A short step between democracy and dictatorship

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Abstract

When preferences are defined over two alternatives and societies are variable, the group formed by the relative majority rule, the unanimity rule, the dictatorial rules and the strongly dictatorial rules is characterized in terms of five axioms: unanimity, reducibility, substitutability, exchangeability and parity. This result is used to provide characterizations of each of these rules by postulating separating axioms, that is, an axiom and its negation. Such axioms identify traits specifically differentiating a type of rule from the other types. For instance, majority differs from strong dictatorship in the existence of a society for which collective indifference should be a less likely outcome than the strict preference of one alternative over the other. As a second example, the difference between majority and strong dictatorship can be traced back to the requirement that the likelihood of collective indifference diminishes with the size of society.

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1. Introduction

Democracy and dictatorship define two focal procedures to make a collective decision. In a democracy, every member of the collective always has the potential to influence the decision. By contrast, in a dictatorship, a given member of the collective always determines the decision. Democracy is typically associated with some form of majority rule. For instance, the (relative) majority rule is the decision rule characteristically adopted by legislatures to make ordinary decisions, which are generally binary decisions: to confront the status quo with a change.

This paper carries out a parallel analysis of majority and dictatorship in the framework of preference aggregation over two alternatives and variable set of individuals. The restriction to two alternatives guarantees that no majority rule concept generates the preference cycles arising in, for example, the Condorcet paradox. Another interesting feature of this restriction is that the preference aggregation problem can be interpreted as a collective decision problem: a preference for alternative α against alternative β can be identified with a vote for α, whereas indifference between α and β can be supposed to represent a blank vote. Hence, the outcome of the preference aggregation problem can be viewed as the outcome of an election: if the collective preference has α preferred to β, α is the chosen alternative; and if the collective preference is indifference, then a tie results.

There are many characterizations of the majority rules for the case of two alternatives. For the relative majority rule, May (1952, p. 682), Fishburn (1973, p. 58; 1983, p. 33) and Llamazares (2006, p. 319) suggest axiomatic characterizations when the set of individuals is held constant, whereas Aşan and Sanver (2002, p. 411), Woeginger (2003, p. 91; 2005, p. 9) and Miroiu (2004, p. 362) consider the case in which the set of individuals is variable, with Xu and Zhong (2010, p. 120) dealing with the case in which the preferences are held fixed and the set of individuals variable. Llamazares (2006) and Houy (2007) provide axiomatizations of absolute majority rules, which are those lying between the relative majority rule and the unanimity rule. As regards rules on the democratic side, this paper restricts attention to those situated at the two extremes of the majority spectrum: the relative majority rule (the less demanding majority concept) and the unanimity rule (the more demanding majority concept).

Though a characterization of a dictatorial rule is probably not interesting by itself, it appears to be worth providing parallel axiomatizations of appealing and non-appealing rules to identify the points of divergence between them. Despite the fact that this exercise does not seem to have attracted much interest, this paper is concerned with parallel characterizations of majority and dictatorial rules. Specifically, the main result is a characterization of the set formed by four types of rules. Two types are taken from the democratic side: the (relative) majority rule and the unanimity rule. The other two, from the dictatorial side: the dictatorial rules and the strongly dictatorial rules. In these two last rules there is a linear order on the set of individuals that establishes a hierarchy. In a strongly dictatorial rule the collective preference assigned to a given set of individuals I always coincides with the preference of the individual that, in the hierarchy, is ranked above the rest of members of I. Such an individual is a strong dictator: whatever his preference, the collective preference agrees with it. The same occurs in a dictatorial rule with the difference that the individual placed highest in the hierarchy does not determine the collective preference when he is indifferent. Instead, the next non-indifferent individual in the hierarchy is the one whose strict preference will coincide with the collective preference.

The main result of the paper, Proposition 3.2, characterizes the set of rules formed by those four types in terms of five axioms, which are interpreted next in the framework of elections. One, unanimity: if all the individuals cast the same vote, that vote constitutes the result of the election. Two, reducibility: the outcome of an election involving n voters can be obtained from a certain election involving $n - 1$ voters. Proposition 3.1 shows these two properties to be necessary and sufficient for a rule to be one of the four selected types when it is assumed that, when there are two individuals, the outcome of the rule already coincides with the outcome of one the selected rules. For instance, if all the elections involving two individuals are resolved using the majority rule, then the majority rule is applied to all the elections if and only if unanimity and reducibility hold. Therefore, axioms that characterize the majority rule for two individuals together with unanimity and reducibility would characterize the majority rule.

Three, substitutability: for every group of individuals and any individual j not in the group, j can replace some member i of the group without altering the outcome. This expresses a property of inter-group anonymity: some j outside the group to which i belongs can always cast the same vote as i and cause no change in the outcome of the election. Four, exchangeability: if the result of the election coincides with none of the votes cast by the individuals, then having two individuals in the group exchanging their votes does not modify the outcome. This expresses a property of intra-group anonymity: members i and j in the given group can permute their votes and induce no change in the outcome of the election. And five, parity: for any group of individuals, when all the possible elections involving that group are considered, the two alternatives must have been chosen the same number of times. This is a property of non-discrimination between the alternatives and is implied by the standard neutrality condition.

The rest of results, Propositions 3.3-3.6, are applications of the main result obtained by identifying separating properties, that is, properties satisfied by some group of the four types of rules and whose negation is satisfied by the rest of types. For example, majority and strong dictatorship satisfy the weak condition *C* stating that in some case in which not all the individuals agree, the result is a tie, whereas, in some other case in which not all the individuals agree, the result is not a tie. Hence, any rule meeting *C* is minimally resolute and open to compromise in non-trivial situations (that is, when opinions differ). Since neither unanimity nor dictatorship satisfies *C*, the group of rules formed by the majority rule and the strongly dictatorial rules become characterized by *C* together with the five previous axioms. Given this result, any additional condition satisfied by majority but not by strong dictatorship, or vice versa, would separate majority from strong dictatorship.

One of such condition is weak relative resoluteness: for some group of individuals, the proportion of cases in which a tie arises is smaller than 1/3 (given parity, this requirement can be replaced by the property that the proportion of cases in which a tie arises is smaller than the proportion of cases in which any of the two alternatives is selected). Accordingly, majority is characterized by *C*, the five axioms and weak relative resoluteness, whereas strong dictatorship is characterized by *C*, the five axioms and the negation of weak relative resoluteness. It is somewhat curious in these two characterizations that the difference between majority and strong dictatorship can be traced back to the decision of whether one of the three outcomes (the tie) should be discriminated and that it is the majority rule that assumes the discriminatory treatment of the tie.

Definitions and main axioms can be found in Section 2, results and additional axioms in Section 3 and the proofs of remarks and propositions in the Appendix.

2. Definitions and main axioms

The members of the set $\mathbb{N}$ of positive integers designate individuals. A society is a finite non-empty subset of $\mathbb{N}$. There are two alternatives: α and β. A preference over $\{\alpha, \beta\}$ is represented by a number from the set $\{-1, 0, 1\}$. If the number is 1, α is preferred to β;

if -1, β is preferred to α; if 0, α is indifferent to β. A preference profile for society I is a function $x_I : I \to \{-1, 0, 1\}$ assigning a preference over $\{\alpha, \beta\}$ to each member of I.

For $n \in \mathbb{N}$, X_n is the set of all preference profiles $x_I : I \to \{-1, 0, 1\}$ such that I has n elements. The set X is the set of all preference profiles $x_I : I \to \{-1, 0, 1\}$ such that I is a society. A member x_I of X can be viewed as an election in which I is the set of voters and, for $i \in I$, x_i represents i's vote: if $x_i = 1$, i votes for candidate α; if $x_i = -1$, i votes for candidate β; and if $x_i = 0$, i casts a blank vote. For $x_I \in X$, $i \in I$ and non-empty $J \subset I$, x_i abbreviates $x_I(i)$, whereas x_J is the restriction of x_I to society J. A preference profile x_I is unanimous if there is $a \in \{-1, 0, 1\}$ such that, for all $i \in I$, $x_i = a$. For $x_I \in X$, $i \in I$ and $j \in \mathbb{N}\backslash\{i\}$, $x_I^{i\leftrightarrow j}$ is the preference profile obtained from x_I by permuting i's preference with j's. Formally, $x_I^{i\leftrightarrow j}$ is the preference profile y_I for I such that: (i) for all $k \in I\backslash\{i, j\}$, $y_k = x_k$; and (ii) $y_i = x_j$ and $y_j = x_i$.

Definition 2.1. A social welfare function is a mapping $f : X \to \{-1, 0, 1\}$.

A social welfare function takes as input the preferences over $\{\alpha, \beta\}$ of all the members of any given society I and outputs a preference over $\{\alpha, \beta\}$, interpreted as the collective preference ascribed to society I. For $x_I \in X$: (i) $f(x_I) = 1$ means that, according to f, α is preferred to β in I; (ii) $f(x_I) = -1$, that β is preferred to α; and (iii) $f(x_I) = 0$, that α is indifferent to β in I. Another interpretation is that f determines the outcome of an election x_I: $f(x_I) = 1$ means that α is the winning candidate; $f(x_I) = -1$ that it is β; and $f(x_I) = 0$ that there is a tie between α and β.

Definition 2.2. The unanimity rule is the social welfare function $\upsilon : X \to \{-1, 0, 1\}$ such that, for all $x_I \in X$: (i) if x_I is unanimous, then $f(x_I) = x_i$, where i is any member of I; and (ii) otherwise, $\upsilon(x_I) = 0$.

Definition 2.3. The (relative) majority rule is the social welfare function $\mu : X \to \{-1, 0, 1\}$ such that, for all $x_I \in X$: (i) if $\sum_{i\in I} x_i > 0$, then $\mu(x_I) = 1$; (ii) if $\sum_{i\in I} x_i < 0$, then $\mu(x_I) = -1$; and (iii) if $\sum_{i\in I} x_i = 0$, then $\mu(x_I) = 0$.

