $\mu(t_0 | I_{n,t})$ approximately equals the inverse of the number of elements in $\text{supp}(t_0 | I_{n,t})$. These two assumptions will be maintained throughout the rest of the paper.¹³

4.1 The Case without Noise

Assume that $2\bar{\varepsilon} < 1/N$, which implies that sales are always detected, since $G - \alpha\bar{\varepsilon}$, the lowest possible price ratio when nobody has sold, is above $G + \alpha(\bar{\varepsilon} - 1/N)$, the highest possible ratio if one type has sold. Also, assume that $1/N < \kappa$, so that sales of the first type do not burst the bubble directly. Consider the following profile of strategies.

Example 1 For all $n \in \{t_0, \dots, t_0 + N - 1\}$ the strategy of a type- n agent is:

$$h_{n,t}(h_{n,t-1}, I_{n,t}) = \begin{cases} 1 & \text{for all } t \in \{1, 2, \dots, t^* - 1\} \\ 0 & \text{for all } t \geq t^*, \end{cases} \quad (5)$$

where $t^* \equiv \min\{n + \tau^*, \hat{t}\}$ and $\hat{t} \equiv \min\{t \mid t \geq 2 \text{ and } (p_{t-1} / p_{t-2}) < G - \alpha\bar{\varepsilon}\}$.

¹² The fact that, even without noise, if $G/R > \Gamma$, agents do not sell immediately after getting the signal does not contradict the AB results. In discrete time, if all agents of the same type sell simultaneously, a positive measure $1/N$ of agents can sell before the crash even if there is no noise. However, as $1/N$ shrinks, Γ does not, implying that as $1/N$ approaches zero, the bubble would have to grow infinitely fast for agents not to sell immediately after the signal.

¹³ These assumptions increase tractability and make it sometimes possible to characterize equilibria analytically, without affecting results qualitatively.

In words, (5) dictates that agents hold the maximum long position until prices reveal that sales have started or τ^* periods have passed since observing the signal. As soon as one of these two conditions is met, they should sell everything and never re-enter the market. If everybody follows this, type- t_0 agents sell at $t_0 + \tau^*$, the price ratio $p_{t_0 + \tau^*} / p_{t_0 + \tau^* - 1}$ reveals these sales, and other types sell at $t_0 + \tau^* + 1$, bursting the bubble.

In *Lemma 1*, I derive conditions under which all agents find it optimal to follow (5), and derive a threshold Γ such that, if $G/R < \Gamma$, the only equilibrium is one where agents sell immediately after getting the signal. Equilibria with $\tau^* \geq 1$ and $G/R < \Gamma$ are ruled out because agents refuse to wait for τ^* periods after observing the signal, given that price growth under Γ fails to compensate for the potential loss in the event of a crash.

Lemma 1 *If $2\bar{\varepsilon} < 1/N < \kappa$ and $G/R < \Gamma$, where $\Gamma = (1 + \sqrt{5})/2$, the strategy profile (5) is an equilibrium if and only if $\tau^* = 0$.*

Proof In equilibrium, it must be that agents want to sell when (5) dictates that they should sell. And they must also want to wait when (5) dictates that they should wait.

For the first part, consider, for any $\tau^* \geq 0$, the decision problem of an agent of arbitrary type n who, at time $n + \tau^*$, sees that there were no sales in previous periods. At this point, she knows that her type must be $n = t_0$. She also knows that other agents of her type are selling, that $p_{n + \tau^*} / p_{n + \tau^* - 1}$ will reveal those sales, and that the crash will happen at time $n + \tau^* + 1$. Obviously, her expected payoff from selling now $p_{n + \tau^*}$ exceeds the (discounted) post-crash price $p_{n + \tau^*} (G/R)^{-(\tau^* + 1)}$ that she will get if she waits. Strategy (5) also dictates that agents should sell once they see that sales have started, which is (weakly) optimal since at that point agents get the post-crash price no matter what they do.¹⁴

For the second part, consider a type- n agent at time $t < n$, i.e. before observing the signal. If $p_{t-1} / p_{t-2} > G - \alpha\bar{\varepsilon}$, $\text{supp}(t_0 | I_{n,t})$ is $\{t - \tau^*, t - \tau^* + 1, t - \tau^* + 2, \dots\}$. Selling now yields p_t , whereas if she waits, the payoff is uncertain. If $t_0 = t - \tau^*$, there will be a crash at $t + 1$, but if $t_0 > t - \tau^*$, there will be no crash at $t + 1$ and she will profit from at least one

¹⁴ Once rational agents see that sales have started, they are indifferent between selling and waiting, since they get the post-crash price in either case. I assume that, in this situation of indifference, agents sell. This is unimportant, however, since strict preference for selling can be easily induced, for example by letting some shares to be sold at the pre-crash price, and the rest at the post-crash price.

more period of appreciation. Given that $\Pr[t_0 = t - \tau^* | t_0 \geq t - \tau^*] = 1 - e^{-\lambda} \approx 0$, a crash at $t + 1$ is so unlikely that our type- n agent would rather wait than sell.¹⁵

Note that if $\tau^* = 0$, selling preemptively is the same thing as selling before observing the signal. Thus, for $\tau^* = 0$, all possible cases have been covered and thus, the existence of an equilibrium with $\tau^* = 0$ that does not need any restrictions on G/R has been established.

More work is needed to rule out equilibria with $\tau^* \geq 1$. If there was such an equilibrium, G/R would have to be high enough to keep agents who have observed the signal from selling before $n + \tau^*$. The key tradeoff in that situation is best illustrated by considering a type- n individual at time $t = n + \tau^* - 1$, with $p_{t-1} / p_{t-2} > G - \alpha \bar{\varepsilon}$. At this point, prices and (5) imply that $\text{supp}(t_0 | I_{n,t}) = \{n-1, n\}$, with both values of t_0 being (roughly) equiprobable. An individual type- n agent understands that, if she sells now, she will get p_t , and if she waits her discounted payoff will be $p_t (G/R)^{-(\tau^*+1)}$ if $t_0 = n-1$, and $p_t G/R$ if $t_0 = n$. Thus, she will deviate from (5) by selling at $n + \tau^* - 1$ if

$$1 > \frac{1}{2} \left(\frac{G}{R} \right)^{-(\tau^*+1)} + \frac{1}{2} \frac{G}{R}. \quad (6)$$

Note two things here. First, since the crash is bigger for longer bubbles, if (6) holds for $\tau^* = 1$, it holds for any $\tau^* \geq 1$. Second, for any τ^* , (6) captures the situation in which incentives to sell preemptively are strongest. At other times $t = n + \tau^* - s$ (with $s \geq 2$) $\text{supp}(t_0 | I_{n,t})$ is given by $\{n-s, \dots, n\}$ and the probability that of a crash at $t+1$ is given by $\Pr[t_0 = t-s | t-s \leq t_0 \leq t] \approx 1/(s+1)$, clearly below $1/2$. Thus, if (6) holds for $\tau^* = 1$, there is no equilibrium where agents follow (5) unless $\tau^* = 0$. A few algebra steps (see appendix A) show that (6) with $\tau^* = 1$ is equivalent to $G/R < \Gamma$, where $\Gamma = (1 + \sqrt{5})/2 \approx 1.618$. ■

4.2 Introducing a Minimal Amount of Noise

Henceforth, I will maintain the restriction that $G/R < \Gamma$, which rules out bubbles without noise. In this subsection, I will show that, if the amount of noise $\bar{\varepsilon}$ is increased and agents

¹⁵ This reasoning implicitly assumes that $t - \tau^* \geq 1$. If this does not hold, though, agents are even less inclined to sell preemptively, since chances of a crash at $t + 1$ are literally zero.

play strategies similar to (5), the model generates arbitrarily long bubbles, even if the amount of noise is minimal in the sense that it can hide sales by at most one type. In particular, I assume $1/N < 2\bar{\varepsilon} < 2/N$, so that price ratios p_t/p_{t-1} fall into one of three categories: *high* if in $[G + \alpha(\bar{\varepsilon} - 1/N), G + \alpha\bar{\varepsilon}]$, *medium* if in $[G - \alpha\bar{\varepsilon}, G + \alpha(\bar{\varepsilon} - 1/N))$ and *low* if below $G - \alpha\bar{\varepsilon}$. High price ratios reveal with certainty that nobody has sold up to (and including) time t , medium ratios are consistent both with nobody having sold and with one type having sold, and low ratios reveal with certainty that at least one type has sold.

As in *Example 1*, the strategies I will consider in *Example 2* are such that agents sell if they see low price growth, but wait after high or medium price growth. If price ratios are high, agents will wait for τ^* periods from the time they observe the signal until they sell, and for intermediate ratios, they will wait for τ^{**} periods.

Example 2 For all $n \in \{t_0, \dots, t_0 + N - 1\}$ the strategy of a type- n agent is:

$$h_{n,t}(h_{n,t-1}, I_{n,t}) = \begin{cases} 1 & \text{for all } t \in \{1, 2, \dots, t^* - 1\} \\ 0 & \text{for all } t \geq t^*, \end{cases} \quad (7)$$

where $t^* \equiv \min\{t_1, t_2, t_3\}$, $\tau^* \geq \tau^{**} \geq 0$ and

$$\begin{aligned} t_1 &\equiv \min\{t \mid t \geq 2 \quad \text{and} \quad p_{t-1}/p_{t-2} < G - \alpha\bar{\varepsilon}\} \\ t_2 &\equiv \min\{t \mid t \geq 2, \quad t \geq n + \tau^{**} \quad \text{and} \quad G - \alpha\bar{\varepsilon} \leq p_{t-1}/p_{t-2} < G + \alpha(\bar{\varepsilon} - 1/N)\} \\ t_3 &\equiv \min\{t \mid t \geq 2, \quad t \geq n + \tau^* \quad \text{and} \quad G + \alpha(\bar{\varepsilon} - 1/N) \leq p_{t-1}/p_{t-2} \leq G + \alpha\bar{\varepsilon}\}. \end{aligned}$$

Following (7), agents maintain the maximum long position until one of the following happens: (i) Prices reveal that sales have begun, (ii) it has been at least τ^{**} periods since the signal and, given prices, sales may or may not have begun, or (iii) it has been τ^* periods since the signal and prices reveal that sales have not begun. As before, once an agent sells, she stays out of the market forever. Figure 1 depicts the type of bubbles generated by these strategies, with $\tau^* > \tau^{**}$.¹⁶ The duration of the bubble is not deterministic, since depending on the realizations of ε_t , sales may start at any time between $t_0 + \tau^{**}$ and $t_0 + \tau^*$. Since the width of this window, $\tau^* - \tau^{**}$, will be important in the coming analysis, let $d \equiv \tau^* - \tau^{**}$.

¹⁶ It is actually impossible to build an equilibrium with strategies as in (7) and $\tau^* < \tau^{**}$. (Proof of this claim is available upon request.)

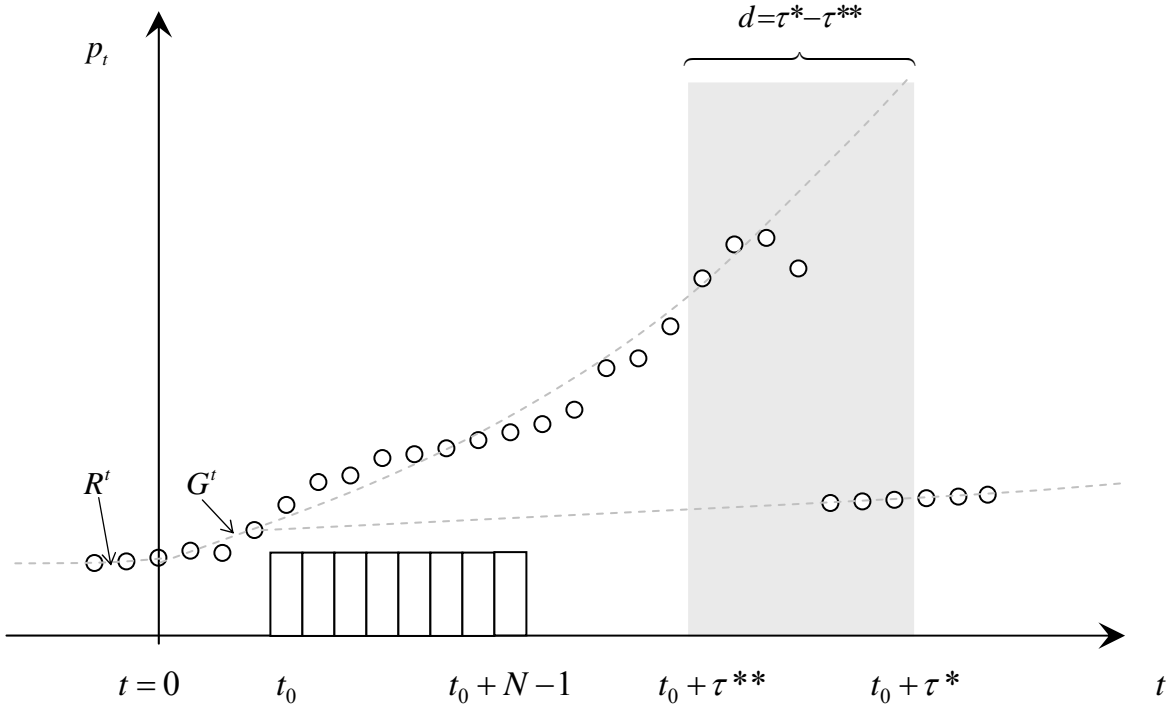


Figure 1 At time $t_0 + \tau^{**}$, type- t_0 agents sell if $p_{t_0+\tau^{**}-1} / p_{t_0+\tau^{**}-2}$ is medium and wait if it is high. They follow this sell-if-medium-wait-if-high rule for d periods, and if price ratios keep being high, at time $t_0 + \tau^*$, they sell even if $p_{t_0+\tau^*-1} / p_{t_0+\tau^*-2}$ is high. If $p_{t_0+\tau^*-1} / p_{t_0+\tau^*-2}$ is high, only type- t_0 agents sell at $t_0 + \tau^*$, whereas if $p_{t_0+\tau^{**}-1} / p_{t_0+\tau^{**}-2}$ is medium, all of the $d+1$ types $t_0, \dots, t_0 + d$ sell.

These strategies have a simple Markov structure, since behavior depends only on how long it has been since observing the signal, and on whether the most recent price ratio is high, medium, or low. Earlier price ratios do affect beliefs, and I take this into account when verifying that strategies are optimal. But fortunately, for appropriate parameter values, agents indeed find it best to follow (7) at all times and for all price histories. To show this, it is useful to classify the situations in which agents may find themselves into four categories, and to prove one lemma for each category. Thus, in Lemma 2, I verify that type- n agents find it optimal to sell at $t = n + \tau^*$ if p_{t-1} / p_{t-2} is high. In Lemma 3, I derive conditions under which type- n agents agree not to sell preemptively at $t < n + \tau^*$ if p_{t-1} / p_{t-2} is high. Lemma 4 verifies that type- n agents want to sell at $t \geq n + \tau^{**}$ if p_{t-1} / p_{t-2} is medium, and Lemma 5 derives conditions ensuring that type- n agents do not want to sell preemptively at time $t < n + \tau^{**}$ if p_{t-1} / p_{t-2} is medium.

Lemma 2 For all $G/R \in (1, \Gamma)$ and $\tau^* \geq 1$, type- n agents find it optimal to sell at $t = n + \tau^*$ if p_{t-1}/p_{t-2} is high.

Proof Only agents of type $n = t_0$ may find themselves still in the market τ^* periods after observing the signal. At this point, they know that their type is $n = t_0$ since p_{t-1}/p_{t-2} could not possibly be high if others had observed the signal before them. At time $t = t_0 + \tau^*$, knowing that other agents of type t_0 are selling, a type- t_0 individual faces the following sell-or-wait trade-off. If she sells at t her payoff will be p_t and if she waits, with probability $\pi \equiv (1/N)/(2\bar{\varepsilon})$ the price ratio p_t/p_{t-1} will be low, precipitating a crash at $t + 1$ (and resulting in a discounted payoff $p_t(G/R)^{-(\tau^*+1)}$), and with probability $1 - \pi$, p_t/p_{t-1} will be medium, in which case $d + 1$ types will sell at time $t + 1$. If $d + 2 < \kappa N$, sales triggered by the medium price will not burst the bubble and it will be possible to sell at the (discounted) price $p_t G/R$. Thus, if $d + 2 < \kappa N$, selling is preferable to waiting if

$$1 > \pi \left(\frac{G}{R} \right)^{-(\tau^*+1)} + (1 - \pi) \frac{G}{R}. \quad (8)$$

Since $\pi \geq 1/2$ this condition is less stringent than (6), and thus holds for all $G/R \in (1, \Gamma)$ and $\tau^* \geq 1$. (For $\tau^* = 0$, (8) holds for all $G/R \in (1, \Gamma)$ if $\pi \geq \Gamma - 1$.) If $d + 2 \geq \kappa N$, medium price ratios trigger enough sales to burst the bubble, and thus (8) becomes $1 > (G/R)^{-(\tau^*+1)}$, which of course holds for any $\tau^* \geq 0$. ■

Lemma 2 shows that, following fast price growth, agents want to sell when (7) dictates that they should. This requires no restrictions beyond $G/R < \Gamma$. But keeping agents from selling before they should is not as easy. Preemptive sales may be tempting because, while one waits, there is a chance that an earlier type is selling and will precipitate a crash. In *Lemma 3*, I show that agents are willing to wait under two conditions. First, the bubble must grow quickly and that it is must be likely that, when one type sells, the price ratio will be intermediate rather than low. The second condition is that d/N should be small relative to κ , so that sales following medium price growth do not burst the bubble.

Lemma 3 If $1 < (1 - \pi/2)G/R + \pi/2(G/R)^{-(\tau^*+1)}$, $d + 2 < \kappa N$, $t < n + \tau^*$ and p_{t-1}/p_{t-2} is high, type- n agents find it optimal not to sell at time t .

Proof See appendix B.

While the proof is in appendix B, here, I sketch the main ideas. First, I argue that preemptive sales are tempting at $t = n + \tau^* - j$ only if all of the $d + 1$ ratios before t are high. From the point of view of type- n agents, only in those cases, there is a positive probability $(j + 1)^{-1}$ that t_0 equals $n - j$. In other cases, the price history rules out $t_0 = n - j$, and thus, type- n agents know that nobody is selling at t , and that a crash at $t + 1$ is impossible. Then, I show that, if type- n agents prefer waiting to selling at $t = n + \tau^* - j$ (with high p_{t-1} / p_{t-2}) for $j = 1$, then, they also do for any $j > 1$. To see this, it is useful to first consider the sell-or-wait trade-off of a type- n agent at time $t = n + \tau^* - 1$ (with high p_{t-1} / p_{t-2}). In this case, $\text{supp}(t_0 | I_{n,t}) = \{n - 1, n\}$.¹⁷ If the agent waits, with probability $\pi / 2$ the bubble will burst at $t + 1$ (this is the scenario where $t_0 = n - 1$ and sales of type- t_0 agents push p_t / p_{t-1} below $G - \alpha \bar{\varepsilon}$), while with probability $1 - \pi / 2$ they will sell at $t + 1$ and profit from one more period of appreciation. Thus, for $j = 1$, waiting is optimal if

$$1 < \frac{1}{2} \left(\pi \left(\frac{G}{R} \right)^{-(\tau^*+1)} + (1 - \pi) \frac{G}{R} \right) + \frac{1}{2} \frac{G}{R}. \quad (9)$$

From here, note that if $j > 1$ the probability of a crash at $t + 1$ falls to $\pi / (j + 1)$, and thus, preemptive selling becomes less tempting.

Finally, note that (9) only applies to cases with $d + 2 < \kappa N$ since, when $t_0 = n - 1$ and p_t / p_{t-1} is medium, sales of type t_0 plus the $d + 1$ types that sell at $t + 1$ do not burst the bubble. If $d + 2 \geq \kappa N$, incentives to sell preemptively are too strong, and agents refuse to wait $\tau^* > 0$ periods. (See the end of the proof in appendix B for details on this.)

In sum, *Lemma 3* precludes preemptive sales if the last price ratio is high, (9) holds and $(d + 2) / N$ is less than κ . For large τ^* , (9) approximates $1 < G / R (1 - \pi / 2)$, which is satisfied by many pairs $(G / R, \pi)$ with $1 < G / R < \Gamma$ and $1/2 < \pi < 1$. The fact that $d + 2$ must be less than κN indicates that long-lived bubbles, i.e. equilibria with a large τ^* , may arise only if τ^{**} is also large. This suggests that the key to generating bubbles will

¹⁷ Again, in the special case $t_0 = 1$, type-1 agents know their type since period 1.

be precluding preemptive sales after medium price ratios. But before discussing that, let us prove *Lemma 4*, by which type- n agents want to sell at $t \geq n + \tau^{**}$ if p_{t-1}/p_{t-2} is medium.

Lemma 4 *For any $\tau^{**} \geq 0$ and any $G/R < \Gamma$, type- n agents find it optimal to sell at time $t \geq n + \tau^{**}$ if p_{t-1}/p_{t-2} is medium.*

Proof Consider the wait-or-sell choice of a type- n agent at $t = n + \tau^{**}$. If, in addition to p_{t-1}/p_{t-2} being medium, p_{t-2}/p_{t-3} is also medium, values of t_0 other than n and $n-1$ can be ruled out, because if t_0 was below $n-1$, at least two types would have sold at $t-1$, making p_{t-1}/p_{t-2} low. Similarly, if p_{t-2}/p_{t-3} is high and p_{t-3}/p_{t-4} is medium, all values of t_0 but n and $n-1$ are inconsistent with p_{t-2}/p_{t-3} being high. In these cases, type- n agents see that, if they are second ($t_0 = n-1$) the bubble will burst next period for sure, and if they are first ($t_0 = n$) the bubble will burst with probability π . Selling is preferable if

$$1 > \frac{1}{2} \left(\frac{G}{R} \right)^{-(\tau^{**}+2)} + \frac{1}{2} \left[\pi \left(\frac{G}{R} \right)^{-(\tau^{**}+1)} + (1-\pi) \frac{G}{R} \right], \quad (10)$$

which holds for any $G/R < \Gamma$, $\pi \in (0.5, 1)$ and $\tau^{**} \geq 0$. While (10) has been derived for price histories implying $\text{supp}(t_0 | I_{n,t}) = \{n-1, n\}$, incentives to sell are even stronger for other histories. If p_{t-1}/p_{t-2} is the first medium price ratio after k consecutive high ratios (with $k \in \{2, \dots, d+1\}$), $\text{supp}(t_0 | I_{n,t})$ is given by $\{n-k, \dots, n\}$, and thus the probability that the crash will not happen next period falls from $(1-\pi)/2$ to $(1-\pi)/(k+1)$.¹⁸ Inequality (10) does not apply to the special case with $n=1$, since, in that case, type-1 agents know their type with certainty. Still, if $\tau^{**} \geq 1$, (10) guarantees that they want to sell at time $1 + \tau^{**}$. Finally, if $t > n + \tau^{**}$, type- n agents know that at least two types (their own and type $n+1$) will sell in the current period, guaranteeing a crash at $t+1$. ■

Since, intuitively, agents should be more inclined to sell after medium than high prices, and since parameter restrictions are needed to preclude preemptive sales after high

¹⁸ In addition, the expected size of the crash increases, since the equivalent of (10) for $k \in \{2, 3, \dots, d+1\}$ is

$$1 > \frac{\left(\frac{G}{R} \right)^{-(\tau^{**}+k+1)} + \dots + \left(\frac{G}{R} \right)^{-(\tau^{**}+2)} + \left[\pi \left(\frac{G}{R} \right)^{-(\tau^{**}+1)} + (1-\pi) \left(\frac{G}{R} \right) \right]}{j+1}.$$

prices, it is not surprising that more restrictions are needed to rule out preemptive sales after medium prices. Indeed, *Lemma 5* rules out such sales only if $\pi G/R > 1$.

Lemma 5 *Suppose that $t < n + \tau^{**}$ and that p_{t-1}/p_{t-2} is medium. If $\pi G/R > 1$, there exist a threshold $\bar{d} > 0$ such that if $\tau^* - \tau^{**} \geq \bar{d}$, type- n agents find it optimal not to sell at t .*

Proof See appendix B.

Once more, I sketch the main ideas here and provide details in Appendix B. The proof proceeds by finding, among all the possible scenarios with $t < n + \tau^{**}$ and a medium p_{t-1}/p_{t-2} , the one where type- n agents are most tempted to sell. Then, I derive conditions under which, even in that worst-case scenario, type- n agents choose to wait. In appendix B, I show that type- n agents are most tempted to sell preemptively when $t = n + \tau^{**} - 1$, $\tau^{**} \geq 1$, p_{t-1}/p_{t-2} is medium, and the $d+1$ price ratios immediately prior to p_{t-1}/p_{t-2} are all high. For these price histories, $\text{supp}(t_0 | I_{n,t})$ contains the $d+3$ elements $\{n-(d+2), \dots, n\}$. In the first $d+1$ cases, i.e. if $n-(d+2) \leq t_0 \leq n-2$, agents with signal $n-2$ or earlier have been delaying their sales because of high price ratios preceding p_{t-1}/p_{t-2} . Since p_{t-1}/p_{t-2} is intermediate, they will sell at time t together with agents of type $n-1$, causing a crash at $t+1$. If $t_0 = n-1$, only agents of type $n-1$ will sell at t , and the bubble will burst at $t+1$ with probability π , and at $t+2$ with probability $1-\pi$. Since, in all of the $d+2$ cases with $t_0 < n$, it would be a good idea to sell preemptively, for waiting to be optimal, the expected payoff W_d for the case $t_0 = n$ must be so large that

$$1 < \frac{\left(\frac{G}{R}\right)^{-(\tau^{**}+2)} + \dots + \left[\pi \left(\frac{G}{R}\right)^{-(\tau^{**}+1)} + (1-\pi) \left(\frac{G}{R}\right) \right] + W_d}{d+3}. \quad (11)$$

W_d depends on G/R and on how long the bubble keeps growing after t . Potentially, this could be up to $d+1$ periods. However, in expected value, the bubble will burst sooner than that, since every period from t to $t+d+1$, there is a probability π that prices will be high and growth will continue, and a probability $1-\pi$ that prices will be intermediate, trigger sales, and cause a crash. Taking this into account, W_d is given by

$$W_d = (1-\pi) \frac{G}{R} \left(1 + \pi \frac{G}{R} + \left(\pi \frac{G}{R} \right)^2 + \dots + \left(\pi \frac{G}{R} \right)^d \right) + \left(\pi \frac{G}{R} \right)^{d+1}. \quad (12)$$

The crucial element of the proof is that, if $\pi G/R > 1$, W_d grows exponentially as d increases. This ensures that (11) will hold if d is big enough. To see this, note that the right-hand-side of (11) is decreasing in τ^{**} . Thus, if (11) holds in the limit as τ^{**} approaches infinity, it also holds for lower τ^{**} . For large τ^{**} , (11) is approximated by

$$d + 3 < (1-\pi) \left(\frac{G}{R} \right) + W_d.$$

If $\pi G/R > 1$, as d increases, the left-hand side grows linearly, and the right-hand side exponentially. Hence, there is a threshold value \bar{d} such (11) holds whenever $d \geq \bar{d}$.