Definition 2.4. A social welfare function $f : X \to \{-1, 0, 1\}$ is a strongly dictatorial rule if there is a linear order $\Rightarrow$ on $\mathbb{N}$ such that, for all $x_I \in X$, $f(x_I) = x_i$, where i is the member of I such that, for all $j \in I\backslash\{i\}$, $i \Rightarrow j$. That individual $i \in I$ will be called strong dictator in I.

In a strongly dictatorial rule there is a hierarchy among individuals such that, for every society I and preference profile for I, the collective preference ascribed to I coincides with the preference of the member of I ranked higher in the hierarchy.

Definition 2.5. A social welfare function $f : X \rightarrow \{-1, 0, 1\}$ is a dictatorial rule if there is a linear order $\Rightarrow$ on $\mathbb{N}$ such that, for all $x_I \in X$: (i) if x_I is unanimous, then $f(x_I) = x_i$, where i is any member of I; and (ii) otherwise, $f(x_I) = x_i$, where i is the member of I such that $x_i \neq 0$ and there is no $j \in I$ such that $x_j \neq 0$ and $j \Rightarrow i$. That individual $i \in I$ will be called dictator in I except for $J = \{j \in I\text{: } x_j = 0\}$.

In a dictatorial rule there is a hierarchy among individuals such that, for every society I and preference profile for I, the collective preference ascribed to I coincides with the preference of the member of I ranked higher in the hierarchy and not indifferent. If no such individual exists, then all the individuals are indifferent and the outcome is indifference.

UNA. *Unanimity*. For all $x_I \in X$, if x_I is unanimous then $f(x_I) = x_i$, where i is any member of I.

UNA states that if all the members of a society have the same preference, then that preference constitutes the collective preference.

For $x_I \in X$, $i \in \mathbb{N}\backslash I$ and $a \in \{-1, 0, 1\}$, (x_I, a^i) designates the member y_J of X such that: (i) $J = I \cup \{i\}$; (ii) for all $j \in I$, $y_j = x_j$; and (iii) $y_i = a$. In words, (x_I, a^i) is the member of X obtained from x_I by adding another individual i with preference a. Expressions like (x_I, a^i, b^j), where $i \notin I$ and $j \notin I$, should be interpreted analogously. For instance, if $I = \{i, j\}$, then (a^i, b^j) stands for the member x_I of X such that $x_i = a$ and $x_j = b$.

RED. *Reducibility*. For all $x_I \in X$, $i \in I$ and $j \in I\backslash\{i\}$, if $x_i \neq x_j$, then, for some $k \in \{i, j\}$, $f(x_I) = f(x_{I\backslash\{i,j\}}, f(x_{\{i,j\}})^k)$.

RED asserts that the result of aggregating n preferences (or the result of an election with n voters) can be obtained by aggregating certain $n-1$ preferences (or by considering an election with $n-1$ voters). Specifically, RED asserts that, when x_I is not unanimous, $f(x_I)$ can be obtained as follows. Choose any two individuals i and j whose preferences x_i and x_j are different. Determine the preference $f(x_{\{i,j\}})$ of the society $\{i, j\}$. Select a representative $k \in \{i, j\}$ of society $\{i, j\}$. Replace, in x_I, the preferences (x_i, x_j) by the preference $f(x_{\{i,j\}})$ and ascribe $f(x_{\{i,j\}})$ to the representative k. Finally, $f(x_I)$ is equal to the result $f(x_{I\setminus\{i,j\}}, f(x_{\{i,j\}})^k)$ of aggregating preferences $x_{\{i,j\}}$ and preference $f(x_{\{i,j\}})$ of individual *k*. RED is another example of axioms based on the idea of ascribing the preference of some subsociety to some of its members, who acts as a representative of the subsociety.

For instance, Aşan and Sanver (2002, p. 411) define weak path independence as $f(x_{I\cup J}) = f(f(x_I)^i, f(x_J)^j)$, where I and J are disjoint societies, $i \in I$, $j \in J$ and $\{f(x_I), f(x_J)\} \neq \{1, -1\}$. Whereas RED reduces a problem of aggregating n preference to one of aggregating $n-1$, weak path independence reduces a problem of aggregating n preferences to one of aggregating just two preferences, for the particular case in which the preference $f(x_I)$ of subsociety I is not the inverse of the preference $f(x_J)$ of subsociety J.

The property of reducibility to subsocieties in Woeginger (2003, p. 90) states that, for every society $I = \{i_1, \dots, i_n\}$ with at least two members, $f(x_I) = f(f(x_{I\setminus\{i_1\}}), \dots, f(x_{I\setminus\{i_n\}}))$, where each $f(x_{I\setminus\{i_r\}})$ is ascribed to some member of $I\setminus\{i_r\}$. This says that $f(x_I)$ can be obtained by first aggregating the preferences of the members of the n subsocieties arising from I by removing just one individual and next aggregating those aggregations. In contrast to RED, reducibility to subsocieties does not ultimately reduce the dimension of the aggregation problem.

Chambers (2008, p. 350) introduces the representative consistency as a property of rules that transform vectors of votes for candidates into a candidate. Adapted to the present framework and combined with UNA, representative consistency implies that, for all $J \subset I$, $f(x_I) = f(x_{I\setminus J}, (f(x_J)^i)_{i\in J})$. This says that the outcome of election x_I coincides with the outcome of any election obtained from x_I by replacing the vote of each voter in any given strict subset J of I with the vote $f(x_J)$, which can be viewed as the representative vote of the group J. RED differs from representative consistency in having the whole set J of voters replaced by a representative voter casting the representative vote and in requiring J to have just two members.

Definition 2.6. For society I, $i \in I$ and $j \in \mathbb{N}\setminus I$, j can replace i in society I (abbreviated "$j \equiv_I i$") if, for every preference profile x_I for I, $f(x_{I\setminus\{i\}}, (x_i)^j) = f(x_I)$.

That j can replace i in I means that the collective preference assigned to I is never modified by substituting j for i, with j adopting the same preference as i. Replaceability is obviously symmetric: $j \equiv_I i$ implies $i \equiv_I j$.

SUB. *Substitutability*. For every society I and $j \in \mathbb{N}\setminus I$, there is $i \in I$ such that $j \equiv_I i$.

SUB states that, in every aggregation problem (or election) for any given society, every individual outside that society can replace some member of the society.

EXC. *Exchangeability*. For all $x_I \in X$, $i \in I$ and $j \in I \setminus \{i\}$, if $f(x_I) \notin \{x_k\}_{k \in I}$, then $f(x_I^{i \leftrightarrow j}) = f(x_I)$.

EXC holds that any two preferences are exchangeable whenever the collective preference is different from all the individual's preferences. If the antecedent is removed from EXC, the result is the standard anonymity condition: preferences are unconditionally exchangeable, that is, the collective preference is unaffected by permuting the preference of any two individuals. EXC and anonymity are actually intra-anonymity conditions, because they involve individuals of a given society. SUB, instead, is an inter-anonymity condition, since it deals with individuals of different societies. Hence, EXC makes the social welfare function robust to certain permutations of preferences, whereas SUB makes it robust to certain permutations of individuals. The requirement that $f(x_I) \notin \{x_k\}_{k \in I}$ in EXC can be viewed as the justification for the exchangeability of preferences: if the preference of no individual represents the collective preference, then it does not matter who holds those individual preferences.

Given a social welfare function f, society I and $a \in \{1, 0, -1\}$, define π_a^I to be the number of preference profiles x_I for I such that $f(x_I) = a$ divided by the number of preference profiles for I. That is, π_1^I is the proportion of elections involving the set I of voters in which the chosen candidate is α; π_{-1}^I is the proportion in which the chosen candidate is β; and π_0^I is the proportion in which, since no candidate is chosen, a tie results).

NEU. *Neutrality*. For all $x_I \in X$, $f(x_I) = -f(-x_I)$, where $-x_I$ is the preference profile for I obtained from x_I by replacing, for all $i \in I$, x_i with $-x_i$.

PAR. *Parity*. For every society I, $\pi_1^I = \pi_{-1}^I$.

NEU is the standard neutrality axiom: if all the individual preferences are reversed, then the collective preference is also reversed. According to PAR, in every society, every possible collective strict preference should be obtained the same total number of times. In terms of elections, if all elections are equally likely, then all the candidates are also equally likely to become the winning candidate. PAR can be motivated by a principle of neutrality: if all elections are possible, then, by symmetry between the two candidates, there is no apparent reason to make a candidate more likely to be the winning candidate. In fact, PAR is implied, but does not imply, NEU.

Remark 2.7. The unanimity rule, the majority rule, every strongly dictatorial rule and every dictatorial rule satisfy UNA, RED, SUB, EXC and NEU (and, therefore, PAR).

3. Results

Let g be the unanimity rule, the majority rule, a strongly dictatorial rule or a dictatorial rule. Suppose that a social welfare function f agrees with g on those preference profiles for societies with just two individuals. Proposition 3.1 next asserts that, for f to agree to g on the whole domain, UNA and RED are necessary and sufficient conditions (the case $g = \upsilon$ should be credited to one of the referees).

Proposition 3.1. Let g be the unanimity rule, the majority rule, a strongly dictatorial rule or a dictatorial rule. If social welfare function $f : X \to \{-1, 0, 1\}$ agrees with g on X_2, then $f = g$ if and only if UNA and RED hold.

Proposition 3.1 suggests that, in the presence of UNA and RED, the difference between the four types of rules considered can be traced back to the way those rules handle preference aggregation for the simplest non-trivial case: societies consisting of two individuals. This facilitates the characterization of any subgroup of the four types of rules, because any such characterization can be obtained by selecting axioms ensuring that the social welfare function agrees on X_2 with some rule in the subgroup. Proposition 3.2 shows that one can dispense with the agreement assumption on X_2 in Proposition 3.1 by adding axioms SUB, EXC and PAR.

Proposition 3.2. A social welfare function $f : X \to \{-1, 0, 1\}$ satisfies UNA, RED, SUB, EXC and PAR if and only if f is the unanimity rule, the majority rule, a strongly dictatorial rule or a dictatorial rule.

The usefulness of Proposition 3.2 lies in facilitating the search for axioms that separate one rule, or group of rules, from the other rules characterized in Proposition 3.2. The following results illustrate this search for separating axioms.

RES. *Minimal resolution*. There is a society I and a preference profile x_I for I such that x_I is not unanimous and $f(x_I) \neq 0$.

COM. *Minimal compromise*. There is a society I and a preference profile x_I for I such that x_I is not unanimous and $f(x_I) = 0$.

UNA solves the preference aggregation problem for unanimous profiles. This justifies considering the aggregation of preference profiles that are not unanimous separately from those that are. In this case, RES and COM appear reasonable. By RES, indifference cannot always be the result of aggregating non-unanimous preferences. And according to COM, indifference must be the outcome of aggregating some non-unanimous preferences. This expresses the idea that there is room for compromise. Otherwise, in the presence of divergent opinions, there is always an alternative that defeats the other alternative.

WRR. *Weak relative resoluteness*. For some society I, $\pi_0^I < ⅓$.

In the presence of PAR, $\pi_0^I < ⅓$ is equivalent to $\pi_0^I < \pi_1^I$. Hence, the presumption that strict preference (the choice of an alternative) is more desirable than indifference (the tie) motivates WRR: of the three outcomes, the less likely should be the tie.

SIR. *Size increases relative resoluteness*. For every two societies I and J having at least two members, $|I| > |J|$ implies $\pi_0^I < \pi_0^J$.

SIR states that, for societies with at least two members, the proportion of cases in which a tie occurs decreases with the size of the society: the larger a society, the smaller the likelihood of a tie.

Proposition 3.3. A social welfare function $f : X \to \{-1, 0, 1\}$ satisfies UNA, RED, SUB, EXC, PAR and

(i) (RES $\wedge$ COM) if and only if f is the majority rule or a strongly dictatorial rule;
(ii) $\neg$(RES $\wedge$ COM) if and only if f is the unanimity rule or a dictatorial rule;
(iii) WRR if and only if f is the majority rule or a dictatorial rule;
(iv) $\neg$WRR if and only if f is the unanimity rule or a strongly dictatorial rule;
(v) SIR if and only if f is the majority rule or a dictatorial rule;
(vi) $\neg$SIR if and only if f is the unanimity rule or a strongly dictatorial rule.

The results in Proposition 3.3 are summarized in the next two tables, where axioms UNA, RED, SUB, EXC and PAR are everywhere assumed. For instance, a social welfare function satisfies UNA, RED, SUB, EXC, PAR and, in addition, RES, COM and WRR if and only if it is the majority rule. On the other hand, a social welfare function satisfies UNA, RED, SUB, EXC, PAR and, in addition, RES, COM and ¬WRR if and only if it is a strongly dictatorial rule.

	RES ∧ COM	¬(RES ∧ COM)
WRR	majority	dictatorship
¬WRR	strong dictatorship	unanimity

	RES ∧ COM	¬(RES ∧ COM)
SIR	majority	dictatorship
¬SIR	strong dictatorship	unanimity

Hence, given UNA, RED, SUB, EXC, PAR, RES and COM, the difference between majority and strong dictatorship can be reduced to having some society averse to collective indifference in the sense that the outcome indifference occurs less frequently than the other two outcomes. Proposition 3.4 expresses a similar idea: majority seems connected to some form of discrimination of indifference as the collective preference.

Proposition 3.4. Let $A \in \{(1), (2), (3)\}$. A social welfare function f satisfies UNA, RED, SUB, EXC, PAR, RES, COM and:

(i) A if and only if f is the majority rule μ;
(ii) $\neg A$ if and only if f is a strongly dictatorial rule.

$$\text{For some society } I,\ \pi_0^I < \pi_1^I. \tag{1}$$

$$\text{There is } n \in \mathbb{N} \text{ such that, for every society } I \text{ with } n \text{ members, } \pi_0^I < \pi_1^I. \tag{2}$$

$$\text{For each } n \in \mathbb{N}\backslash\{1, 2\}, \text{ there is a society with } n \text{ members such that } \pi_0^I < \pi_1^I. \tag{3}$$

The results in Proposition 3.3 jointly characterize two types of rules. But no characterization brings together rules of the same family. Proposition 3.5 fills that gap by providing joint characterizations of unanimity and majority, on the one hand, and dictatorship and strong dictatorship, on the other. It is no wonder that a sufficiently strong anonymity requirement can be used as a separating property.

WAN. *Weak anonymity*. For every $x_I \in X$, if I has at least two members, then there are $i \in I$ and $j \in I\backslash\{i\}$ such that $f(x_I^{i\leftrightarrow j}) = f(x_I)$.

Proposition 3.5. A social welfare function $f : X \to \{-1, 0, 1\}$ satisfies UNA, RED, SUB, EXC, PAR and

(i) WAN if and only if f is the majority rule or the unanimity rule;
(ii) $\neg$WAN if and only if f is a strongly dictatorial rule or a dictatorial rule.

The following table combines Proposition 3.5 with Proposition 3.3(i) and 3.3(ii).

	RES $\wedge$ COM	$\neg$(RES $\wedge$ COM)
WAN	majority	unanimity
$\neg$WAN	strong dictatorship	dictatorship

Appendix

Proof of Remark 2.7. Unanimity rule υ. It is not difficult to verify that υ satisfies UNA, SUB (every individual can replace any other individual in every society), EXC ($\upsilon(x_I) = \upsilon(x_I^{i\leftrightarrow j})$ always holds) and NEU. As regards RED, if $x_i \neq x_j$ in x_I, then neither $x_{\{i,j\}}$ nor x_I is unanimous, so $\upsilon(x_I) = 0 = \upsilon(x_{\{i,j\}})$. If $I\backslash\{i, j\} = \varnothing$ or, for all $k \in I\backslash\{i, j\}$, $x_k = 0$, then $\upsilon(x_{I\backslash\{i,j\}}, \upsilon(x_{\{i,j\}})^k) = 0$ because $(x_{I\backslash\{i,j\}}, \upsilon(x_{\{i,j\}})^k)$ is unanimous. If, for some $k \in I\backslash\{i, j\}$, $x_k \neq 0$, then $(x_{I\backslash\{i,j\}}, \upsilon(x_{\{i,j\}})^k)$ is not unanimous and, accordingly, $\upsilon(x_{I\backslash\{i,j\}}, \upsilon(x_{\{i,j\}})^k) = 0$.

Majority rule μ. It is not difficult to verify that μ satisfies UNA, SUB (every individual can replace any other individual in every society), EXC ($\mu(x_I) = \mu(x_I^{i\leftrightarrow j})$ is always the case) and NEU. With respect to RED, let $x_i \neq x_j$ in x_I. Then, for some $a \in \{-1, 1\}$, either $\{x_i, x_j\} = \{a, 0\}$ or $\{x_i, x_j\} = \{a, -a\}$. In the first case, the passage from x_I to $(x_{I\backslash\{i,j\}}, \mu(x_{\{i,j\}})^k)$ essentially amounts to removing a 0, which does not alter the result of the majority rule. In the second case, in passing from x_I to $(x_{I\backslash\{i,j\}}, \mu(x_{\{i,j\}})^k)$, a pair of preferences $(-1, 1)$ is replaced by a preference 0, which does neither alter the result generated by the majority rule.

Strongly dictatorial rule. Let σ be a strongly dictatorial rule with associated linear order $\Rightarrow$. It is plain that UNA and NEU hold. EXC is trivially satisfied because it is never the case that $\sigma(x_I) \notin \{x_i\}_{i\in I}$. As regards RED, let $x_i \neq x_j$ in x_I and $k \in \{i, j\}$. Let r be the strong dictator in I. If $r \in \{i, j\}$, then, by setting $k = r$, $\sigma(x_{I\backslash\{i,j\}}, \sigma(x_{\{i,j\}})^k) = \sigma(x_{I\backslash\{i,j\}}, (x_r)^r)$

$= x_r = \sigma(x_I)$, because I, $\{i, j\}$ and $I\setminus\{i, j\} \cup \{r\}$ have the same strong dictator. If $r \notin \{i, j\}$, then, given that I and $I\setminus\{i, j\} \cup \{k\}$ have the same strong dictator, $\sigma(x_I) = \sigma(x_{I\setminus\{i,j\}}, \sigma(x_{\{i,j\}})^k)$. Concerning SUB, let I be a society, i the strong dictator in I and $j \in \mathbb{N}\setminus I$. If I is a singleton, then, since UNA holds, j can replace the only member of I. Otherwise, $j \Rightarrow i$ implies that j can replace i, whereas $i \Rightarrow j$ implies that j can replace any $k \in I\setminus\{i\}$.