In sum, *Lemmas 2* and *4* require no restrictions beyond $G/R < \Gamma$ in order to ensure that agents want to sell when (7) dictates so. In contrast, by *Lemmas 3* and *5*, agents will, in some situations, refuse to wait as prescribed by (7), unless $d < \kappa N - 2$, (9), and (11) hold. It remains to see whether all conditions can be simultaneously satisfied, which is not obvious, given that (9) tends to hold when d and π are small, while (11) tends to hold in the opposite case. Fortunately, *Proposition 1* establishes that the inequalities are compatible. In fact, for parameters within a given region, bubbles arise for arbitrarily large values of τ^* and τ^{**} .

Proposition 1 *Suppose that $\pi \in (1/2, 1)$, $G/R < \Gamma$, $d < \kappa N - 2$, $1 < G/R(1 - \pi/2)$ and $\pi G/R > 1$. If agents' strategies are given by (7), and $\pi \in (1/\Gamma, (2\Gamma - 2)/\Gamma)$, there exist values of G/R for which bubbles of arbitrary duration arise. Precisely, for parameters in this region, given any integer $w > 0$, there exist equilibria with $\tau^* > \tau^{**} > w$.*

Proof Given that $G/R < \Gamma$, (8) and (10) hold for all $\tau^* \geq 1$ and $\tau^{**} \geq 0$. Since $1/\pi < G/R$, there exists a \bar{d} such that (11) holds for all $d \geq \bar{d}$. And if $2/(2 - \pi) \leq G/R$, (9) holds for all $\tau^* \geq 0$. Combining these inequalities, we get

$$\max\{1/\pi, 2/(2 - \pi)\} < G/R < \Gamma. \quad (13)$$

The set of pairs $(G/R, \pi)$ that satisfies (13) is not empty, given that

$$\max \left\{ \frac{1}{\pi}, \frac{2}{2-\pi} \right\} = \begin{cases} \frac{1}{\pi} & \text{if } \frac{1}{2} \leq \pi \leq \frac{2}{3} \\ \frac{2}{2-\pi} & \text{if } \frac{2}{3} \leq \pi \leq 1, \end{cases}$$

and thus, for $\pi \in (1/\Gamma, (2\Gamma - 2)/\Gamma) \approx (0.618, 0.764)$, $\max\{1/\pi, 2/(2-\pi)\}$ is below Γ . ■

For any pair $(G/R, \pi)$ satisfying (13), bubbles can be constructed as follows. Let \bar{d} be the first integer for which $d + 3 < (1 - \pi)(G/R) + W_d$. Given κ , N can always be made so large that $\bar{d} < \kappa N - 2$. Also, π and N imply a value of $\bar{\varepsilon}$, namely $\bar{\varepsilon} = (2\kappa N)^{-1}$. Because $2/(2-\pi) \leq G/R$, τ^* can grow indefinitely without ever violating (9), and thus, for each d with $\bar{d} \leq d < \kappa N - 2$, there are as many equilibria as integers $k \geq 0$, since all pairs $(\tau^{**}, \tau^*) = (k, k + d)$ satisfy equilibrium requirements.

Note that, while τ^* can be arbitrarily large, this does not mean that the equilibria constructed above rely on the possibility that bubbles may last forever. There is always a well-defined maximum duration of the bubble, which is $\tau^* + 2$ periods. An exogenous maximum duration of the bubble $\bar{\tau}$ could be added to the model, and would have no effect as long as it was greater than $\tau^* + 2$.

It should also be noted that, as previously mentioned, multiplicity is unavoidable. For the parameter region described in Proposition 1, there are equilibria with very long bubbles, equilibria with very short bubbles, and anything in between. For other parameter values, it is possible to reduce the extent of multiplicity by lowering the maximum τ^* that can be supported in equilibrium. For example, if G/R is below $2/(2-\pi)$, (9) only holds for $\tau^* \leq -[\ln(2 - (2-\pi)G/R) - \ln \pi] / \ln(G/R) - 1$. Unfortunately, one cannot rule out equilibria where agents sell immediately, or very soon after observing the signal. Thus, $(\tau^{**}, \tau^*) = (0, 1)$ is an equilibrium whenever $G/R < \Gamma$, and for many combinations of parameter values $(\tau^{**}, \tau^*) = (0, 0)$ is also an equilibrium.

To illustrate these sets of equilibria, consider an example with $G/R = 1.6$, $N = 100$, $\bar{\varepsilon} = 1/150$ (implying $\pi = 3/4$), and $\kappa = 1/4$. For these values, $(\tau^{**}, \tau^*) = (k, k + d)$, with $k \geq 0$ and $8 \leq d < 23$ satisfy all the relevant inequalities. Further, $(\tau^{**}, \tau^*) = (0, j)$, for all $j \in \{0, 1, \dots, d - 1\}$ are also equilibria. These are parameters for which bubbles can be short,

long, or intermediate. A maximum τ^* exists if $\bar{\varepsilon}$ is reduced by a little bit, so that $\pi = 0.75 + 10^{-13}$. Then, (9) only holds for $\tau^* \leq 61$, and no equilibria with $\tau^* > 61$ exist.

In conclusion, noise greatly increases the model's ability to generate bubbles. With $G/R < \Gamma$, the only equilibrium in *Example 1* is one where agents sell as soon as they see the signal, whereas in *Example 2*, even though noise can only hide sales by one type, bubbles of unbounded duration arise.¹⁹

4.3 Re-entering the Market after Selling

Thus far, only trigger strategies have been admissible. This has been useful to simplify the analysis, but is unrealistic. Except for markets such real estate, where high transaction costs preclude frequent trading, in other asset markets, such as stock markets and foreign exchange markets, frequent trading is very common.

Moreover, restricting attention to trigger strategies may not be innocuous. For instance, in the situation captured by (11), a type- n agent weighs selling vs. waiting at time $t = n + \tau^{**} - 1$, having observed a medium p_{t-1}/p_{t-2} , and $d+1$ consecutive high price ratios $p_{t-2}/p_{t-3}, \dots, p_{t-(d+2)}/p_{t-(d+3)}$ before. Waiting at t implies getting the post-crash price with probability $(d+1+\pi)/(d+3)$, gaining one more period of appreciation with probability $(1-\pi)/(d+3)$, and gaining W_d with probability $1/(d+3)$. In (11), waiting is optimal because, even though selling is likely to avoid losses, selling also implies potentially missing out on the huge gain W_d . Allowing reentry reduces this opportunity cost, and may tilt the balance in favor of selling preemptively, since a type- n agent who sold at t and then saw that p_t/p_{t-1} was high, could reenter the market at $t+1$, hence foregoing only part of W_d . In this case, which would happen with probability $\pi/(d+3)$, she would earn an expected discounted payoff W_{d-1} . Thus, the equivalent of (11) for allowed reentry would be

¹⁹ It is interesting to note that, if noise cannot even hide one sale, i.e. if $1/N > 2\bar{\varepsilon}$, it is impossible to build a version of *Example 2* where agents wait τ^* if $\varepsilon_i \in [\bar{\varepsilon}(1-2\pi), \bar{\varepsilon}]$ and τ^{**} if $\varepsilon_i \in [-\bar{\varepsilon}, \bar{\varepsilon}(1-2\pi))$, i.e. wait τ^* periods if prices are in the top π fraction of $[-\bar{\varepsilon}, \bar{\varepsilon}]$ and τ^{**} for prices in the lower $1-\pi$ fraction of $[-\bar{\varepsilon}, \bar{\varepsilon}]$. Attempts to generate bubbles in this fashion would fail, because type- n agents' decisions at time $n + \tau^* - 1$, would be governed by (6) instead of (9), and thus, τ^* could only be zero.

$$d + 3 - \pi + \pi W_{d-1} < \left(\frac{G}{R}\right)^{-(\tau^{**}+2)} + \cdots + \left[\pi \left(\frac{G}{R}\right)^{-(\tau^{**}+1)} + (1-\pi) \left(\frac{G}{R}\right) \right] + W_d. \quad (14)$$

Like (11), (14) holds for all τ^{**} if it holds in the limit as τ^{**} approaches infinity, in which case it can be rewritten as

$$d + 3 - \pi < (1-\pi) \left(\frac{G}{R}\right) + W_d - \pi W_{d-1},$$

which, using (12), is equivalent to

$$d + 3 - \pi < (1-\pi) \left(\frac{G}{R}\right) + (\pi G)^d (G - \pi) + (1-\pi)^2 G \frac{(\pi G)^d - 1}{\pi G - 1}. \quad (15)$$

As in (11), the left-hand side grows linearly with d and if $\pi G/R > 1$, the right-hand side grows exponentially. Thus, there is a positive integer $\bar{d} > \bar{d}$ such that (15), and hence (14), hold for all $d \geq \bar{d}$. However, once reentry is possible, equilibria with $\bar{d} \leq d < \bar{d}$ vanish.²⁰

Beyond the fact that the minimum d is lengthened from \bar{d} to \bar{d} , there are no new requirements that equilibria with bubbles must satisfy once reentry is allowed. Agents who, following (7), sell at $t = t^*$, never want to reenter because they know that the bubble will burst in one or two periods. Also, once reentry is allowed, it is still optimal to wait at $t = n + \tau^* - j$ (with $j \geq 1$) following a high price ratio. (See appendix C for details.)

In conclusion, reentry makes a quantitative difference, but not a qualitative one. The mechanism protecting bubbles from preemptive sales under forbidden reentry continues to work, and bubbles with arbitrarily large τ^* still arise. In the example with $G/R = 1.6$, $N = 100$, $\pi = 3/4$, and $\kappa = 1/4$, $\bar{d} = 16$, pairs $(\tau^{**}, \tau^*) = (k, k + d)$, with $k \geq 0$ and $16 \leq d < 23$ satisfy all inequalities, but equilibria from *Example 2* with large τ^{**} and $8 \leq d < 16$ vanish, as they are not “reentry proof”.

5 A More General Specification of Noise

The equilibria presented thus far feature simple strategies, and generate bubbles in an environment that, given its minimal amount of noise, is arguably the least favorable for the emergence of bubbles. But minimal-noise environments are special in the sense that sales

²⁰ Note that (15) keeps type- n agents from selling at $t = n + \tau^{**} - j$, with medium p_{t-1}/p_{t-2} also in cases with $j > 1$ and/or less than $d + 1$ high price ratios before p_{t-1}/p_{t-2} . Both of these changes decrease the probability of a crash at $t + 1$, making preemptive selling less tempting.

by one type have a large impact on the probability of a crash. This constitutes an important departure from the continuous-time AB model, where each type has measure zero. Extending the model to allow for more noise is thus desirable in order to reduce the weight of each type. Moreover, increasing $\bar{\varepsilon}$ makes it possible to relax some of the more stringent parameter restrictions previously needed. In particular, in section 4, long and reentry-proof bubbles arise only if $G/R > 3/2$, which seems implausible unless periods are quite long. The extension of the model is also of interest because it shows a new class of equilibria with strategies that condition behavior on more than the last price ratio, but are still tractable, because the price history is summarized by just one variable. Moreover, the degree of equilibrium multiplicity in the extended model is less than in the minimal-noise case, where, even for parameters imposing no upper bound on bubble duration, the equilibrium where agents sold as soon as they got the signal could not be ruled out. In the analysis I present next, there will be parameter values for which, in equilibrium, bubble durations below a certain (positive) threshold can be ruled out.

From here onwards, I assume that $2\bar{\varepsilon} = (\bar{z} + 1)/N$, so that noise can hide sales of up to \bar{z} types.^{21 22} Previous price categories *high*, *medium*, *low* must be extended, since a price ratio p_t/p_{t-1} will now fall into one of $\bar{z} + 2$ different categories $0, 1, \dots, \bar{z} + 1$. If p_t/p_{t-1} is consistent with anywhere from 0 to z types (for $z \in \{0, \dots, \bar{z}\}$) having sold by the end of time t , this is denoted by $c(t) = z$ and if p_t/p_{t-1} reveals that sales have begun, this is denoted by $c(t) = \bar{z} + 1$. More precisely, the function $c : \{1, 2, \dots\} \rightarrow \{0, \dots, \bar{z} + 1\}$ is defined by:

$$c(t) = \begin{cases} z & \text{if } G + \alpha(\bar{\varepsilon} - (z+1)/N) < p_t/p_{t-1} \leq G + \alpha(\bar{\varepsilon} - z/N) \text{ for } z \in \{0, \dots, \bar{z}\} \\ \bar{z} + 1 & \text{if } p_t/p_{t-1} \leq G - \alpha\bar{\varepsilon}. \end{cases} \quad (16)$$

Unfortunately, a straightforward extension of (7) with $\bar{z} + 1$ waiting times $\tau_0^*, \dots, \tau_{\bar{z}}^*$ (and $\tau_z^* \geq \tau_{z+1}^* \geq 0$ for $z \in \{0, \dots, \bar{z} - 1\}$) is unlikely to be optimal. Such a strategy would have type- n agents sell as soon as $c(t-1) = \bar{z} + 1$ or $c(t-1) = z$ and $t \geq n + \tau_z^*$ for some $z \in \{0, \dots, \bar{z}\}$. Thus, at time $t_0 + \tau_0^*$, with $c(t-1) = 0$, type- t_0 agents would be

²¹ I assume that \bar{z} is an integer. Removing this assumption complicates formulae without adding insight. Also, it is technically possible that $\bar{z} + 1$ types might sell and the price ratio would not fall below, but instead equal $G - \alpha\bar{\varepsilon}$. However, I ignore this, because it is a measure zero event.