Dictatorial rule δ. Let δ be a dictatorial rule with associated linear order $\Rightarrow$. As with strongly dictatorial rules, it should not be difficult to see that UNA and NEU hold. EXC is trivially satisfied because it is never the case that $\delta(x_I) \notin \{x_i\}_{i\in I}$. With respect to RED, let $x_i \neq x_j$ in x_I. This means that $\delta(x_I) \neq 0$. Let J be the set of indifferent individuals in I and r the dictator in I except for J, so $\delta(x_I) = x_r$. If $r \in \{i, j\}$, then, without loss of generality, suppose $r = i$. Hence, $\delta(x_{I\setminus\{i,j\}}, \delta(x_{\{i,j\}})^i) = \delta(x_{I\setminus\{j\}})$. If $x_j = 0$, then $\delta(x_{I\setminus\{j\}}) = x_r = \delta(x_I)$ because i dictator in I except for J implies i dictator in $I\setminus\{j\}$ except for $J\setminus\{j\}$. And if $x_j \neq 0$, then $\delta(x_{I\setminus\{j\}}) = x_r = \delta(x_I)$ because i dictator in I except for J implies i dictator in $I\setminus\{j\}$ except for J. If $r \notin \{i, j\}$, then there are two cases. Case 1: $0 \in \{x_i, x_j\}$. Without loss of generality, suppose $x_i = 0$. Then $\delta(x_{I\setminus\{i,j\}}, \delta(x_{\{i,j\}})^j) = \delta(x_{I\setminus\{i,j\}}, (x_j)^j) = \delta(x_{I\setminus\{i\}}) = \delta(x_I)$, the last step because r dictator in I except for J implies r dictator in $I\setminus\{i\}$ except for $J\setminus\{i\}$. Case 2: $0 \notin \{x_i, x_j\}$. Without loss of generality, suppose that $i \Rightarrow j$, so i is the dictator in $\{i, j\}$. Thus, $\delta(x_{I\setminus\{i,j\}}, \delta(x_{\{i,j\}})^i) = \delta(x_{I\setminus\{i,j\}}, (x_i)^i) = \delta(x_{I\setminus\{j\}}) = \delta(x_I)$, the last step because r dictator in I except for J implies r dictator in $I\setminus\{j\}$ except for J. Finally, to prove that SUB holds, let I be a society and $j \in \mathbb{N}\setminus I$. If I is a singleton, then, since UNA holds, j can replace the only member of I. Otherwise, consider the listing of the members of I such that k appears before r in the list if and only of $k \Rightarrow i$. Let s be the member of I appearing last in the listing. If, for all $k \in I$, $k \Rightarrow j$, j can replace s in I. Otherwise, let i be the first member in the listing such that $j \Rightarrow i$. Then j can replace i in I.■

Proof of Proposition 3.1. By Remark 2.7, if g is the unanimity rule, the majority rule, a strongly dictatorial rule or a dictatorial rule, then UNA and RED hold. So it remains to be proved that, if $f = g$ on X_2 and both UNA and RED hold, then $f = g$. Let g be one of the rules referred to in Proposition 3.1. By UNA, $f = g$ on X_1. Taking the fact that $f = g$ on $X_1 \cup X_2$ as the base case of an induction argument, choose $n \geq 3$ and assume that $f = g$ on $X_1 \cup \ldots \cup X_{n-1}$. It must be shown that $f = g$ on X_n. To this end, let $x_I \in X_n$.

Case 1: $g \in \{\upsilon, \mu\}$. If x_I is unanimous, then, by UNA, $f(x_I) = \mu(x_I)$. If x_I is not unanimous, then there are two cases. Case 1a: for some $i \in I$, $x_i = 0$. Since x_I is not unanimous, there are $a \in \{-1, 1\}$ and $j \in I\setminus\{i\}$ such that $x_j = a$. By RED, for some $k \in \{i, j\}$, $f(x_I) = f(x_{I\setminus\{i,j\}}, f(x_{\{i,j\}})^k)$. By the induction hypothesis, $f(x_{I\setminus\{i,j\}}, f(x_{\{i,j\}})^k) = g(x_{I\setminus\{i,j\}},$

$g(x_{\{i,j\}})^k)$ and $f(x_{\{i,j\}}) = g(x_{\{i,j\}}) = g(0^i, a^j)$. Therefore, $f(x_I) = g(x_{I\setminus\{i,j\}}, g(0^i, a^j)^k)$. Case 1a1: $g = \mu$. In this case, $f(x_I) = \mu(x_{I\setminus\{i,j\}}, \mu(0^i, a^j)^k) = \mu(x_{I\setminus\{i,j\}}, a^k) = \mu(x_{I\setminus\{i,j\}}, a^j) = \mu(x_{I\setminus\{i\}}) = \mu(x_I)$, the last step due to $x_i = 0$. Case 1a2: $g = \upsilon$. Hence, $f(x_I) = \upsilon(x_{I\setminus\{i,j\}}, \upsilon(0^i, a^j)^k) = \upsilon(x_{I\setminus\{i,j\}}, 0^k)$. As x_I is not unanimous, $\upsilon(x_I) = 0$, so it must be shown that $f(x_I) = 0$. If $x_{I\setminus\{i,j\}} = (0, \dots, 0)$, then, by UNA, $\upsilon(x_{I\setminus\{i,j\}}, 0^k) = 0$. If $x_{I\setminus\{i,j\}} \neq (0, \dots, 0)$, then $(x_{I\setminus\{i,j\}}, 0^k)$ is not unanimous and, accordingly, $\upsilon(x_{I\setminus\{i,j\}}, 0^k) = 0$. Case 1b: for all $i \in I$, $x_i \neq 0$. By the assumption that x_I is not unanimous, there are $i \in I$ and $j \in I\setminus\{i\}$ such that $x_i = 1$ and $x_j = -1$. By RED, for some $k \in \{i, j\}$, $f(x_I) = f(x_{I\setminus\{i,j\}}, f(x_{\{i,j\}})^k) = f(x_{I\setminus\{i,j\}}, f(1^i, -1^j)^k)$. By the induction hypothesis, $f(x_{I\setminus\{i,j\}}, f(1^i, -1^j)^k) = g(x_{I\setminus\{i,j\}}, 0^k)$. If $g = \mu$, then $\mu(x_{I\setminus\{i,j\}}, 0^k) = \mu(x_{I\setminus\{i,j\}}) = \mu(x_I)$. If $g = \upsilon$, then $\upsilon(x_{I\setminus\{i,j\}}, 0^k) = 0 = \upsilon(x_I)$.

Case 2: g is a strongly dictatorial rule σ. Let $i \in I$ be the strong dictator in I and $j \in I\setminus\{i\}$ the strong dictator in $I\setminus\{i\}$. Case 2a: $x_i \neq x_j$. By RED, for some $k \in \{i, j\}$, $f(x_I) = f(x_{I\setminus\{i,j\}}, f(x_{\{i,j\}})^k)$. By the induction hypothesis, $f(x_{I\setminus\{i,j\}}, f(x_{\{i,j\}})^k) = \sigma(x_{I\setminus\{i,j\}}, \sigma(x_{\{i,j\}})^k) = \sigma(x_{I\setminus\{i,j\}}, (x_i)^k)$, the last step because i strong dictator in I implies i strong dictator in $\{i, j\}$. If $k = i$, then $\sigma(x_{I\setminus\{i,j\}}, (x_i)^k) = x_i$, thanks to the fact that i strong dictator in I implies i strong dictator in $I\setminus\{j\}$. If $k = j$, then $\sigma(x_{I\setminus\{i,j\}}, (x_i)^k) = x_i$ because j is the strong dictator in $I\setminus\{i\}$. In sum, $f(x_I) = x_i = \sigma(x_I)$. Case 2b: $x_i = x_j$. If there is no $r \in I\setminus\{i, j\}$ such that $x_r = x_j$, then, by UNA, $f(x_I) = x_i = \sigma(x_I)$. If $r \in I\setminus\{i, j\}$ is such that $x_r \neq x_j$, then, by RED, for some $k \in \{r, j\}$, $f(x_I) = f(x_{I\setminus\{r,j\}}, f(x_{\{r,j\}})^k)$. By the induction hypothesis, the fact that j strong dictator in $I\setminus\{i\}$ implies j strong dictator in $\{j, k\}$ and the fact that i strong dictator in I implies i strong dictator $I\setminus\{r, j\} \cup \{k\}$, $f(x_{I\setminus\{r,j\}}, f(x_{\{r,j\}})^k) = \sigma(x_{I\setminus\{r,j\}}, \sigma(x_{\{r,j\}})^k) = \sigma(x_{I\setminus\{r,j\}}, (x_j)^k) = x_i = \sigma(x_I)$.

Case 3: g is a dictatorial rule δ. Let $i \in I$ be the dictator in I except for $\varnothing$ and $j \in I\setminus\{i\}$ the dictator in $I\setminus\{i\}$ also except for $\varnothing$. Therefore, i is the dictator in $I\setminus\{j\}$ except for any $J \subset I\setminus\{j\}$ such that $i \notin J$ and j is the dictator in $I\setminus\{i\}$ except for any $J \subset I\setminus\{i\}$ such that $j \notin J$ (for clarity, the "except for…" proviso will be omitted). Case 3a: $x_i \neq 0$ and $x_j \neq 0$. The proof is as in case 2, with the conclusion being that $f(x_I) = x_i = \delta(x_I)$. Case 3b: $x_i = 0$ and $x_j \neq 0$. By RED, for some $k \in \{i, j\}$, $f(x_I) = f(x_{I\setminus\{i,j\}}, f(x_{\{i,j\}})^k)$. By the induction hypothesis, $f(x_{I\setminus\{i,j\}}, f(x_{\{i,j\}})^k) = \delta(x_{I\setminus\{i,j\}}, \delta(x_{\{i,j\}})^k) = \delta(x_{I\setminus\{i,j\}}, (x_j)^k)$. If $k = i$, then $\delta(x_{I\setminus\{i,j\}}, (x_j)^k) = x_j = \delta(x_I)$, because i is the dictator in $I\setminus\{j\}$. If $k = j$, then $\delta(x_{I\setminus\{i,j\}}, (x_j)^k) = x_j = \delta(x_I)$, because j is the dictator in $I\setminus\{i\}$. Case 3c: $x_i \neq 0$ and $x_j = 0$. By RED, for some $k \in \{i, j\}$, $f(x_I) = f(x_{I\setminus\{i,j\}}, f(x_{\{i,j\}})^k)$. By the induction hypothesis, $f(x_{I\setminus\{i,j\}}, f(x_{\{i,j\}})^k) = \delta(x_{I\setminus\{i,j\}}, \delta(x_{\{i,j\}})^k) = \delta(x_{I\setminus\{i,j\}}, (x_i)^k) = x_i$: if $k = i$, the last step because i is the dictator in $I\setminus\{j\}$; and if $k = j$, the last step follows from the fact that j is the dictator in $I\setminus\{i\}$. Consequently, $f(x_I) = x_i = \delta(x_I)$. Case 3d: $x_i = x_j = 0$. Let $J = \{k \in I\colon x_k \neq 0\}$. If $J = \varnothing$, then, by UNA, $f(x_I) = 0 = \delta(x_I)$. If $J \neq \varnothing$, then let $r \in J$ be the dictator in J except for $\varnothing$. Since $\delta(x_I) = x_r$, it must be shown that $f(x_I) = x_r$. By RED, for some $k \in \{i, r\}$, $f(x_I) = f(x_{I\setminus\{i,r\}}, f(x_{\{i,r\}})^k)$. By the induction hypothesis, $f(x_{I\setminus\{i,r\}}, f(x_{\{i,r\}})^k) = \delta(x_{I\setminus\{i,r\}}, \delta(x_{\{i,r\}})^k) = \delta(x_{I\setminus\{i,r\}}, (x_r)^k)$. If $k = i$, then $\delta(x_{I\setminus\{i,r\}}, (x_r)^k) = x_r$ follows from the fact that i dictator in I implies i dictator in $I\setminus\{r\}$. And if $k = r$, then $\delta(x_{I\setminus\{i,r\}}, (x_r)^k) = \delta(x_{J\setminus\{r\}}, (x_r)^k) = x_r$ because r is the dictator in J.■

The proof of Proposition 3.2 relies on Lemmas 1–7 stated next. For three given individuals i, j and k, let $i \equiv j$ abbreviate $i \equiv_{jk} j$, so $i \equiv j$ means that i can replace j in the society $\{j, k\}$.

Lemma 1. SUB implies that (a) $i \equiv j$ and $i \equiv k$, or (b) $j \equiv i$ and $j \equiv k$, or (c) $k \equiv i$ and $k \equiv j$.

Proof. By SUB, $i \equiv j$ or $i \equiv k$. If $i \equiv j$, then, by SUB, $k \equiv i$ or $k \equiv j$. If $i \equiv k$, then, by SUB, $j \equiv i$ or $j \equiv k$. Therefore, four cases arise: (i) $i \equiv j$ and $k \equiv i$; (ii) $i \equiv j$ and $k \equiv j$; (iii) $i \equiv k$ and $j \equiv i$; and (iv) $i \equiv k$ and $j \equiv k$. But (i) and (iii) represent the same case, since $i \equiv j$ is equivalent to $j \equiv i$ and $k \equiv i$ is equivalent to $i \equiv k$.■

Lemma 2. Let $f : X \rightarrow \{-1, 0, 1\}$ be a social welfare function satisfying RED and SUB. Let $a \in \{-1, 0 ,1\}$, $b \in \{-1, 0, 1\}\setminus\{a\}$, $i \in \mathbb{N}$ and $j \in \mathbb{N}\setminus\{i\}$ satisfy $f(a^i, b^j) = f(b^i, a^j)$. Then, for all $k \in \mathbb{N}\setminus\{i, j\}$, $f(a^k, b^j) = f(b^k, a^j) = f(a^i, b^j) = f(a^i, b^k) = f(b^i, a^k)$.

Proof. Suppose $f(a^i, b^j) = f(b^i, a^j)$, where $a \neq b$. By Lemma 1, there are three cases. Case 1: $i \equiv j$ and $i \equiv k$. In this case, $i \equiv k$ and $f(a^i, b^j) = f(b^i, a^j)$ imply $f(a^k, b^j) = f(a^i, b^j) = f(b^i, a^j) = f(b^k, a^j)$. Moreover, $f(a^k, b^j) = f(b^k, a^j)$ and $i \equiv j$ imply $f(a^k, b^i) = f(a^k, b^j) = f(b^k, a^j) = f(b^k, a^i)$. Case 2: $j \equiv i$ and $j \equiv k$. Now, $j \equiv k$ implies $f(a^i, b^k) = f(a^i, b^j) = f(b^i, a^j) = f(b^i, a^k)$. Since $f(a^i, b^k) = f(b^i, a^k)$, $j \equiv i$ implies $f(a^j, b^k) = f(a^i, b^k) = f(b^i, a^k) = f(b^j, a^k)$. Case 3: $k \equiv i$ and $k \equiv j$. It follows from $k \equiv i$ and $f(a^i, b^j) = f(b^i, a^j)$ that $f(a^k, b^j) = f(a^i, b^j) = f(b^i, a^j) = f(b^k, a^j)$. As $f(a^i, b^j) = f(b^i, a^j)$, $k \equiv j$ implies $f(a^i, b^k) = f(a^i, b^j) = f(b^i, a^j) = f(b^i, a^k)$.■

Lemma 3. Let $f : X \rightarrow \{-1, 0, 1\}$ be a social welfare function satisfying RED and SUB. Then, for all $i \in \mathbb{N}$ and $j \in \mathbb{N}\setminus\{i\}$, it cannot be that $f(1^i, -1^j) = 1 \neq f(0^i, -1^j)$, $f(1^i, 0^j) = 0$ and $f(-1^i, 0^j) = -1$.

Proof. Assume $f(1^i, -1^j) = 1 \neq f(0^i, -1^j)$, $f(1^i, 0^j) = 0$ and $f(-1^i, 0^j) = -1$. Let $k \in \mathbb{N}\setminus\{i, j\}$. By Lemma 1, there are three cases. Case 1: $i \equiv j$ and $i \equiv k$. By RED, there are $r \in \{i, j\}$ and $s \in \{j, k\}$ such that $f(f(-1^i, 0^j)^r, 1^k) = f(-1^i, 0^j, 1^k) = f(f(0^j, 1^k)^s, -1^i)$. Since $i \equiv k$, $f(0^j, 1^k) = f(0^j, 1^i) = 0$. Hence, $f(-1^i, 0^j) = -1$ implies $f(-1^r, 1^k) = f(0^s, -1^i)$. On the one hand, $r \in \{i, j\}$ and $i \equiv j$ imply $f(-1^r, 1^k) = f(-1^j, 1^k)$. Given this, $i \equiv k$ implies $f(-1^j, 1^k) = f(-1^j, 1^i) = 1$. To sum up, $f(-1^r, 1^k) = 1$. On the other hand, if $s = j$, then $f(0^s, -1^i) = f(0^j, -1^i) =$

$-1 \neq 1 = f(-1^r, 1^k)$: contradiction. And if $s = k$, then $i \equiv j$ implies $f(0^s, -1^i) = f(0^k, -1^i) = f(0^k, -1^j)$, whereas $i \equiv k$ implies that $f(0^k, -1^j) = f(0^i, -1^j) \neq 1 = f(-1^r, 1^k)$: contradiction.

Case 2: $j \equiv i$ and $j \equiv k$. By RED, there are $r \in \{i, j\}$ and $s \in \{j, k\}$ such that $f(f(1^i, -1^j)^r, 0^k) = f(1^i, -1^j, 0^k) = f(f(-1^j, 0^k)^s, 1^i)$. It follows from $j \equiv i$ that $f(-1^j, 0^k) = f(-1^i, 0^k)$, and from $j \equiv k$ that $f(-1^i, 0^k) = f(-1^i, 0^j) = -1$. Since $f(1^i, -1^j) = 1$, $f(1^r, 0^k) = f(-1^s, 1^i)$. On the one hand, $r \in \{i, j\}$ and $j \equiv i$ imply $f(1^r, 0^k) = f(1^i, 0^k)$. Given this, $j \equiv k$ implies $f(1^i, 0^k) = f(1^i, 0^j) = 0$, so $f(1^r, 0^k) = 0$. On the other hand, $s \in \{j, k\}$ and $j \equiv k$ imply $f(-1^s, 1^i) = f(-1^j, 1^i) = 1 \neq 0 = f(1^r, 0^k)$: contradiction

Case 3: $k \equiv i$ and $k \equiv j$. By RED, there are $r \in \{i, k\}$ and $s \in \{j, k\}$ such that $f(f(1^i, -1^k)^r, 0^j) = f(1^i, 0^j, -1^k) = f(f(0^j, -1^k)^s, 1^i)$. Since $k \equiv j$, $f(1^i, -1^k) = f(1^i, -1^j) = 1$. And since $k \equiv i$, $f(0^j, -1^k) = f(0^j, -1^i) = -1$. Therefore, $f(1^r, 0^j) = f(-1^s, 1^i)$. On the one hand, $r \in \{i, k\}$ and $k \equiv i$ imply $f(1^r, 0^j) = f(1^i, 0^j) = 0$. And, on the other, $s \in \{j, k\}$ and $k \equiv j$ imply $f(-1^s, 1^i) = f(-1^j, 1^i) = 1$: contradiction.■

Lemma 4. Let $f : X \rightarrow \{-1, 0, 1\}$ be a social welfare function satisfying RED and SUB. Then, for all $i \in \mathbb{N}$ and $j \in \mathbb{N}\backslash\{i\}$, it cannot be that $f(1^i, -1^j) = f(0^i, 1^j) = 1$, $f(-1^i, 1^j) = f(0^i, -1^j) = -1$ and $f(1^i, 0^j) = f(-1^i, 0^j) = 0$.