²² To rule out crashes before prices reveal that sales have begun, I also assume that $(\bar{z} + 1)/N < \kappa$.

supposed to sell, knowing that no other types would be selling. With the likelihood of a crash at $t+1$ given by $1/(\bar{z}+1)$, for large \bar{z} , selling would probably be suboptimal. This contrasts with *Example 2*, where, if a type- t_0 agent knew that other type- t_0 agents were selling at t , she willingly went along with the sale, since the probability that p_t/p_{t-1} was low was more than fifty percent.²³ Strategies conditioning sales only on the most recent price ratio are thus unlikely to be followed, as agents only want to sell at t if a crash at $t+1$ is likely. Hence, I propose strategies where agents sell if sales have begun, or if it has been τ^* periods since observing the signal and $\bar{z}+1$ types could be out of the market by the end of the period.^{24 25}

Example 3 *The strategy of a type- n agent is:*

$$h_{n,t}(h_{n,t-1}, I_{n,t}) = \begin{cases} 1 & \text{for all } t \in \{1, 2, \dots, t^*-1\} \\ 0 & \text{for all } t \geq t^*, \end{cases} \quad (17)$$

with $t^* \equiv \min\{t_1, t_2\}$, $t_1 \equiv \min\{t \geq 2 \mid c(t-1) = \bar{z}+1\}$, and

$t_2 \equiv \min\{t \geq 1 \mid t \geq n + \tau^* \text{ and, given } I_{n,t}, \bar{z}+1 \text{ types could be out of the market by the end of } t\}$.

To determine whether $\bar{z}+1$ types could be out of the market by the end of period t , agents of all types imagine that they are type- m agents, with $m = t - \tau^*$, and count the number of elements in $\text{supp}(t_0 \mid I_{m,m+\tau^*})$. (Agents do this even if their signal implies that type- m agents cannot exist.) Let the number of elements in this support be denoted by x_t . If x_t equals $\bar{z}+1$, from the point of view of type- m agents, types $m - \bar{z}, \dots, m$ could be selling at t , and thus, type- m agent would sell. If $x_t < \bar{z}+1$, type- m agents would know that $\bar{z}+1$ could not possibly be out by the end of time t , and thus, they would wait. x_t plays a crucial role in coordinating sales, since it summarizes the price history in the same way for all types.

²³ Another contingency where type- n agents may not want to follow this strategy is if $t = n + \tau^*$, $c(t-1) = \bar{z}$, $c(t-2) = 0$, and $c(t-3) = \bar{z}$. In this circumstance, type- n agents would know that at most two types, n and $n-1$, could be selling at t and again, for \bar{z} large enough, the probability of a crash at $t+1$ would be low.

²⁴ For tractability reasons, I will restrict attention to the case where reentry is not allowed. Intuitively, I speculate that allowing for reentry would, as in section 4.3, change results quantitatively but not qualitatively. However, I have not allowed for reentry in this example.

²⁵ Later in the section, I will show that this strategy profile can be an equilibrium only if $\bar{z} \leq 42$. I will explain why, and in footnote 30, suggest an extension that might generate equilibria for any \bar{z} .

To illustrate how x_t evolves as prices are observed, it is useful to consider a simple example. Let $\bar{z} = 2$, $\kappa = 1/3$, $N = 10$, $t_0 = 20$, $\tau^* = 10$, and suppose that at $t = 29$, $x_{29} = 3$. Since $x_{29} = 3$, all types think that, if type-19 agents exist, $\text{supp}(t_0 | I_{19,29}) = \{17, 18, 19\}$. Following (17), type-20 agents wait and see what p_{29} / p_{28} reveals. Since there are no sales at $t = 29$, one of the following three (equiprobable) continuations will follow:

1. If $c(29) = 0$, at $t = 30$ everybody knows that nobody sold at $t = 29$. Type-20 agents learn that t_0 is 20, and x_{30} is 1. There are no sales $t = 30$ or $t = 31$, since for any $c(30) \in \{0, 1, 2\}$, type-21 agents have $\text{supp}(t_0 | I_{21,31}) = \{20, 21\}$, implying $x_{31} = 2$. At $t = 32$, regardless of whether $c(31)$ is 0, 1, or 2, $\text{supp}(t_0 | I_{22,32}) = \{20, 21, 22\}$, and thus $x_{32} = 3$. Types 20, 21 and 22 sell at $t = 32$ and the bubble bursts at $t = 33$.
2. If $c(29) = 1$, at $t = 30$, agents think that, either one type or none sold at $t = 29$, implying $\text{supp}(t_0 | I_{20,30}) = \{19, 20\}$, and $x_{30} = 2$. Type-20 agents wait, and from here:
 - a. If $c(30) = 0$, agents learn that nobody sold at $t = 29$. Type-21 agents have $\text{supp}(t_0 | I_{21,31}) = \{20, 21\}$. With $x_{31} = 2$, as in (1), there are no sales at $t = 31$, types 20, 21 and 22 sell at $t = 32$, and the bubble bursts at $t = 33$.
 - b. If $c(30) = 1$ or $c(30) = 2$, type-21 agents have $\text{supp}(t_0 | I_{21,31}) = \{19, 20, 21\}$. Thus, $x_{31} = 3$, and types 20 and 21 sell at $t = 31$. With probability $2/3$, this causes a crash at $t = 32$, and with probability $1/3$, $c(31)$ is 2, $x_{32} = 3$, and type-22 agents sell at $t = 32$, causing a crash at $t = 33$.
3. If $c(29) = 2$, at $t = 30$, all types think that 0, 1, or 2 types could have sold at $t = 29$. Thus, $\text{supp}(t_0 | I_{20,30}) = \{18, 19, 20\}$, and $x_{30} = 3$. Type-20 agents sell. Then:
 - a. If $c(30) = 1$, $\text{supp}(t_0 | I_{21,31}) = \{20, 21\}$, implying $x_{31} = 2$, so type-21 agents do not sell at $t = 31$. The bubble bursts at $t = 32$ with probability $1/3$ (since type 20 has sold, $c(31)$ could be 3), and with probability $2/3$, $c(31) \in \{1, 2\}$ and $x_{32} = 3$, making types 21 and 22 sell at $t = 32$ and causing a crash at $t = 33$.

- b. If $c(30) = 2$, $x_{31} = 3$, and type-21 agents sell at time 31. This causes a crash at $t = 32$ with probability $2/3$, while with probability $1/3$, $c(31)$ is 2, x_{32} is 3, type-22 agents sell at $t = 32$ and the bubble bursts at $t = 33$.
- c. If $c(30) = 3$, the bubble bursts at $t = 31$.

In sum, x_t is determined by $L(t) \equiv \max\{s \mid s < t \text{ and } x_s = \bar{z} + 1\}$, i.e., the last time that x was $\bar{z} + 1$, and prices since then $p_{L(t)} / p_{L(t)-1}, \dots, p_{t-1} / p_{t-2}$. Precisely,

$$x_t = \underline{c}(t-1) + t - L(t), \quad (18)$$

where $\underline{c}(t-1) = \min\{c(s) \mid L(t) \leq s \leq t-1\}$ is the maximum number of types that (given prices observed between $L(t)$ and t) could have left the market by the end of period $L(t)$, and $t - L(t)$ is how long it has been since the last time that x was $\bar{z} + 1$.²⁶ If $c(L(t)) = 0$, $p_{L(t)} / p_{L(t)-1}$ reveals that nobody sold at or before time $L(t)$, implying $x_{L(t)+1} = 1$. In that case, regardless of the next price ratios, x grows by one every period until time t . In general, for $c(L(t)) = z$, $x_{L(t)+1}$ is $z + 1$. If $c(s) \geq z$ for $L(t) + 1 \leq s \leq t - 1$, x grows by one every period until time t , but if for some $s \in \{L(t) + 1, \dots, t - 1\}$, $c(s) = y < z$, prices reveal that z types could not have sold by time $L(t)$, and \underline{c} is adjusted downward to y .

Optimality conditions for *Example 3* are simple. An equilibrium for a certain τ^* exists if G/R exceeds a lower bound $\underline{\gamma}(\tau^*)$, to entice agents to wait when they should wait, without exceeding an upper bound $\bar{\gamma}(\tau^*)$, as excessive potential gains would make agents wait when they should sell.

To determine $\bar{\gamma}(\tau^*)$, note that, among situations where type- n agents shall sell, waiting is most tempting if $t = n + \tau^*$ and $x_t = \bar{z} + 1$.²⁷ In this circumstance, a type- n agent who sells along with other type- n agents earns p_t , whereas if she deviates from (17) and waits, she obtains an expected discounted payoff which I label $p_t F(0, \bar{z} + 1)$. $\bar{\gamma}(\tau^*)$ is the

²⁶ x_t is well defined by (18) only if past values of x pin down $L(t)$ and $\underline{c}(t)$. Where there are no past values, I set $x_t = t$ for $t = 1, \dots, \bar{z} + 1$, and let (18) define x_t for $t \geq \bar{z} + 2$.

²⁷ If $x_{n+\tau^*} < \bar{z} + 1$, type- n agents delay selling until $x_{n+\tau^*+q} = \bar{z} + 1$, which will happen for some $q \in \{1, \dots, \bar{z}\}$. When selling at time $n + \tau^* + q$, type- n agents know with certainty that, in addition to them, q other types $n + 1, \dots, n + q$ are selling. The higher q , the higher the likelihood of a crash at $n + \tau^* + q + 1$, and the lower the number of periods that the bubble can keep growing.

value of G/R for which $F(0, \bar{z} + 1)$ equals one. $F(0, \bar{z} + 1)$ is computed using a backward-induction algorithm (see appendix D), which uses the fact that, in the best-case scenario for a deviating type- n agent, she would sell at time $n + \tau^* + \bar{z}$, earning a (discounted) payoff $p_t(G/R)^{\bar{z}}$. If the bubble survives up to this point, $x_{n+\tau^*+\bar{z}}$ can only be $\bar{z} + 1$. Moving back as $j = \bar{z}, \dots, 2$, the algorithm uses payoffs at $n + \tau^* + j$ for $x_{n+\tau^*+j} \in \{j+1, \dots, \bar{z} + 1\}$, to find payoffs at $n + \tau^* + j - 1$ for $x_{n+\tau^*+j-1} \in \{j, \dots, \bar{z} + 1\}$, taking into account that at each time and for each x , the deviating type- n agent would make an optimal sell-or-wait choice. Finally, $F(0, \bar{z} + 1)$ is a weighted average of payoffs at $n + \tau^* + 1$ for $x_{n+\tau^*+1} \in \{2, \dots, \bar{z} + 1\}$ and payoffs if the bubble bursts at $n + \tau^* + 1$.

$\tau^* \setminus \bar{z}$	1	10	20	37	38	42	43
1	3.90057	1.31948	1.18558	1.114090	1.111689	1.103107	1.101184
10	4	1.35242	1.19838	1.119004	1.116394	1.107095	1.105019
25	4	1.35429	1.20079	1.120955	1.118312	1.108887	1.106780
50	4	1.35431	1.20095	1.121331	1.118697	1.109300	1.107197
100	4	1.35431	1.20095	1.121354	1.118721	1.109332	1.107233
500	4	1.35431	1.20095	1.121354	1.118721	1.109332	1.107233

Table 1. $\bar{\gamma}(\tau^*)$ for selected values of \bar{z} and τ^* .

Numerical results, displayed in Table 1 for selected \bar{z} and τ^* , show that $\bar{\gamma}(\tau^*)$ is increasing in τ^* .²⁸ Moreover, $\bar{\gamma}(\tau^*)$ is concave and converges to a value $\bar{\gamma}(\infty)$ as τ^* approaches infinity. Let $\bar{\gamma}(500) \approx \bar{\gamma}(\infty)$, since, for any $\bar{z} \leq 50$, $\bar{\gamma}(10^6) - \bar{\gamma}(500) < 10^{-15}$.

A similar procedure determines $\underline{\gamma}(\tau^*)$, the lowest G/R such that type- n agents wait when they should. While $t < n + \tau^*$, preemptive selling is most tempting at $n + \tau^* - 1$ with $x_{n+\tau^*-1} = \bar{z} + 1$.²⁹ A type- n agent in this position chooses selling, which yields p_t ,

²⁸ To see why, note that $F(0, \bar{z} + 1)$ decreases with τ^* and increases with G/R . The former is obvious, since a rise in τ^* only increases potential losses. But a rise in G/R increases both gains and losses, so $F(0, \bar{z} + 1)$ could conceivably be fall with G/R , especially for low τ^* . To ensure that $\bar{\gamma}(\tau^*)$ is correct, I verify that for G/R near $\bar{\gamma}(\tau^*)$, $F(0, \bar{z} + 1)$ rises with G/R , and that $F(0, \bar{z} + 1) > 1$ if $G/R > \bar{\gamma}(\tau^*)$.

²⁹ At time $t = n + \tau^* - j$ with $x_t = \bar{z} + 1$, $\text{supp}(t_0 | I_{n,t})$ is given by $\{n - j - (\bar{z} + 1), \dots, n - j, \dots, n\}$. If t_0 is greater than $n - j$, nobody will sell at time t , and the bubble cannot possibly burst at time $t + 1$. Thus, the greater j the lower the incentive to sell preemptively.

versus waiting, which yields $p_t \hat{F}(-1, \bar{z} + 1)$. As before, $\hat{F}(-1, \bar{z} + 1)$ is computed via backward induction (see Appendix D for details). But there is a major difference in how $\bar{\gamma}(\tau^*)$ and $\underline{\gamma}(\tau^*)$ are defined, because, when computing $\hat{F}(-1, \bar{z} + 1)$, payoffs at times $n + \tau^* + j$ for $j \in \{0, \dots, \bar{z}\}$ and $x_{n+\tau^*+j} \in \{j, \dots, \bar{z} + 1\}$ are on the equilibrium path. Thus, agents shall wait, not just at $n + \tau^* - 1$ with $x_{n+\tau^*-1} = \bar{z} + 1$, but also at times $n + \tau^* + j$ with $j + 1 \leq x_{n+\tau^*+j} \leq \bar{z}$ for $j = \{0, \dots, \bar{z} - 1\}$. $\underline{\gamma}(\tau^*)$ is therefore G/R such that the smaller of $\hat{F}(-1, \bar{z} + 1)$ and $\min\{\hat{F}(j, x_{n+\tau^*+j}) \mid 0 \leq j \leq \bar{z} - 1, j + 1 \leq x_{n+\tau^*+j} \leq \bar{z}\}$ equals one, where $p_{n+\tau^*+j} \hat{F}(j, x_{n+\tau^*+j})$ is the payoff of a type- n who waits at time $n + \tau^* + j$ with $x_{n+\tau^*+j}$.

$\tau^* \setminus \bar{z}$	1	10	20	37	38	42	43
1	1.40037	1.17391	1.14186	1.113140	1.111907	1.107233	1.106158
10	1.64398	1.21141	1.15104	1.114914	1.113535	1.108430	1.107264
25	1.64575	1.21674	1.15365	1.115708	1.114277	1.109011	1.107809
50	1.64575	1.21701	1.15396	1.115880	1.114441	1.109151	1.107943
100	1.64575	1.21701	1.15397	1.115892	1.114453	1.109162	1.107954
500	1.64575	1.21701	1.15397	1.115892	1.114453	1.109162	1.107954

Table 2. $\underline{\gamma}(\tau^*)$ for selected values of \bar{z} and τ^* .

Results for $\underline{\gamma}(\tau^*)$, displayed in Table 2, show that $\underline{\gamma}(\tau^*)$ is also an increasing and concave function of τ^* , converging in the limit to $\underline{\gamma}(\infty)$. Again, I let $\underline{\gamma}(500) \approx \underline{\gamma}(\infty)$.

Comparing tables 1 and 2, it becomes clear that, as \bar{z} grows, $\bar{\gamma}(\tau^*)$ falls faster than $\underline{\gamma}(\tau^*)$. To understand why, let $\tilde{\gamma}(\tau^*)$ denote G/R such that $\hat{F}(-1, \bar{z} + 1)$ equals one, and observe that, for low values of \bar{z} , selling preemptively is most tempting at $n + \tau^* - 1$ with $x_{n+\tau^*-1} = \bar{z} + 1$, and thus, while \bar{z} is low, $\tilde{\gamma}(\tau^*) = \underline{\gamma}(\tau^*)$ for any τ^* . Since the gains an agent might forego by selling preemptively at $n + \tau^* - 1$ grow quickly with \bar{z} , $\tilde{\gamma}(\tau^*)$ falls rapidly. But for \bar{z} above a certain threshold (12 or 13, depending on τ^*), $\underline{\gamma}(\tau^*)$ no longer equals $\tilde{\gamma}(\tau^*)$. That is, as \bar{z} grows, incentives to sell preemptively at some t between $n + \tau^*$ and $n + \tau^* + \bar{z} - 1$ with $x_t = \bar{z}$ become stronger than those at $n + \tau^* - 1$ with $x_{n+\tau^*-1} = \bar{z} + 1$. For example, if $\bar{z} = 50$, selling preemptively is most tempting at

$n + \tau^* + 31$ with $x_{n+\tau^*+31} = 50$, because, even though the probability of a crash in this situation is less than at $n + \tau^* - 1$ with $x_{n+\tau^*-1} = \bar{z} + 1$, 19 periods of potential gains are less enticing than 50. Since $\bar{\gamma}(\tau^*)$ falls faster than $\underline{\gamma}(\tau^*)$, the following equilibria arise:

- I. If $\bar{z} \leq 37$, $\bar{\gamma}(\tau^*)$ exceeds $\underline{\gamma}(\tau^*)$ for any $\tau^* \geq 1$. That is, for any $\tau^* \geq 0$, there exists a G/R such that all equilibrium conditions are met.
- II. For $38 \leq \bar{z} \leq 42$, it is only possible to have equilibria for τ^* above a certain threshold, since $\bar{\gamma}(1)$ no longer exceeds $\underline{\gamma}(1)$, but $\bar{\gamma}(\infty)$ does exceed $\underline{\gamma}(\infty)$.
- III. If $\bar{z} \geq 43$, $\bar{\gamma}(\tau^*) < \underline{\gamma}(\tau^*)$ for all $\tau^* \geq 1$, ruling out bubbles altogether.³⁰

Given parameters $(\bar{z}, G/R)$, multiple values of τ^* satisfy equilibrium conditions. While $\bar{z} \leq 34$, $\underline{\gamma}(\infty) < \bar{\gamma}(1)$, and thus, for G/R between $\bar{\gamma}(1)$ and $\bar{\gamma}(\infty)$, there is a lower, but no upper bound on τ^* . For G/R between $\underline{\gamma}(1)$ and $\underline{\gamma}(\infty)$, there is an upper, but no lower bound on τ^* , and for G/R between $\underline{\gamma}(\infty)$ and $\bar{\gamma}(1)$, any $\tau^* \geq 0$ meets equilibrium conditions. If $35 \leq \bar{z} \leq 42$, $\bar{\gamma}(1) < \underline{\gamma}(\infty)$, and for G/R between these two values, there are both upper and lower limits on τ^* .³¹ If $\underline{\gamma}(\infty) < G/R < \bar{\gamma}(\infty)$, there is only a lower bound on τ^* , and for G/R in $(\underline{\gamma}(1), \bar{\gamma}(1))$ (a nonempty interval only if $35 \leq \bar{z} \leq 37$), there is only an upper limit on τ^* . Since, for any $\bar{z} \leq 42$, there are values of G/R such that τ^* below a certain positive threshold are ruled out, *Example 3* rationalizes bubbles in a stronger sense than the minimal-noise case.

6. Conclusion

This paper builds on models of greater fool's bubbles (Allen, et al. (1993), Conlon (2004), and especially Abreu and Brunnermeier (2003)), extending them by introducing noisy prices, and price responsiveness to selling pressure. These features make it possible to circumvent the main critiques of these models, which assumed either that exact parameter proportions held, or that prices were to some extent independent of sales. I show that the mechanism generating bubbles in the aforementioned papers does not depend on these

³⁰ While strategies given by (17) generate bubbles only for $\bar{z} \leq 42$, I speculate that equilibria could arise for greater values of \bar{z} if (17) was modified to let agents sell when it has been τ^* periods since the signal and the number of that types that could have sold by the end of the period surpasses some percentage of $\bar{z} + 1$.

³¹ For instance, if $\bar{z} = 42$ and $G/R = 1.109159407$, in equilibrium, τ^* can only be between 34 and 63.

assumptions, since, with multidimensional uncertainty, bubbles arise even though prices are fully endogenous, and these bubbles are robust to small changes in parameters. Showing that models of bubbles are robust should contribute towards increasing their use to study applied questions, such as optimal monetary policy in the presence of bubbles.

Another potential avenue for future work is to further refine models in order to bridge some of the gaps that still separate them from data. For example, as Brunnermeier (2001) points out, in models, bubbles burst abruptly, while in reality, they often deflate gradually. A version of this model where the noisy component was not bounded might generate such a gradual decline, since, for any price growth rate, some agents would still think that prices could rebound, and after the peak, agents would become gradually convinced that the growth is over. Such an environment would probably be even more conducive towards generating bubbles, since the slower crash would reduce the threat of large sudden losses.

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APPENDIX A — Derivation of Γ

Start with (6) for $\tau^* = 1$, set $x = G/R$ and perform algebra steps as follows

$$\begin{aligned} 2 > x^{-2} + x &\Leftrightarrow 2x^2 > 1 + x^3 \Leftrightarrow 0 > 1 + x^3 - 2x^2 \Leftrightarrow 0 > 1 + x(x^2 - 2x) \\ &\Leftrightarrow 0 > 1 + x(x^2 - 2x + 1 - 1) \Leftrightarrow 0 > 1 + x(x^2 - 2x + 1) - x \Leftrightarrow x - 1 > x(x - 1)(x - 1) \end{aligned}$$

Clearly, $x = 1$ is a root of the polynomial $1 + x^3 - 2x^2$ and for $x \neq 1$, we have

$$1 > x(x - 1) \Leftrightarrow 0 > x^2 - x - 1 \Leftrightarrow \frac{1 - \sqrt{5}}{2} < x < \frac{1 + \sqrt{5}}{2}.$$

Thus, the other two roots are $(1 - \sqrt{5})/2$, and $\Gamma = (1 + \sqrt{5})/2$.

APPENDIX B — Proof of Lemmas 3 and 5

Proof of Lemma 3

To show that type- n agents do not wish to sell at time $t = n + \tau^* - j$ (with $j \geq 1$) if p_{t-1}/p_{t-2} is high, first note that the incentive for such preemptive sales comes from the possibility that t_0 could be $n - j$. If $t_0 = n - j$, type- t_0 agents would sell at time t and, with probability π , these sales would precipitate a crash at $t + 1$. If $t_0 = n - j$ can be ruled out, type- n agents know that nobody is selling at t , and have no incentive to sell because they know they can sell for a higher expected price at $t + 1$. Price histories for which $t_0 = n - j$ is either impossible or a zero-probability event can be divided into the following groups:

- (I) If at least one of the price ratios $p_{t-2}/p_{t-3}, \dots, p_{t-(d+1)}/p_{t-(d+2)}$ is medium, $t_0 = n - j$ is inconsistent with p_{t-1}/p_{t-2} being high. To see this, observe that, if $p_{t-i}/p_{t-(i+1)}$ is medium for some $i \in \{2, \dots, d+1\}$ and t_0 is $n - j$, type- t_0 agents would have sold at $t - i + 1$, and p_{t-1}/p_{t-2} could not possibly be high. In fact, only if $j = 1$ and $i = 2$, p_{t-1}/p_{t-2} could be medium. Otherwise, the bubble would have burst before time t , or p_{t-1}/p_{t-2} would be low, precipitating the crash at t .
- (II) Even if all of the $d + 1$ ratios $p_{t-1}/p_{t-2}, \dots, p_{t-(d+1)}/p_{t-(d+2)}$ are high, if $j > N - 1$, it is impossible that t_0 is $n - j$, since type- n agents know that t_0 cannot be smaller than $n - (N - 1)$.

- (III) If all of the $d + 1$ ratios $p_{t-1}/p_{t-2}, \dots, p_{t-(d+1)}/p_{t-(d+2)}$ are high, and $j \leq N - 1$, t_0 could be $n - j$. But if $j > \tau^*$, type- n agents have not observed the signal, and thus, since $\lambda \approx 0$, they assign zero probability to the event that $t_0 = n - j$.

Thus, a type- n agent's sell-or-wait tradeoff at time $t = n + \tau^* - j$ is nontrivial if all of the $d + 1$ price ratios $p_{t-1}/p_{t-2}, \dots, p_{t-(d+1)}/p_{t-(d+2)}$ are high, $j \leq N - 1$ and $j \leq \tau^*$. To understand the problem faced by a type- n agent in these cases, it is useful to consider the sell-or-wait trade-off for $\tau^* \geq 1$ and $j = 1$. Our type- n agent knows that, with probability $1/2$, her type was second to observe the signal, i.e., $t_0 = n - 1$, and with probability $1/2$, her type was first, i.e., $t_0 = n$. If her type was second, the first type is selling at t . This will lead to a low p_t/p_{t-1} and a crash at $t + 1$ with probability π , and with probability $1 - \pi$ to a medium p_t/p_{t-1} , in which case—given the assumption that $d + 2 < \kappa N$ —she will be able to sell at $t + 1$ having earned a high rate of return for one more period. If her type was first, nobody is selling at t , and she will, for sure, sell at $t + 1$ after one more period of price growth. Thus, for $j = 1$, waiting is optimal if inequality (9) holds.