Proof. Assume $f(1^i, -1^j) = f(0^i, 1^j) = 1$, $f(-1^i, 1^j) = f(0^i, -1^j) = -1$ and $f(1^i, 0^j) = f(-1^i, 0^j) = 0$. Let $k \in \mathbb{N}\backslash\{i, j\}$. By Lemma 1, there are three cases. Case 1: $i \equiv j$ and $i \equiv k$. By RED, there are $r \in \{i, k\}$ and $s \in \{i, j\}$ such that $f(f(0^i, -1^k)^r, 1^j) = f(0^i, 1^j, -1^k) = f(f(0^i, 1^j)^s, -1^k)$. On the one hand, $i \equiv j$ implies $f(0^i, -1^k) = f(0^j, -1^k)$, whereas $i \equiv k$ implies $f(0^j, -1^k) = f(0^j, -1^i) = 0$, so $f(f(0^i, -1^k)^r, 1^j) = f(0^r, 1^j)$. On the other hand, $f(0^i, 1^j) = 1$ implies $f(f(0^i, 1^j)^s, -1^k) = f(1^s, -1^k)$. Thus, $f(0^r, 1^j) = f(1^s, -1^k)$. As $r \in \{i, k\}$ and $i \equiv k$, $f(0^r, 1^j) = f(0^i, 1^j) = 1$. In addition, $s \in \{i, j\}$ and $i \equiv j$ imply $f(1^s, -1^k) = f(1^j, -1^k)$. Since $i \equiv k$, $f(1^j, -1^k) = f(1^j, -1^i) = -1 \neq 1 = f(0^r, 1^j)$: contradiction.

Case 2: $j \equiv i$ and $j \equiv k$. By RED, there are $r \in \{i, j\}$ and $s \in \{j, k\}$ such that $f(f(-1^i, 0^j)^r, 1^k) = f(-1^i, 0^j, 1^k) = f(f(0^j, 1^k)^s, -1^i)$. As $f(-1^i, 0^j) = 0$, $f(f(-1^i, 0^j)^r, 1^k) = f(0^r, 1^k)$. Moreover, $j \equiv i$ implies $f(0^j, 1^k) = f(0^i, 1^k)$, whereas $j \equiv k$ implies $f(0^i, 1^k) = f(0^i, 1^j) = 1$. In view of this, $f(f(0^j, 1^k)^s, -1^i) = f(1^s, -1^i)$. Summing up, $f(0^r, 1^k) = f(1^s, -1^i)$. Given that $j \equiv i$ and $r \in \{i, j\}$, $f(0^r, 1^k) = f(0^i, 1^k)$. And since $j \equiv k$, $f(0^i, 1^k) = f(0^i, 1^j) = 1$. On the other hand, $j \equiv k$ and $s \in \{j, k\}$ imply $f(1^s, -1^i) = f(1^j, -1^i) = -1$: contradiction.

Case 3: $k \equiv i$ and $k \equiv j$. By RED, there are $r \in \{i, k\}$ and $s \in \{j, k\}$ such that $f(f(1^i, 0^k)^r, -1^j) = f(1^i, -1^j, 0^k) = f(f(-1^j, 0^k)^s, 1^i)$. Given that $k \equiv i$, $f(-1^j, 0^k) = f(-1^j, 0^i) = -1$. Since k

$\equiv j$, $f(1^i, 0^k) = f(1^i, 0^j) = 0$. As a result, $f(0^r, -1^j) = f(-1^s, 1^i)$. It follows from $r \in \{i, k\}$ and $k \equiv i$ that $f(0^r, -1^j) = f(0^i, -1^j) = -1$. Similarly, $s \in \{j, k\}$ and $k \equiv j$ imply $f(-1^s, 1^i) = f(-1^j, 1^i) = 1 \neq -1 = f(0^r, -1^j)$: contradiction.■

Lemma 5. Let $f : X \to \{-1, 0, 1\}$ be a social welfare function satisfying RED and SUB. Then, for all $i \in \mathbb{N}$ and $j \in \mathbb{N}\backslash\{i\}$, it cannot be that $f(1^i, -1^j) = f(1^i, 0^j) = 1$, $f(-1^i, 1^j) = f(0^i, -1^j) = -1$ and $f(0^i, 1^j) = f(-1^i, 0^j) = 0$.

Proof. Assume $f(1^i, -1^j) = f(1^i, 0^j) = 1$, $f(-1^i, 1^j) = f(0^i, -1^j) = -1$ and $f(0^i, 1^j) = f(-1^i, 0^j) = 0$. Let $k \in \mathbb{N}\backslash\{i, j\}$. By Lemma 1, there are three cases. Case 1: $i \equiv j$ and $i \equiv k$. By RED, there are $r \in \{i, k\}$ and $s \in \{i, j\}$ such that $f(f(1^i, -1^k)^r, 0^j) = f(1^i, 0^j, -1^k) = f(f(1^i, 0^j)^s, -1^k)$. It follows from $i \equiv j$ that $f(1^i, -1^k) = f(1^j, -1^k)$ and from $i \equiv k$ that $f(1^j, -1^k) = f(1^j, -1^i) = -1$. Hence, $f(f(1^i, -1^k)^r, 0^j) = f(-1^r, 0^j)$. Since $r \in \{i, k\}$ and $i \equiv k$, $f(-1^r, 0^j) = f(-1^i, 0^j) = 0$. As a result, $0 = f(f(1^i, 0^j)^s, -1^k) = f(1^s, -1^k)$. But $s \in \{i, j\}$ and $i \equiv j$ imply $f(1^s, -1^k) = f(1^j, -1^k)$, whereas $i \equiv k$ implies $f(1^j, -1^k) = f(1^j, -1^i) = -1$: contradiction.

Case 2: $j \equiv i$ and $j \equiv k$. By RED, there are $r \in \{i, j\}$ and $s \in \{j, k\}$ such that $f(f(0^i, 1^j)^r, -1^k) = f(0^i, 1^j, -1^k) = f(f(1^j, -1^k)^s, 0^i)$. That is, $f(0^r, -1^k) = f(f(1^j, -1^k)^s, 0^i)$. On the one hand, $r \in \{i, j\}$ and $j \equiv i$ imply $f(0^r, -1^k) = f(0^i, -1^k)$, whereas $j \equiv k$ implies $f(0^i, -1^k) = f(0^i, -1^j) = -1$. On the other hand, $j \equiv i$ implies $f(1^j, -1^k) = f(1^i, -1^k)$, whereas $j \equiv k$ implies $f(1^i, -1^k) = f(1^i, -1^j) = 1$. Hence, $f(f(1^j, -1^k)^s, 0^i) = f(1^s, 0^i)$. As $s \in \{j, k\}$ and $j \equiv k$, $f(1^s, 0^i) = f(1^j, 0^i) = 0$. Summing up, $-1 = f(0^r, -1^k) = f(f(1^j, -1^k)^s, 0^i) = 0$: contradiction.

Case 3: $k \equiv i$ and $k \equiv j$. By RED, there are $r \in \{i, j\}$ and $s \in \{j, k\}$ such that $f(f(-1^i, 0^j)^r, 1^k) = f(-1^i, 0^j, 1^k) = f(f(0^j, 1^k)^s, -1^i)$. Thus, $f(0^r, 1^k) = f(f(0^j, 1^k)^s, -1^i)$. Since $k \equiv i$, $f(0^j, 1^k) = f(0^j, 1^i) = 1$. Therefore, $f(0^r, 1^k) = f(1^s, -1^i)$. As $k \equiv j$ and $s \in \{j, k\}$, $f(1^s, -1^i) = f(1^j, -1^i) = -1$. Accordingly, $f(0^r, 1^k) = -1$. If $r = j$, then $k \equiv i$ implies $f(0^r, 1^k) = f(0^j, 1^k) = f(0^j, 1^i) = 1$: contradiction. If $r = i$, then $k \equiv j$ implies $f(0^r, 1^k) = f(0^i, 1^k) = f(0^i, 1^j) = 0$: contradiction.■

Lemma 6. Let $f : X \to \{-1, 0, 1\}$ be a social welfare function satisfying UNA, RED, SUB, EXC and PAR. Let $i \in \mathbb{N}$ and $j \in \mathbb{N}\backslash\{i\}$. If $f(1^i, -1^j) = 0$, then, for all preference profiles $x_{\{i, j\}}$ for $\{i, j\}$, $f(x_{\{i, j\}}) = \mu(x_{\{i, j\}})$ or, for all preference profiles $x_{\{i, j\}}$ for $\{i, j\}$, $f(x_{\{i, j\}}) = \upsilon(x_{\{i, j\}})$.

Proof. Suppose $f(1^i, -1^j) = 0$. By EXC, $f(-1^i, 1^j) = 0$. By UNA, for all $a \in \{-1, 0, 1\}$, $f(a^i, a^j) = a$. Therefore, for every preference profile $x_{\{i, j\}}$ for $\{i, j\}$ in the set $\{(1^i, -1^j),$

$(-1^i, 1^j)$, $(1^i, 1^j)$, $(-1^i, -1^j)$, $(0^i, 0^j)\}$, $f(x_{\{i,j\}}) = \mu(x_{\{i,j\}}) = \upsilon(x_{\{i,j\}})$. The remaining four preference profiles for $\{i, j\}$ are $(1^i, 0^j)$, $(0^i, 1^j)$, $(-1^i, 0^j)$ and $(0^i, -1^j)$.

- Part 1: for all $k \in \mathbb{N}\backslash\{i, j\}$ and $a \in \{-1, 1\}$, $f(a^k, -a^i) = f(a^k, -a^j) = 0$. This follows from $f(1^i, -1^j) = f(-1^i, 1^j) = 0$ and Lemma 2.

- Part 2: for all $a \in \{-1, 1\}$, $f(a^i, 0^j) \in \{0, a\}$. Suppose not: $f(a^i, 0^j) = -a$, where $a \in \{-1, 1\}$. Let $k \in \mathbb{N}\backslash\{i, j\}$. By RED, there are $r \in \{i, j\}$ and $s \in \{i, k\}$ such that $f(f(a^i, 0^j)^r, -a^k) = f(a^i, 0^j, -a^k) = f(f(a^i, -a^k)^s, 0^j)$. Since $f(a^i, 0^j) = -a$, by UNA, $f(f(a^i, 0^j)^r, -a^k) = f(-a^r, -a^k) = -a$. As a consequence, $f(f(a^i, -a^k)^s, 0^j) = -a$. By Part 1, $f(a^i, -a^k) = 0$. By UNA, $f(f(a^i, -a^k)^s, 0^j) = f(0^s, 0^j) = 0$: contradiction.