Fortunately, even though (9) captures the simple case with $j = 1$, it is sufficient to rule out preemptive sales for arbitrary $j \geq 1$. To see why, consider a more general scenario with $t = n + \tau^* - j$, $d + 2 < \kappa N$, $j \leq \min\{\tau^*, N - 1\}$, and high ratios $p_{t-i}/p_{t-(i+1)}$ for all $i \in \{1, 2, \dots, d + 1\}$. Further, let θ_j denote the expected discounted payoff that a type- n agent obtains if she follows (7) at all times (e.g., θ_1 is the right hand-side of (9)). To show that $\theta_1 > 1$ implies $\theta_j > 1$ for all $j > 1$, I will first show that $\theta_1 > 1$ implies $\theta_2 > 1$, and then generalize. If $t = n + \tau^* - 2$, there are three values of t_0 that type- n agents cannot rule out, $n - 2$, $n - 1$, and n . If t_0 is $n - 2$, type- t_0 agents sell at time t , p_t/p_{t-1} will be low with probability π and medium with probability $1 - \pi$. If t_0 is not $n - 2$, p_t/p_{t-1} will be medium with probability $1 - \pi$ and high with probability π . In the former case, (7) dictates that type- n agents sell at $t + 1$, having gained one more period of appreciation.³² In the latter case, type- n agents will have the option to sell and enjoy that period of appreciation,

³² Assuming $d \geq 1$. The possibility that $j > d + 1$ is discussed later in the proof.

but they can also keep waiting, which will be optimal, because they will find themselves in a situation with $t+1 = n + \tau^* - 1$ and all of the last $d+1$ price ratios being high, and by assumption, $\theta_1 > 1$. Formally, we have

$$\begin{aligned} \theta_2 = & \frac{1}{3} \left[\pi \left(\frac{G}{R} \right)^{-(\tau^*+1)} + (1-\pi) \frac{G}{R} \right] \\ & + \frac{2}{3} \left\{ (1-\pi) \frac{G}{R} + \pi \left[\frac{1}{2} \left(\pi \frac{G}{R} \left(\frac{G}{R} \right)^{-(\tau^*+1)} + (1-\pi) \left(\frac{G}{R} \right)^2 \right) + \frac{1}{2} \left(\frac{G}{R} \right)^2 \right] \right\}. \end{aligned}$$

Factoring G/R out of the curly bracket and since θ_1 is the right-hand side of (9), we have

$$\begin{aligned} \theta_2 = & \frac{1}{3} \left[\pi \left(\frac{G}{R} \right)^{-(\tau^*+1)} + (1-\pi) \frac{G}{R} \right] + \frac{2}{3} \frac{G}{R} \{ (1-\pi) + \pi \theta_1 \} \\ & > \frac{1}{3} \left[\pi \left(\frac{G}{R} \right)^{-(\tau^*+1)} + (1-\pi) \frac{G}{R} \right] + \frac{2}{3} \frac{G}{R} = \frac{G}{R} \left(1 - \frac{\pi}{3} \right) + \frac{\pi}{3} \left(\frac{G}{R} \right)^{-(\tau^*+1)} > \theta_1 > 1. \end{aligned}$$

Generalizing to arbitrary j , the support of t_0 contains the $j+1$ values $\{n-j, \dots, n\}$. If $t_0 = n-j$, with probability π there will be a crash at $t+1$ and with probability $1-\pi$ the agent will gain one period of appreciation. If $t_0 = n-k$, for $k \in \{1, \dots, j-1\}$, nobody sells at t , type- n agents sell as soon as the price ratio becomes medium, and there is a chance that the bubble will burst before they sell, since if price ratios keep being high, at time $n-k + \tau^*$, agents of type $t_0 = n-k$ will sell, precipitating a crash in the next period with probability π . Finally, if $t_0 = n$, nobody sells before type n , and type- n agents will reap appreciation gains for up to j periods. Thus, for general values of j , θ_j is given by

$$\begin{aligned} \theta_j = & \frac{1}{j+1} \left[\pi \left(\frac{G}{R} \right)^{-(\tau^*+1)} + (1-\pi) \frac{G}{R} \right] + \frac{1}{j+1} \left[(1-\pi) \frac{G}{R} + \pi \frac{G}{R} \left(\pi \left(\frac{G}{R} \right)^{-(\tau^*+1)} + (1-\pi) \frac{G}{R} \right) \right] + \\ & + \dots + \frac{1}{j+1} \left[(1-\pi) \frac{G}{R} \left(1 + \pi \frac{G}{R} + \dots + \left(\pi \frac{G}{R} \right)^{j-1} \right) + \pi^j \left(\frac{G}{R} \right)^{j-1-(\tau^*+1)} \right] \\ & + \frac{1}{j+1} \left[(1-\pi) \frac{G}{R} \left(1 + \pi \frac{G}{R} + \dots + \left(\pi \frac{G}{R} \right)^{j-1} \right) + \left(\pi \frac{G}{R} \right)^j \right], \end{aligned}$$

which can be rewritten as

$$\theta_j = \frac{1}{j+1} \left[\pi \left(\frac{G}{R} \right)^{-(\tau^{*+1})} + (1-\pi) \frac{G}{R} \right] + \frac{j}{j+1} (1-\pi) \frac{G}{R}$$

$$+ \frac{j}{j+1} \pi \frac{G}{R} \frac{\left[\pi \left(\frac{G}{R} \right)^{-(\tau^{*+1})} + (1-\pi) \frac{G}{R} + \dots + (1-\pi) \frac{G}{R} \left(1 + \dots + \left(\pi \frac{G}{R} \right)^{j-2} \right) + \left(\pi \frac{G}{R} \right)^{j-1} \right]}{j}.$$

Substituting θ_{j-1} for its value yields

$$\theta_j = \frac{1}{j+1} \left[\pi \left(\frac{G}{R} \right)^{-(\tau^{*+1})} + (1-\pi) \frac{G}{R} \right] + \frac{j}{j+1} \frac{G}{R} \left\{ (1-\pi) + \pi \theta_{j-1} \right\} \quad (19)$$

And using the fact that $\theta_{j-1} > 1$ we conclude that

$$\theta_j > \frac{1}{j+1} \left[\pi \left(\frac{G}{R} \right)^{-(\tau^{*+1})} + (1-\pi) \frac{G}{R} \right] + \frac{j}{j+1} \frac{G}{R} = \frac{G}{R} \left(1 - \frac{\pi}{j+1} \right) + \frac{\pi}{j+1} \left(\frac{G}{R} \right)^{-(\tau^{*+1})} > \theta_1 > 1.$$

I have implicitly assumed $j \leq d+1$. This is innocuous, since if $j > d+1$, type- n agents will not sell at time $t+1$ even if p_t / p_{t+1} is medium. However, since they have option to sell in that circumstance, their expected payoff cannot be smaller than if they sold then. Therefore, they have no incentive to sell preemptively at t .

Finally, note that the assumption that $d+2 < \kappa N$ is crucial, since (9) is correct only if, when $t_0 = n-1$ and p_t / p_{t-1} is medium, sales of type t_0 plus the $d+1$ types that sell at $t+1$ do not surpass κ . Otherwise, type- n agents refuse to wait until $n + \tau^*$ to sell. If $d+1 \leq \kappa N < d+2$, medium prices burst the bubble if and only if $t_0 = n-1$, and thus (9) would have to be replaced by (6), which fails for all $\tau^* \geq 1$. If $d+1 > \kappa N$ agents are even less willing to wait, since the probability of a crash increases further from $1/2$ to $1-\pi/2$. ■

Proof of Lemma 5

First, I show that preemptive sales are most tempting when $t = n + \tau^{**} - 1$, $\tau^{**} \geq 1$, p_{t-1} / p_{t-2} is medium, and the $d+1$ (or more) price ratios immediately prior to p_{t-1} / p_{t-2} are all high. For these price histories, $\text{supp}(t_0 | I_{n,t})$ contains the $d+3$ elements $\{n-(d+2), \dots, n\}$. Three scenarios are possible:

I. **More than one type will have sold by the end of time t , causing a crash at $t+1$.**

- (i) If $t_0 = n - (d + 2)$, type- t_0 agents sold at $t - 1 = t_0 + \tau^*$, ε_{t-1} was high enough to make p_{t-1} / p_{t-2} intermediate, and $d + 1$ types $n - (d + 1), \dots, n - 1$ will sell at t .
- (ii) If $t_0 = n - l$, for $l \in \{2, \dots, d + 1\}$, nobody sold at $t - 1$, but a low ε_{t-1} made p_{t-1} / p_{t-2} intermediate and will trigger sales by the l types $n - l, \dots, n - 1$.

II. Only type $n - 1$ will sell at t , and the bubble will burst at time $t + 1$ with probability π . and at time $t + 2$ with probability $1 - \pi$. This happens if $t_0 = n - 1$.

III. Nobody will sell at time t , and the bubble will continue to grow, at least for one more period and at most for $d + 2$ more periods. This happens if $t_0 = n$.

In scenarios I and II, selling is preferable to waiting, and the opposite is true in III. Next, I argue that, when $\text{supp}(t_0 | I_{n,t}) = \{n - (d + 2), \dots, n\}$, the probability of III is smallest. I do this by considering, in turn, what happens when $\tau^{**} < 1$, $t = n + \tau^{**} - j$, for $j \geq 2$, or at least one of the $d + 1$ price ratios immediately prior to p_{t-1} / p_{t-2} is intermediate.

- (i) $\tau^{**} \geq 1$, implies $t \geq n$, i.e. that type- n agents have observed their signal at time $t = n + \tau^{**} - 1$. If $t < n$, type- n agents have no significant incentive to sell preemptively, because $\text{supp}(t_0 | I_{n,t}) = \{n - (d + 2), n - (d + 1), \dots\}$, and since $\lambda \approx 0$, all values are roughly equiprobable. Thus, cases I and II have probability zero, and the bubble will continue to grow with probability one.
- (ii) If $t = n + \tau^{**} - j$, for $j \geq 2$, $\text{supp}(t_0 | I_{n,t}) = \{n - (d + j + 1), \dots, n - 1, n\}$. Cases I and II occur for the same number of values of t_0 as before, but the probability that nobody will sell at t grows, since case III applies to all $t_0 \in \{n - (j - 1), n\}$.
- (iii) If the number of consecutive high ratios preceding p_{t-1} / p_{t-2} is $k < d + 1$, $\text{supp}(t_0 | I_{n,t}) = \{n - (k + 1), \dots, n\}$. Case I(a) is no longer possible, and if $k < d$, the number of possible values of t_0 for which case I(b) arises is reduced. Thus, the probability of III increases.

Having argued that incentives to sell are strongest when $\text{supp}(t_0 | I_{n,t}) = \{n - (d + 2), \dots, n\}$, it follows that, if preemptive sales are ruled out in that case, they are also ruled out in all

other scenarios with $t < n + \tau^{**}$ and intermediate p_{t-1} / p_{t-2} . If $\text{supp}(t_0 | I_{n,t}) = \{n - (d + 2), \dots, n\}$, as discussed in the text, waiting is preferable to selling if (11) holds. The expected gain W_d on the right-hand side of (11) is the average of all payoffs—weighted by their probability—that are possible if $n = t_0$ and all agents follow equilibrium strategies. If $n = t_0$, nobody sells at t , p_t / p_{t-1} is medium with probability $1 - \pi$ (triggering sales of type t_0 at $t + 1$) and high with probability π , in which case type- t_0 agents wait until next period, and again, sell if the price is medium and wait if it is high. Continuing in this fashion, type- n agents can benefit from up to $d + 1$ periods of appreciation. Thus, W_d is given by

$$W_d = (1 - \pi) \frac{G}{R} + \pi \left((1 - \pi) \left(\frac{G}{R} \right)^2 + \pi \left(\dots + \pi \left((1 - \pi) \left(\frac{G}{R} \right)^{d+1} + \pi \left(\frac{G}{R} \right)^{d+1} \right) \right) \right),$$

which, rearranging terms, can be written as (12). To see when (11) is satisfied, note that, holding d constant, the right-hand-side is a decreasing function of τ^{**} , implying that, if (11) holds in the limit as τ^{**} approaches infinity, it will also hold for lower values of τ^{**} . As τ^{**} becomes arbitrarily large, (11) approximates

$$1 < \frac{(1 - \pi) \left(\frac{G}{R} \right) + W_d}{d + 3},$$

which, if $\pi G / R \neq 1$, is the same as

$$d + 3 < (1 - \pi) \left(\frac{G}{R} \right) \left(1 + \frac{\left(\frac{\pi G}{R} \right)^{d+1} - 1}{\pi \frac{G}{R} - 1} \right) + \left(\frac{\pi G}{R} \right)^{d+1}.$$

If $\pi G / R > 1$, as d increases, the left hand side increases linearly, while the right hand side grows exponentially. This implies existence of \bar{d} such that whenever $d \geq \bar{d}$, (11) holds.

$\pi G / R > 1$ turns out to be not only sufficient, but also necessary for (11) to hold. If $\pi G / R \leq 1$, (11) fails for all τ^{**} and all d .³³ In that case, τ^{**} can only be zero, meaning

³³ Proof of this claim is available from the author upon request.

that $d = \tau^*$ and thus that (9) applies only to $\tau^* < \kappa N - 2$. Thus, if $\pi G / R \leq 1$, and agents play strategies given by (7), the overvaluation is corrected before (or at the latest at the same time as) a mass κ of rational agents become aware of it, and thus, according to the definition by AB, bubbles do not arise. ■

Appendix C – Details of Section 4.3

Here, I show that, even after reentry is allowed, (9) is still sufficient to rule out sales by type- n agents at time $t = n + \tau^* - j$ for $j \geq 1$, if p_{t-1} / p_{t-2} is high. It is only necessary to consider cases where all of the $d + 1$ price ratios $p_{t-1} / p_{t-2}, \dots, p_{t-(d+1)} / p_{t-(d+2)}$ are high, since, as discussed in the proof of lemma 3, if this does not hold, there is no incentive to sell preemptively.

Let us thus examine the decision of a type- n agent at $t = n + \tau^* - j$ with $j \geq 1$, and all of $d + 1$ price ratios before t being high. If $j = 1$, and the type- n agent sold preemptively at t , she would not reenter at $t + 1$ even if p_t / p_{t-1} was high, since other type- n agents would be selling and by (8), it would be better to stay out of the market. If $j > 1$, a type- n agent who had sold preemptively at t would reenter at $t + 1$ if p_t / p_{t-1} was high, since reentering would yield an expected payoff $\theta_{j-1} > 1$, where, as in the proof of lemma 3, θ_j denotes the expected discounted payoff that a type- n agent obtains if she follows (7) at all times. But even though selling and reentering is preferable to selling preemptively without the option to reenter, it is still better to not sell preemptively in the first place. To see why, note that θ_j exceeds the payoff from exiting and reentering if

$$\theta_j > \frac{1}{j+1} + \frac{j}{j+1} [(1-\pi) + \pi\theta_{j-1}].$$

Recalling (9), we have

$$\frac{1}{j+1} \left[\pi \left(\frac{G}{R} \right)^{-(\tau^*+1)} + (1-\pi) \frac{G}{R} \right] + \frac{j}{j+1} \frac{G}{R} ((1-\pi) + \pi\theta_{j-1}) > \frac{1}{j+1} + \frac{j}{j+1} [(1-\pi) + \pi\theta_{j-1}].$$

Simplifying, and rearranging terms, we get

$$\left[\pi \left(\frac{G}{R} \right)^{-(\tau^*+1)} + (1-\pi) \frac{G}{R} \right] + j \left(\frac{G}{R} - 1 \right) ((1-\pi) + \pi\theta_{j-1}) > 1$$

Using the fact that $\theta_j > 1$ for all $j \geq 1$, we have

$$\left[\pi \left(\frac{G}{R} \right)^{-(\tau^*+1)} + (1-\pi) \frac{G}{R} \right] + j \left(\frac{G}{R} - 1 \right) \left((1-\pi) + \pi \theta_{j-1} \right) > \left[\pi \left(\frac{G}{R} \right)^{-(\tau^*+1)} + (1-\pi) \frac{G}{R} \right] + j \left(\frac{G}{R} - 1 \right)$$

and the right-hand side is greater than one, since

$$\begin{aligned} \left[\pi \left(\frac{G}{R} \right)^{-(\tau^*+1)} + (1-\pi) \frac{G}{R} \right] + j \left(\frac{G}{R} - 1 \right) > 1 &\Leftrightarrow \left[\pi \left(\frac{G}{R} \right)^{-(\tau^*+1)} + (1-\pi) \frac{G}{R} \right] + j \frac{G}{R} > 1 + j \\ &\Leftrightarrow \left(1 - \frac{\pi}{j+1} \right) \frac{G}{R} + \frac{\pi}{j+1} \left(\frac{G}{R} \right)^{-(\tau^*+1)} > \theta_1 > 1. \end{aligned}$$

In words, selling preemptively is preferable to waiting only if $t_0 = n - j$. For all other values of t_0 selling preemptively (and reentering at $t+1$ if p_t / p_{t-1} is high) implies missing out on a fraction of the expected appreciation gains. If (9) holds, this opportunity cost is always large enough to deter preemptive sales. ■

Appendix D – Details of Section 5

Algorithm to compute $F(0, \bar{z} + 1)$ and $\bar{\gamma}$.