- Part 3: for all $a \in \{-1, 1\}$, $f(a^j, 0^i) \in \{0, a\}$. Permute i and j in the proof of Part 2.

- Part 4: $f(1^i, 0^j) = 1$ implies that, for all preference profiles $x_{\{i,j\}}$ for $\{i, j\}$, $f(x_{\{i,j\}}) = \mu(x_{\{i,j\}})$. Suppose $f(1^i, 0^j) = 1$. By Part 3, $f(0^i, 1^j) \in \{0, 1\}$. If $f(0^i, 1^j) = 1$, then, by PAR, $f(-1^i, 0^j) = f(0^i, -1^j) = -1$. As a result, for all preference profiles $x_{\{i,j\}}$ for $\{i, j\}$, $f(x_{\{i,j\}}) = \mu(x_{\{i,j\}})$. Hence, the proof amounts to obtaining a contradiction from the other possibility, $f(0^i, 1^j) = 0$. To this end, assume $f(0^i, 1^j) = 0$. By Part 2, $f(-1^i, 0^j) \in \{0, -1\}$. Case 1: $f(-1^i, 0^j) = -1$. By PAR, $f(0^i, -1^j) = 0$. By Lemma 1, there are three possibilities. Case 1a: $i \equiv j$ and $i \equiv k$. By RED, there are $r \in \{i, j\}$ and $s \in \{i, k\}$ such that $f(f(1^i, -1^j)^r, -1^k) = f(1^i, -1^j, -1^k) = f(f(1^i, -1^k)^s, -1^j)$. By Part 1, $f(1^i, -1^j) = 0 = f(1^i, -1^k)$. Therefore, $f(0^r, -1^k) = f(0^s, -1^j)$. Since $r \in \{i, j\}$ and $i \equiv j$, $f(0^r, -1^k) = f(0^j, -1^k)$. It then follows from $i \equiv k$ that $f(0^j, -1^k) = f(0^j, -1^i) = -1$. On the other hand, $s \in \{i, k\}$ and $i \equiv k$ imply $f(0^s, -1^j) = f(0^i, -1^j) = 0$: contradiction.

Case 1b: $j \equiv i$ and $j \equiv k$. By RED, there are $r \in \{i, j\}$ and $s \in \{j, k\}$ such that $f(f(-1^i, 1^j)^r, -1^k) = f(-1^i, 1^j, -1^k) = f(f(1^j, -1^k)^s, -1^i)$. By Part 1, $f(1^i, -1^j) = 0 = f(1^j, -1^k)$. Therefore, $f(0^r, -1^k) = f(0^s, -1^i)$. Since $r \in \{i, j\}$ and $j \equiv i$, $f(0^r, -1^k) = f(0^i, -1^k)$. It then follows from $j \equiv k$ that $f(0^i, -1^k) = f(0^i, -1^j) = 0$. On the other hand, $s \in \{j, k\}$ and $j \equiv k$ imply $f(0^s, -1^i) = f(0^j, -1^i) = -1$: contradiction.

Case 1c: $k \equiv i$ and $k \equiv j$. By RED, there are $r \in \{i, k\}$ and $s \in \{j, k\}$ such that $f(f(-1^i, 1^k)^r, -1^j) = f(-1^i, -1^j, 1^k) = f(f(-1^j, 1^k)^s, -1^i)$. By Part 1, $f(1^i, -1^k) = 0 = f(1^j, -1^k)$. Therefore, $f(0^r, -1^j) = f(0^s, -1^i)$. Since $r \in \{i, k\}$ and $k \equiv i$, $f(0^r, -1^j) = f(0^i, -1^j) = 0$. On the other hand, $s \in \{j, k\}$ and $j \equiv k$ imply $f(0^s, -1^i) = f(0^j, -1^i) = -1$: contradiction.

Case 2: $f(-1^i, 0^j) = 0$. By PAR, $f(0^i, -1^j) = -1$. By Lemma 1, there are three possibilities, which are handled in exactly the same way as the three possibilities in case 1, with the only difference that $f(0^j, -1^i) = 0$ (instead of -1) and $f(0^i, -1^j) = -1$ (instead of 0).

• Part 5: $f(1^i, 0^j) = 0$ implies that, for all preference profiles $x_{\{i,j\}}$ for $\{i, j\}$, $f(x_{\{i,j\}}) = \upsilon(x_{\{i,j\}})$. Suppose $f(1^i, 0^j) = 0$. By Part 3, $f(0^i, 1^j) \in \{0, 1\}$. If $f(0^i, 1^j) = 0$, then, by Part 2, $f(-1^i, 0^j) \in \{-1, 0\}$. If $f(-1^i, 0^j) = -1$, then, by PAR, $f(0^i, -1^j) = 1$, which contradicts Part 3. In view of this, $f(-1^i, 0^j) = 0$ and, by PAR, $f(0^i, -1^j) = 0$. The conclusion is then that, for all preference profiles $x_{\{i,j\}}$ for $\{i, j\}$, $f(x_{\{i,j\}}) = \upsilon(x_{\{i,j\}})$. Accordingly, the proof amounts to reaching a contradiction from the remaining possibility, $f(0^i, 1^j) = 1$. So assume $f(0^i, 1^j) = 1$. By Part 2, $f(-1^i, 0^j) \in \{0, -1\}$. Case 1: $f(-1^i, 0^j) = 0$. By PAR, $f(0^i, -1^j) = -1$. The present case 1 is symmetric to case 1 in Part 4, the only difference being that, in the previous case, the value of f in profiles $(1^i, 0^j)$, $(0^i, 1^j)$, $(-1^i, 0^j)$ and $(0^i, -1^j)$ coincided with i's preferences, whereas now it coincides with j's. So just rewrite the proof of case 1 in Part 4 permuting i and j. Case 2: $f(-1^i, 0^j) = -1$. By PAR, $f(0^i, -1^j) = -0$. The remark made in case 2 of Part 4 applies: the rewritten proof for case 1 in this part is valid for case 2, with the only difference that now $f(0^j, -1^i) = -1$ (instead of 0) and $f(0^i, -1^j) = 0$ (instead of -1).■

Lemma 7. Let $f : X \to \{-1, 0, 1\}$ be a social welfare function satisfying UNA, RED, SUB, EXC and PAR. Let $i \in \mathbb{N}$ and $j \in \mathbb{N}\backslash\{i\}$. If, for some $a \in \{-1, 1\}$, $f(a^i, -a^j) = a$, then: (i) i is a strong dictator in $\{i, j\}$; or (ii) i is a dictator in $\{i, j\}$ except for $\varnothing$, with j being a dictator in $\{i, j\}$ except for $\{i\}$.

Proof. Consider the case $a = 1$ (to prove the case $a = -1$, replace "i" with "j" and "1" with "-1" in the proof presented next). Suppose $f(1^i, -1^j) = 1$. With $I = \{i, j\}$, there are another 8 preference profiles for I. By UNA, $f(1^i, 1^j) = 1$, $f(0^i, 0^j) = 0$ and $f(-1^i, -1^j) = -1$. Therefore, only five profiles rest to be considered: $(-1^i, 1^j)$, $(1^i, 0^j)$, $(0^i, 1^j)$, $(-1^i, 0^j)$ and $(0^i, -1^j)$. By EXC, $f(-1^i, 1^j) = 0$ would imply $f(1^i, -1^j) = 0$, contradicting the assumption that $f(1^i, -1^j) = 1$. As a consequence, $f(-1^i, 1^j) \in \{1, -1\}$. Let $k \in \mathbb{N}\backslash\{i, j\}$.

Case 1: $f(-1^i, 1^j) = 1$. If $f(1^i, 0^j) = 1$, by PAR, $f(0^i, 1^j) = f(-1^i, 0^j) = f(0^i, -1^j) = -1$. But, by EXC, $f(0^i, 1^j) = -1$ implies $f(1^i, 0^j) = -1$: contradiction. As a result, $f(1^i, 0^j) \in \{-1. 0\}$. Case 1a: $f(1^i, 0^j) = -1$. By EXC, $f(0^i, 1^j) = -1$. By RED, there are $r \in \{i, j\}$ and $s \in \{j, k\}$ such that $f(f(1^i, 0^j)^r, -1^k) = f(1^i, 0^j, -1^k) = f(f(0^j, -1^k)^s, 1^i)$. Since $f(1^i, 0^j) = -1$, by UNA, $f(f(1^i, 0^j)^r, -1^k) = f(-1^r, -1^k) = -1$. Therefore, $f(f(0^j, -1^k)^s, 1^i) = -1$. Case 1a1: $f(0^j, -1^k) = 1$. By UNA, $f(f(0^j, -1^k)^s, 1^i) = f(1^s, 1^i) = 1$: contradiction. Case 1a2: $f(0^j, -1^k) = -1$. In this case, $f(f(0^j, -1^k)^s, 1^i) = f(-1^s, 1^i)$. Lemma 2 and $f(1^i, -1^j) = f(-1^i, 1^j) = 1$ imply $f(1^i,$

$-1^k) = 1$. This and $s \in \{j, k\}$ imply $f(-1^s, 1^i) = 1$. As a result, $f(f(0^j, -1^k)^s, 1^i) = 1$: contradiction. Case 1a3: $f(0^j, -1^k) = 0$. By Lemma 2, $f(1^i, -1^j) = f(-1^i, 1^j) = 1$ and $f(1^i, 0^j) = f(0^i, 1^j) = -1$ imply $f(1^j, -1^k) = f(-1^j, 1^k) = 1$ and $f(1^j, 0^k) = f(0^j, 1^k) = -1$. Hence, by PAR, $f(0^j, -1^k) = 0$ implies $f(-1^j, 0^k) = 0$. By Lemma 2, any individual can replace both j and k in any preference profile for $\{j, k\}$. This makes f symmetric with respect to the preference profiles for societies with three individuals.