As in previous sections, selling is (weakly) optimal in the period of the crash, since, at that point, agents get the post-crash price no matter what. Type- n agents should also sell if $t \geq n + \tau^*$ and $x_t = \bar{z} + 1$. If the first period in which these conditions are met is $n + \tau^* + \bar{z}$, type- n agents sell knowing that $t_0 = n$, and that $\bar{z} + 1$ types $t_0, \dots, t_0 + \bar{z}$ are selling. Since this guarantees a crash at $n + \tau^* + \bar{z} + 1$, selling is obviously optimal. But, in general, the first period with $x_t = \bar{z} + 1$ and $t \geq n + \tau^*$ may be anywhere from $n + \tau^*$ to $n + \tau^* + \bar{z}$. If type- n agents sell at $n + \tau^* + q$, with $x_{n+\tau^*+q} = \bar{z} + 1$ and $q \in \{0, \dots, \bar{z}\}$, they know that $q + 1$ types $t_0, \dots, t_0 + q$ are selling. As q rises, a crash at $n + \tau^* + q + 1$ becomes more likely, and the number of periods that the bubble could keep growing falls. Since incentives to sell are weakest for $q = 0$, if agents want to sell then, they also do for $q > 0$.

At time $n + \tau^*$, with $x_{n+\tau^*} = \bar{z} + 1$, if a type- n agent sells, she obtains a payoff p_t , whereas waiting yields an expected payoff denoted by $p_t F(0, \bar{z} + 1)$. In equilibrium, it must be that $F(0, \bar{z} + 1) < 1$. To compute $F(0, \bar{z} + 1)$, it is necessary to compute payoffs at every information sets that a type- n agent could visit if she does not sell at time $n + \tau^*$.

Each information set is denoted by $(j, x_{n+\tau^*+j})$, where $j = t - (n + \tau^*)$ is how long it has been since $n + \tau^*$, and $x_{n+\tau^*+j}$ is the value of x in that period.

At information set $(0, \bar{z} + 1)$, t_0 could be anywhere from $n - \bar{z}$ to n . If t_0 is $n - \bar{z}$, the bubble is guaranteed to burst at $n + \tau^* + 1$. But for higher values of t_0 , the crash need not happen at $n + \tau^* + 1$, and a type- n agent who waits may profit from extra periods of appreciation. In fact, if $t_0 = n$ and price ratios are high, the bubble may keep growing for up to \bar{z} periods. More precisely, unless the bubble bursts at $n + \tau^* + 1$, a type- n agent who waits at $(0, \bar{z} + 1)$ will move to information set $(1, x_{n+\tau^*+1})$, for some $x_{n+\tau^*+1} \in \{2, \dots, \bar{z} + 1\}$. The lowest $x_{n+\tau^*+1}$ can be is 2, because other type- n agents will be out of the market by the end of period $n + \tau^*$, making $c(p_{n+\tau^*} / p_{n+\tau^*-1}) = 0$, and hence $x_{n+\tau^*+1} = 1$, impossible. The probability of moving from $(0, \bar{z} + 1)$ to $(1, 2)$ is $1/(\bar{z} + 1)^2$, which is the probability that $t_0 = n$ and $c(p_{n+\tau^*} / p_{n+\tau^*-1}) = 1$. The next-best scenario has $c(p_{n+\tau^*} / p_{n+\tau^*-1}) = 2$ and $x_{n+\tau^*+1} = 3$. The probability of moving to $(1, 3)$ is $2/(\bar{z} + 1)^2$, since $c(p_{n+\tau^*} / p_{n+\tau^*-1}) = 2$ is compatible with $t_0 = n$ and $t_0 = n - 1$. In general, the probability that $x_{n+\tau^*+1}$ equals $2, \dots, \bar{z} + 1$, is $(x_{n+\tau^*+1} - 1)/(\bar{z} + 1)^2$. In all these cases, waiting means earning one more period of appreciation, or more, if it is again optimal to wait at $(1, x_{n+\tau^*+1})$. On the other hand, with probability $(\bar{z} + 2)/(2\bar{z} + 2)$, $p_{n+\tau^*} / p_{n+\tau^*-1}$ will fall below $G - \alpha\bar{\varepsilon}$ and the crash will occur at $n + \tau^* + 1$. Of course, the lower t_0 , the higher the likelihood of a crash at $n + \tau^* + 1$ and the bigger the loss incurred during the crash. Thus, $F(0, \bar{z} + 1)$ is given by

$$F(0, \bar{z} + 1) = \sum_{x_{n+\tau^*+1}=2}^{\bar{z}+1} \frac{x_{n+\tau^*+1} - 1}{(\bar{z} + 1)^2} \frac{G}{R} \max\{1, F(1, x_{n+\tau^*+1})\} + \sum_{t_0=n-\bar{z}}^n \frac{(n-t_0)+1}{(\bar{z} + 1)^2} \left(\frac{G}{R}\right)^{-(n-t_0+\tau^*+1)}. \quad (20)$$

Note that a type- n agent who waits at $(0, \bar{z} + 1)$ makes another sell-or-wait decision at information set $(1, x_{n+\tau^*+1})$. Selling yields $p_{n+\tau^*+1} \approx p_{n+\tau^*} G$, and waiting yields an expected payoff $p_{n+\tau^*+1} F(1, x_{n+\tau^*+1})$. Selling is optimal if and only if $1 \geq F(1, x_{n+\tau^*+1})$.

Since, in equilibrium, $F(0, \bar{z} + 1)$ must be below 1, when a type- n agent makes her sell-or-wait choice at $(0, \bar{z} + 1)$, she (correctly) anticipates that, if she waits and finds

herself at information set $(1, \bar{z} + 1)$, she will sell. Indeed, at $(1, \bar{z} + 1)$, a type- n agent who had deviated from (17) by waiting at $(0, \bar{z} + 1)$ would know that two types, n and $n + 1$, would be out of the market by the end of period $n + \tau^* + 1$. This implies that $F(1, \bar{z} + 1)$ must be below $F(0, \bar{z} + 1)$, since potential for appreciation is lower, and the probability of a crash is greater. In fact, by the same reasoning, $F(0, \bar{z} + 1) < 1$ implies $F(j, \bar{z} + 1) < 1$ for all $j \in \{1, \dots, \bar{z}\}$.

Unfortunately, if $x_{n+\tau^*+1} \leq \bar{z}$, $F(0, \bar{z} + 1) < 1$ no longer implies $F(1, x_{n+\tau^*+1}) < 1$. In these cases, it is necessary to compute $F(1, x_{n+\tau^*+1})$. To obtain $F(1, x_{n+\tau^*+1})$, however, I must know $F(2, x_{n+\tau^*+2})$ for $x_{n+\tau^*+2} \in \{3, \dots, \bar{z}\}$. In turn, to obtain $F(2, x_{n+\tau^*+2})$, I must know $F(3, x_{n+\tau^*+3})$ for $x_{n+\tau^*+3} \in \{4, \dots, \bar{z}\}$, and so forth. Fortunately, there is a last period from which to iterate backwards, since a type- n agent who deviates from (17) by not selling at $(0, \bar{z} + 1)$ will sell no later than $n + \tau^* + \bar{z}$. The deviating agent will continue riding the bubble up to this point if $t_0 = n$, and if at all the information sets she visits before $n + \tau^* + \bar{z}$, waiting is preferable to selling, i.e. if $F(j, x_{n+\tau^*+j}) > 1$ for all $j \in \{1, \dots, \bar{z} - 1\}$.³⁴ Note that, if the bubble has not burst by period $n + \tau^* + \bar{z}$, $x_{n+\tau^*+\bar{z}}$ can only be $\bar{z} + 1$, and thus type- n agents know that the \bar{z} types $n + 1, \dots, n + \bar{z}$ will be selling at time $n + \tau^* + \bar{z}$, guaranteeing a crash at $n + \tau^* + \bar{z} + 1$. Thus, $F(\bar{z}, \bar{z} + 1) = (G/R)^{-(\tau^* + \bar{z} + 1)} < 1$, selling is optimal at $(\bar{z}, \bar{z} + 1)$, and the deviating type- n agents' payoff is $p_{n+\tau^*+\bar{z}} \approx p_{n+\tau^*} G^{\bar{z}}$. From here, the backward induction begins by considering sell-or-wait choices at $n + \tau^* + \bar{z} - 1$ for a hypothetical type- n agent who did not sell at $(0, \bar{z} + 1)$. If the bubble has survived up to this point, $x_{n+\tau^*+\bar{z}-1}$ can only be \bar{z} or $\bar{z} + 1$. Since $F(0, \bar{z} + 1) < 1$ implies $F(\bar{z} - 1, \bar{z} + 1) < 1$, selling is optimal at information set $(\bar{z} - 1, \bar{z} + 1)$. If $x_{n+\tau^*+\bar{z}-1} = \bar{z}$, on the other hand, the type- n agent would know that $t_0 = n$, and that other types would not be selling at time $n + \tau^* + \bar{z} - 1$. Thus, waiting would bring

³⁴ The simplest example of such a path is when $c(p_{n+\tau^*} / p_{n+\tau^*-1}) = 1$, i.e., $x_{n+\tau^*+1} = 2$. This reveals to the deviating type- n agent the fact that $t_0 = n$. Every period from $n + \tau^* + 1$ to $n + \tau^* + \bar{z} - 1$ there is a $1/(\bar{z} + 1)$ chance that the price ratio will fall below $G - \alpha\bar{\varepsilon}$. But if $(G/R) \cdot (\bar{z}/(\bar{z} + 1)) > 1$, the deviating type- n agent is willing to take this chance and wait until period $n + \tau^* + \bar{z}$. Unless $p_t / p_{t-1} < G - \alpha\bar{\varepsilon}$ for some $t \in \{n + \tau^* + 1, \dots, n + \tau^* + \bar{z} - 1\}$, x will increase by one unit every period, until $x_{n+\tau^*+\bar{z}} = \bar{z} + 1$.

one more period of appreciation with probability $\bar{z}/(\bar{z}+1)$ and a crash with probability $(\bar{z}+1)^{-1}$ (since other type- n agents did leave the market at time $n+\tau^*$). Hence,

$$F(\bar{z}-1, \bar{z}) = \frac{\bar{z}}{\bar{z}+1} \frac{G}{R} \max\{1, F(\bar{z}, \bar{z}+1)\} + \frac{1}{\bar{z}+1} \left(\frac{G}{R}\right)^{-(\tau^*+\bar{z})} = \frac{\bar{z}}{\bar{z}+1} \frac{G}{R} + \frac{1}{\bar{z}+1} \left(\frac{G}{R}\right)^{-(\tau^*+\bar{z})}.$$

Given payoffs at $(\bar{z}-1, \bar{z})$ and $(\bar{z}-1, \bar{z}+1)$, it is now possible to iterate backward one more period to information sets $(\bar{z}-2, x_{n+\tau^*+\bar{z}-2})$, for $x_{n+\tau^*+\bar{z}-2} \in \{\bar{z}-1, \bar{z}, \bar{z}+1\}$. Again, since $F(\bar{z}-2, \bar{z}+1) < F(0, \bar{z}+1) < 1$, selling is optimal at $(\bar{z}-2, \bar{z}+1)$. And to find $F(\bar{z}-2, \bar{z}-1)$, as was the case with $F(\bar{z}-1, \bar{z})$, the fact that $x_{n+\tau^*+\bar{z}-2} = \bar{z}-1$ implies that our deviating type- n agent would know that $t_0 = n$. Waiting at $(\bar{z}-2, \bar{z}-1)$ would thus bring a crash with probability $(\bar{z}+1)^{-1}$ and, otherwise, the agent would move on to information set $(\bar{z}-1, \bar{z})$ with probability $\bar{z}/(\bar{z}+1)$. The more interesting information set is $(\bar{z}-2, \bar{z})$, where type- n agents do not know whether t_0 is n or $n-1$, because for all $t \in \{n+\tau^*, \dots, n+\tau^*+\bar{z}-3\}$, it was always the case that $c(p_t/p_{t-1}) \geq 2$. Type- n agents may still learn that $t_0 = n$, since there is a probability $1/2 \cdot (1+\bar{z})^{-1}$ that $c(p_{n+\tau^*+\bar{z}-2}/p_{n+\tau^*+\bar{z}-3}) = 1$.³⁵ In that event, type- n agents would move to $(\bar{z}-1, \bar{z})$. The more likely scenario, which occurs with probability $\bar{z}-1/\bar{z}+1$, is that $c(p_{n+\tau^*+\bar{z}-2}/p_{n+\tau^*+\bar{z}-3})$ will be between 2 and \bar{z} , in which case the agent will transition to $(\bar{z}-1, \bar{z}+1)$. Finally, with probability $3/2 \cdot (1+\bar{z})^{-1}$ $c(p_{n+\tau^*+\bar{z}-2}/p_{n+\tau^*+\bar{z}-3})$ will be $\bar{z}+1$, precipitating a crash at time $n+\tau^*+\bar{z}-1$.