Consider the 27 preference profiles for $I = \{i, j, k\}$. By UNA, RED and the symmetry between i, j and k, f assigns the value 1 to the seven profiles in which at least two individuals have preference 1 (for instance, $f(1^i, -1^j, 1^k) = f(f(1^i, -1^j)^i, 1^k) = f(1^i, 1^k) = 1$) and to the three profiles in which two individuals have preference -1 and the remaining individual has preference 1 (for instance, $f(1^i, -1^j, -1^k) = f(f(1^i, -1^j)^i, -1^k) = f(1^i, -1^k) = 1$). Further, f assigns value 0 to also ten profiles: the seven profiles in which at least two individuals have preference 0 (for instance, $f(1^i, 0^j, 0^k) = f(f(1^i, 0^j)^i, 0^k) = f(-1^i, 0^k) = 0$) and the three profiles in which two individuals have preference -1 and the remaining individual has preference 0 (for instance, $f(0^i, -1^j, -1^k) = f(f(0^i, -1^j)^i, -1^k) = f(0^i, -1^k) = 0$). Finally, f assigns value -1 to the profile in which all the individuals have preference -1 and to the six profiles in which all the individuals have different preferences (for instance, $f(0^i, -1^j, 1^k) = f(0^i, 1^k) = -1$). Summarizing, $\pi_1{}^I = 10/27$ and $\pi_{-1}{}^I = 7/27$, which contradicts PAR.

Case 1b: $f(1^i, 0^j) = 0$. By EXC, $f(0^i, 1^j) \neq -1$. By PAR, $f(0^i, 1^j) \neq 1$. Thus, $f(0^i, 1^j) = 0$. By PAR, $f(-1^i, 0^j) = f(0^i, -1^j) = -1$. This cannot be by Lemma 3.

Case 2: $f(-1^i, 1^j) = -1$. Case 2a: $f(1^i, 0^j) = -1$. By EXC, $f(0^i, 1^j) = -1$. By PAR, $f(-1^i, 0^j) = f(0^i, -1^j) = 1$. This and Lemma 2 imply $f(0^j, -1^k) = 1$. By RED, there are $r \in \{i, j\}$ and $s \in \{j, k\}$ such that $f(f(1^i, 0^j)^r, -1^k) = f(1^i, 0^j, -1^k) = f(f(0^j, -1^k)^s, 1^i)$. Since $f(1^i, 0^j) = -1$, by UNA, $f(f(1^i, 0^j)^r, -1^k) = f(-1^r, -1^k) = -1$. On the other hand, by UNA and $f(0^j, -1^k) = 1$, $f(f(0^j, -1^k)^s, 1^i) = f(1^s, 1^i) = 1$: contradiction.

Case 2b: $f(1^i, 0^j) = 0$. By EXC, $f(0^i, 1^j) \neq -1$ and, accordingly, $f(0^i, 1^j) \in \{0, 1\}$. Case 2b1: $f(0^i, 1^j) = 0$. By PAR, $\{f(-1^i, 0^j), f(0^i, -1^j)\} = \{1, -1\}$, which contradicts EXC. Case 2b2: $f(0^i, 1^j) = 1$. By PAR, $\{f(-1^i, 0^j), f(0^i, -1^j)\} = \{0, -1\}$. Lemma 3 proves that $f(-1^i, 0^j) = -1$ and $f(0^i, -1^j) = 0$ cannot be, whereas Lemma 4 shows that $f(-1^i, 0^j) = 0$ and $f(0^i, -1^j) = -1$ cannot be.

Case 2c: $f(1^i, 0^j) = 1$. By EXC, $f(0^i, 1^j) \neq -1$ and, hence, $f(0^i, 1^j) \in \{0, 1\}$. Case 2c1: $f(0^i, 1^j) = 0$. By PAR, $\{f(-1^i, 0^j), f(0^i, -1^j)\} = \{0, -1\}$. By Lemma 5, $f(-1^i, 0^j) = 0$ and $f(0^i,$

$-1^j) = -1$ cannot be. On the other hand, if $f(-1^i, 0^j) = -1$ and $f(0^i, -1^j) = 0$, then i is a strong dictator in $\{i, j\}$. Case 2c2: $f(0^i, 1^j) = 1$. By PAR, $f(-1^i, 0^j) = f(0^i, -1^j) = -1$. This makes i a dictator in $\{i, j\}$ except for $\varnothing$ and makes j a dictator in $\{i, j\}$ except for $\{i\}$.■

Proof of Proposition 3.2. By Remark 2.7, the four types of rules satisfy UNA, RED, SUB, EXC and PAR. Conversely, let f satisfy UNA, RED, SUB, EXC and PAR. Let $i \in \mathbb{N}$ and $j \in \mathbb{N}\backslash\{i\}$. Case 1: $f(1^i, -1^j) = 0$. By Lemma 6, either for all preference profiles $x_{\{i,j\}}$ for $\{i, j\}$, $f(x_{\{i,j\}}) = \upsilon(x_{\{i,j\}})$ or, for all preference profiles $x_{\{i,j\}}$ for $\{i, j\}$, $f(x_{\{i,j\}}) = \mu(x_{\{i,j\}})$. In both cases, f is symmetric on the set of preference profiles for $\{i, j\}$. In view of this, by Lemma 2, for all $k \in \mathbb{N}$ and $r \in \mathbb{N}\backslash\{k\}$, $k \equiv r$ and $r \equiv k$. Hence, either $f = \upsilon$ on X_2 or $f = \mu$ on X_2.

Case 2: $f(1^i, -1^j) = 1$. If, for some $r \in \mathbb{N}$ and $s \in \mathbb{N}\backslash\{r\}$, $f(1^r, -1^s) = 0$, then, by case 1, $f(1^i, -1^j) = 0$, contradicting the assumption that $f(1^i, -1^j) = 1$. Therefore, for all $r \in \mathbb{N}$ and $s \in \mathbb{N}\backslash\{r\}$, $f(1^r, -1^s) \neq 0$. Hence, by Lemma 7, for all $r \in \mathbb{N}$ and $s \in \mathbb{N}\backslash\{r\}$, either there is a dictator in $\{r, s\}$ or there is a strong dictator in $\{r, s\}$. Define binary relation $\Rightarrow$ on $\mathbb{N}$ to be such that $r \Rightarrow s$ if and only if r is a dictator or a strong dictator in $\{r, s\}$. Consequently, for all $r \in \mathbb{N}$ and $s \in \mathbb{N}\backslash\{r\}$, either $r \Rightarrow s$ or $s \Rightarrow r$. Given this, for $\Rightarrow$ to be a linear order it is enough for $\Rightarrow$ to be transitive. To show this, suppose not: $r \Rightarrow s$, $s \Rightarrow t$ and $t \Rightarrow r$. By SUB, $t \equiv r$ or $t \equiv s$. If $t \equiv r$, then $r \Rightarrow s$ implies $t \Rightarrow s$, which contradicts $s \Rightarrow t$. And if $t \equiv s$, then $r \Rightarrow s$ implies $r \Rightarrow t$, which contradicts $t \Rightarrow r$.

The proof concludes by showing that $\Rightarrow$ is the linear order generating either a dictatorial rule or a strongly dictatorial rule. That is, either there is a dictator in every society with two members or there is a strong dictator in every such society. To prove this, it is enough to show that if r is a dictator (strong dictator) in $\{r, s\}$, then, for all $t \in \mathbb{N}\backslash\{r, s\}$, there is some dictator (strong dictator) in $\{r, t\}$ and $\{s, t\}$. Hence, every society $\{r, s\}$ has a dictator (in which case f is dictatorial) or every such society has a strong dictator (in which case f is strongly dictatorial).

Let $r\,D\,s$ abbreviate "r is a dictator in $\{r, s\}$" (the following reasoning is also valid if $r\,D\,s$ means "r is a strong dictator in $\{r, s\}$"). Suppose $r\,D\,s$. Choose $t \in \mathbb{N}\backslash\{r, s\}$. It must be shown that ($t\,D\,r$ or $r\,D\,t$) and ($t\,D\,s$ or $s\,D\,t$). By Lemma 1, there are three cases. Case 2a: $r \equiv s$ and $r \equiv t$. Then $r\,D\,s$ and $r \equiv t$ imply $t\,D\,s$. This and $r \equiv s$ imply $t\,D\,r$. As a result, $t \Rightarrow r \Rightarrow s$. Case 2b: $s \equiv r$ and $s \equiv t$. Then $r\,D\,s$ and $s \equiv t$ imply $r\,D\,t$. This and $s \equiv r$ imply $s\,D\,t$. In this case, $r \Rightarrow s \Rightarrow t$. Case 2c: $t \equiv r$ and $t \equiv s$. Then $r\,D\,s$ and $t \equiv r$ imply $t\,D\,s$. In addition, $r\,D\,s$ and $t \equiv s$ imply $r\,D\,t$. Now, $r \Rightarrow t \Rightarrow s$.■

Proof of Proposition 3.3. Proposition 3.3 is an immediate consequence of Proposition 3.2 given that: (i) the majority rule satisfies RES, COM, WRR and SIR; (ii) the unanimity rule satisfies ¬RES, ¬WRR and ¬SIR; (iii) every dictatorial rule satisfies ¬COM, WRR and SIR; and (iv) every strongly dictatorial rule satisfies RES, COM, ¬WRR and ¬SIR.■

Proof of Proposition 3.4. Proposition 3.4 is an immediate consequence of Proposition 3.3(i) given that the majority rule satisfies (1), (2) and (3), and given that any strongly dictatorial rule satisfies ¬(1), ¬(2) and ¬(3).■

Proof of Proposition 3.5. Proposition 3.5 is an immediate consequence of Proposition 3.2 given that: (i) the majority rule and the unanimity rules satisfy WAN; and (ii) every dictatorial rule and every strongly dictatorial rule satisfy ¬WAN.■

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