For general $j \in \{2, \dots, \bar{z}\}$, knowing $F(j, j+1)$, $F(j, j+2)$, \dots , $F(j, \bar{z})$, and the fact that $F(j, \bar{z}+1) < 1$, an iteration of the algorithm finds $F(j-1, x_{n+\tau^*+j-1})$, for $x_{n+\tau^*+j-1} \in \{j, \dots, \bar{z}\}$. (Again, selling at $(j-1, \bar{z}+1)$ is optimal, since $F(0, j+1) < 1$ implies $F(j-1, \bar{z}+1) < 1$.) For $x_{n+\tau^*+j-1} \in \{j, \dots, \bar{z}\}$, note that, at $(j-1, x_{n+\tau^*+j-1})$, the support of t_0 for a type- n agent is $\{n-(x_{n+\tau^*+j-1}-j), \dots, n\}$. Given price ratios p_t/p_{t-1} for $t \in \{n+\tau^*, \dots, n+\tau^*+j-2\}$, $\underline{c}(n+\tau^*+j-1)$ equals $1+x_{n+\tau^*+j-1}-j$, implying that prices

³⁵ Here, $1/2$ is the probability that $t_0 = n$, conditioning on the fact that $\text{supp}(t_0 | I_{n,t}) = \{n-1, n\}$, and $1/(1+\bar{z})$ is the probability that $c(p_{n+\tau^*+\bar{z}-2}/p_{n+\tau^*+\bar{z}-3}) = 1$ if $t_0 = n$.

are consistent with up to $1 + x_{n+\tau^*+j-1} - j$ types having left the market by the end of $n + \tau^*$. Waiting at $(j-1, x_{n+\tau^*+j-1})$ can bring three kinds of outcomes. The first, possible only if $x_{n+\tau^*+j-1} > j$, is the one where $c(p_{n+\tau^*+j-1}/p_{n+\tau^*+j-2})$ falls below $\underline{c}(n + \tau^* + j - 1)$, ruling out some elements from the support of t_0 , and implying a transition from $(j-1, x_{n+\tau^*+j-1})$ to $(j, x_{n+\tau^*+j})$ for some $x_{n+\tau^*+j} \in \{j+1, \dots, x_{n+\tau^*+j-1}\}$. For a given $x_{n+\tau^*+j}$ in this set, the probability of this is $(x_{n+\tau^*+j} - j)/((1 + x_{n+\tau^*+j-1} - j)(\bar{z} + 1))$. The second outcome, which has probability $(\bar{z} - (x_{n+\tau^*+j-1} - j))/(\bar{z} + 1)$, is the one where $p_{n+\tau^*+j-1}/p_{n+\tau^*+j-2}$ reveals nothing new, the support of t_0 stays as it is, and the agent transitions from $(j-1, x_{n+\tau^*+j-1})$ to $(j, x_{n+\tau^*+j-1} + 1)$. The third outcome is a crash, and it is also always possible. In fact, for each $t_0 \in \{n - (x_{n+\tau^*+j-1} - j), \dots, n\}$, the probability of a crash is $(1 + n - t_0)/(\bar{z} + 1)$. In sum,

$$\begin{aligned}
F(j-1, x_{n+\tau^*+j-1}) = & \sum_{x_{n+\tau^*+j}=j+1}^{x_{n+\tau^*+j-1}} \frac{x_{n+\tau^*+j} - j}{1 + x_{n+\tau^*+j-1} - j} \frac{1}{\bar{z} + 1} \frac{G}{R} \max\{1, F(j, x_{n+\tau^*+j})\} \\
& + \frac{\bar{z} - (x_{n+\tau^*+j-1} - j)}{\bar{z} + 1} \frac{G}{R} \max\{1, F(j, x_{n+\tau^*+j-1} + 1)\} \\
& + \sum_{t_0=n-(x_{n+\tau^*+j-1}-j)}^n \frac{1}{1 + x_{n+\tau^*+j-1} - j} \frac{1 + n - t_0}{\bar{z} + 1} \left(\frac{G}{R}\right)^{-(n+\tau^*+j-t_0)}.
\end{aligned} \tag{21}$$

Iterate to find $F(1, x_{n+\tau^*+1})$ for $2 \leq x_{n+\tau^*+1} \leq \bar{z}$, and use (20) to obtain $F(0, \bar{z} + 1)$.

Algorithm to compute $\hat{F}(-1, \bar{z} + 1)$ and $\underline{\gamma}$

Type- n agents shall not sell all information sets $(t - (n + \tau^*), x_t)$ with $t < n + \tau^*$ for any $x_t \in \{1, \dots, \bar{z} + 1\}$. Among these information sets, preemptive selling is most tempting at $(-1, \bar{z} + 1)$. Let p_t be the payoff from selling, and $p_t \hat{F}(-1, \bar{z} + 1)$, the payoff from waiting. At $(-1, \bar{z} + 1)$, the support of t_0 for type- n agents is given by $\{n - (\bar{z} + 1), \dots, n\}$. If type- n wait, the bubble could burst at $n + \tau^*$, or with probability $x_{n+\tau^*}/((\bar{z} + 1)(\bar{z} + 2))$, they will transition to information set $(0, x_{n+\tau^*})$ for some $x_{n+\tau^*} \in \{1, \dots, \bar{z} + 1\}$. Thus,

$$\hat{F}(-1, \bar{z} + 1) = \sum_{x_{n+\tau^*}=1}^{\bar{z}} \frac{x_{n+\tau^*}}{(\bar{z} + 1)(\bar{z} + 2)} \frac{G}{R} \hat{F}(0, x_{n+\tau^*}) + \frac{1}{\bar{z} + 2} \frac{G}{R} + \sum_{t_0=n-(\bar{z}+1)}^n \frac{n - t_0}{(\bar{z} + 1)(\bar{z} + 2)} \left[\frac{G}{R}\right]^{-(n+\tau^*-t_0)} \tag{22}$$

If $x_{n+\tau^*} \in \{1, \dots, \bar{z}\}$, type- n agents shall wait at $(0, x_{n+\tau^*})$. A general specification of payoffs at those information sets would be $\max\{1, \hat{F}(0, x_{n+\tau^*})\}$, but since the sets are on the equilibrium path, $\hat{F}(j, x_{n+\tau^*+j})$ must exceed 1 for all $j \in \{0, \dots, \bar{z}-1\}$ and $x_{n+\tau^*+j} \in \{j+1, \dots, \bar{z}\}$. (Indeed, G/R must be high enough to make waiting optimal in all these cases, in addition to $(-1, \bar{z}+1)$.) The second term in the sum on the right-hand side of (22) is the payoff at $(0, \bar{z}+1)$, reached with probability $1/(\bar{z}+2)$, and where type- n agents sell, which is optimal, since $F(0, \bar{z}+1) \leq 1$ and $F(j, \bar{z}+1) = \hat{F}(j, \bar{z}+1)$ for $j \in \{0, \dots, \bar{z}\}$.

Again, using backward induction, we compute $\hat{F}(j, x_{n+\tau^*+j})$ for all $j \in \{0, \dots, \bar{z}-1\}$ and $x_{n+\tau^*+j} \in \{j+1, \dots, \bar{z}\}$. It is known that $\hat{F}(j, \bar{z}+1) = F(j, \bar{z}+1) < 1$ for all $j = 0, 1, \dots, \bar{z}$. Thus, we begin at time $n + \tau^* + \bar{z} - 1$, where $x_{n+\tau^*+\bar{z}-1}$ can only be \bar{z} or $\bar{z}+1$. If $x_{n+\tau^*+\bar{z}-1} = \bar{z}$, type- n agents would know that $t_0 = n$, and that nobody would be out of the market by the end of $n + \tau^* + \bar{z} - 1$. Thus, waiting would, for sure, bring one more period of appreciation. Given payoffs at $(\bar{z}-1, \bar{z})$ and $(\bar{z}-1, \bar{z}+1)$, we can find $(\bar{z}-2, x_{n+\tau^*+\bar{z}-2})$, for $x_{n+\tau^*+\bar{z}-2} \in \{\bar{z}-1, \bar{z}, \bar{z}+1\}$. At $(\bar{z}-2, \bar{z}+1)$, selling clearly is optimal, and at $(\bar{z}-2, \bar{z}-1)$, type- n agents know that $t_0 = n$, and thus, waiting will, for sure, take type- n agents to $(\bar{z}-1, \bar{z})$. At information set $(\bar{z}-2, \bar{z})$, type- n agents do not know whether t_0 is n or $n-1$, because $c(t) \geq 1$ for all $t \in \{n + \tau^*, \dots, n + \tau^* + \bar{z} - 3\}$. If they wait, type- n agents may learn that $t_0 = n$, since there is a probability $1/2 \cdot (1 + \bar{z})^{-1}$ that $c(n + \tau^* + \bar{z} - 2)$ will be 0, taking them to $(\bar{z}-1, \bar{z})$. More likely, with probability $\bar{z}/\bar{z}+1$, $c(n + \tau^* + \bar{z} - 2)$ will be between 1 and \bar{z} , and agents will transition to $(\bar{z}-1, \bar{z}+1)$. And with probability $1/2 \cdot (1 + \bar{z})^{-1}$, $c(n + \tau^* + \bar{z} - 2)$ will be $\bar{z}+1$, bringing a crash at time $n + \tau^* + \bar{z} - 1$.

For general $j \in \{1, \dots, \bar{z}\}$, an iteration of the algorithm uses $\hat{F}(j, j+1), \dots, \hat{F}(j, \bar{z})$, and the fact that selling is optimal at $(j, \bar{z}+1)$, to find $\hat{F}(j-1, j), \dots, \hat{F}(j-1, \bar{z})$. (Again, selling is optimal at $(j-1, \bar{z}+1)$.) For $x_{n+\tau^*+j-1} \in \{j, \dots, \bar{z}\}$, at $(j-1, x_{n+\tau^*+j-1})$, the support of t_0 for a type- n agent is $\{n - (x_{n+\tau^*+j-1} - j), \dots, n\}$. Given price ratios p_t / p_{t-1} for $t \in \{n + \tau^* - 1, \dots, n + \tau^* + j - 2\}$, $\underline{c}(n + \tau^* + j - 2)$ is given by $x_{n+\tau^*+j-1} - (j-1)$. In other

words, prices are consistent with up anywhere between 0 and $x_{n+\tau^*+j-1} - (j-1)$ types having sold by the end of $n + \tau^* - 1$. Waiting at $(j-1, x_{n+\tau^*+j-1})$ can bring three kinds of outcomes. The first, possible only if $x_{n+\tau^*+j-1} > j$, is the one where $c(n + \tau^* + j - 1) < \underline{c}(n + \tau^* + j - 2)$, eliminating some values from the support of t_0 , and taking agents from $(j-1, x_{n+\tau^*+j-1})$ to $(j, x_{n+\tau^*+j})$ for some $x_{n+\tau^*+j} \in \{j+1, \dots, x_{n+\tau^*+j-1}\}$. For each $x_{n+\tau^*+j}$ in this set, the probability of this happening is $(x_{n+\tau^*+j} - j) / ((1 + x_{n+\tau^*+j-1} - j)(\bar{z} + 1))$. The second kind, which always has positive probability $(\bar{z} - (x_{n+\tau^*+j-1} - j)) / (\bar{z} + 1)$, is the one where $c(n + \tau^* + j - 1) \geq \underline{c}(n + \tau^* + j - 2)$, in which case the support of t_0 stays as it is, and agents transition from $(j-1, x_{n+\tau^*+j-1})$ to $(j, x_{n+\tau^*+j-1} + 1)$. The third outcome is a crash, and it is also always possible. In fact, for each $t_0 \in \{n - (x_{n+\tau^*+j-1} - j), \dots, n\}$, the probability of a crash is $(1 + n - t_0) / (\bar{z} + 1)$. In sum,

$$\begin{aligned} \hat{F}(j-1, x_{n+\tau^*+j-1}) = & \sum_{x_{n+\tau^*+j}=j+1}^{x_{n+\tau^*+j-1}} \frac{x_{n+\tau^*+j} - j}{1 + x_{n+\tau^*+j-1} - j} \frac{1}{\bar{z} + 1} \frac{G}{R} \hat{F}(j, x_{n+\tau^*+j}) \\ & + \left(1 - \frac{x_{n+\tau^*+j-1} - j}{\bar{z} + 1}\right) \frac{G}{R} \max\{1, \hat{F}(j, x_{n+\tau^*+j-1} + 1)\} \\ & + \sum_{t_0=n-(x_{n+\tau^*+j-1}-j)}^n \frac{1}{1 + x_{n+\tau^*+j-1} - j} \frac{n - t_0}{\bar{z} + 1} \left(\frac{G}{R}\right)^{-(n+\tau^*+j-t_0)}. \end{aligned} \quad (23)$$

Iterate to find $\hat{F}(0, x_{n+\tau^*+1})$ for $1 \leq x_{n+\tau^*+1} \leq \bar{z}$, and use (22) to get $\hat{F}(-1, \bar{z} + 1)$. (See, also, the supplemental computer codes that generate the results in tables 1 and 2.)